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THE MKEPD FRAMEWORK: ENHANCING FINANCIAL RISK ANALYSIS WITH MULTIVARIATE DISTRIBUTION MODELING

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ABSTRACT
This study aims to formulate a new probability distribution, called the Kumaraswamy Exponential Pareto distribution (KEPD), from the Exponential Pareto distribution (EP). This distribution was designed to be suitable for fitting real-life data by utilizing the Kumaraswamy family to create a novel continuous probability distribution approach. This study derived some properties of this new distribution and conducted a simulation study using different parameter combinations. The results of the simulation study demonstrated the impact of additional parameters on the suggested distribution. In real-life data applications, the suggested distribution exhibits a better fit than the existing Kumaraswamy Exponentiated Pareto Distribution (KEPD), Exponential Pareto Distribution (EP), and Exponential Distribution (Exp). The ability to capture the heavy tails and skewness inherent in financial data allows for more accurate and robust modeling of financial risks and returns, providing a valuable tool for financial analysts and risk managers. The improved fit over traditional distributions underscores the potential of KEPD in enhancing financial modeling techniques, contributing to more effective decision-making in financial markets.

KEYWORDS: Kumaraswamy, Exponential Pareto, financial modeling techniques, Simulation, Properties, Pareto, financial risks
MSC: 92B05, 93D20

RESUMEN
Este estudio tiene como objetivo formular una nueva distribución de probabilidad, denominada distribución exponencial de Pareto de Kumaraswamy (KEPD), a partir de la distribución exponencial de Pareto (EP). Esta distribución fue diseñada para ser adecuada para ajustar datos de la vida real utilizando la familia Kumaraswamy para crear un nuevo enfoque de distribución de probabilidad continua. En este estudio se derivaron algunas propiedades de esta nueva distribución y se realizó un estudio de simulación utilizando diferentes combinaciones de parámetros. Los resultados del estudio de simulación demostraron el impacto de parámetros adicionales en la distribución sugerida. En aplicaciones de datos de la vida real, la distribución sugerida exhibe un mejor ajuste que la distribución de Pareto exponenciada de Kumaraswamy (KEPD) existente,
Distribución exponencial de Pareto (EP) y distribución exponencial (exp). La capacidad de capturar las colas pesadas y la asimetría inherentes a los datos financieros permite un modelado más preciso y sólido de los riesgos financieros y los rendimientos, lo que proporciona una herramienta valiosa para los analistas financieros y los gestores de riesgos. El ajuste mejorado con respecto a las distribuciones tradicionales subraya el potencial de KEPD para mejorar las técnicas de modelado financiero, contribuyendo a una toma de decisiones más efectiva en los mercados financieros.

PALABRAS CLAVE: Kumaraswamy, Pareto exponencial, técnicas de modelización financiera, Simulación, Propiedades, Pareto, riesgos financieros

1. INTRODUCTION

Statistical distributions hold a fundamental position in both theoretical and practical applications, serving as tools to depict and describe real-world occurrences. As a result of this, statistical distributions and their attributes hold significant significance in numerous domains, including biology, chemistry, and physics, engineering (such as computer science), and social sciences (including economics and political science). [1] Researchers still develop and investigate novel distributions because they want to have more flexibility when fitting data, even though many distributions have been developed and examined over the years.. [6] introduced a novel probability distribution for variables employed in hydrological contexts with lower and upper bounds. This distribution is part of Kumaraswamy's double bounded distribution family, characterized by two positive shape parameters, denoted as 'a' and 'b.' It finds its application in probability and statistics on the closed interval [0, 1]. In many instances, finite-range distributions are employed to represent data in studies related to reliability and life testing.

To broaden the scope beyond traditional distributions such as normal, Weibull, and gamma, [2], introduced a novel family of generalized distributions, denoted by the prefix "Kw," which can be applied to any continuous baseline G distribution. Among the various distributions within this family, the Kw-normal, Kw-Weibull, and Kw-gamma distributions are some noteworthy examples that have been investigated. The constraint of these distributions having support within the range of 0 to 1 was a limitation when generating different classes of distributions in both the beta and Kw-generated families.

A parent continuous distribution with cdf $F(x)$ and pdf $G(x)$ must be considered. The KwG (Kumaraswamy Generalized) distribution can be generated by applying the quantile function to interval $(0, 1)$, as described by [3]. The cumulative distribution function (CDF) F(x) for the Kw-G distribution is defined as:

$$F(x) = 1 - \{1 - G(x)^a\}^b \tag{1}$$

Indeed, in the Kumaraswamy Generalized (Kw-G) distribution, the parameters 'a' and 'b' are both > 0, and they play a crucial role in introducing skewness and controlling the tail weights of the distribution. In addition, the density function for this family of distributions is straightforward and easily expressed as:

$$f(x) = abg(x)G(x)^{a-1}\{1 - G(x)^a\}^{b-1} \tag{2}$$

The Pareto distribution proved valuable for accommodating right-skewed data during the fitting process. Data from the actual world, which may be bimodal or left-skewed, are significantly more complex. Before the 1990s, several generalizations were created to increase the adaptability of Pareto distribution.

[10] presents a set of four generalized normal distribution families within the T-X framework. These distribution families, referred to as T-normal families, are derived from the quantile functions of (i) the standard exponential, (ii) standard log-logistic, (iii) standard logistic, and (iv) standard extreme value distributions.

. [7] use of quantile functions to define the W function. Jones, [13], [11] and [12] studied distributions with four parameters. The distribution is explored for several characteristics, revealing that it is unimodal and exhibits either a unimodal or decreasing hazard rate. Formulas for the mean, mean deviation, variance, skewness, kurtosis, and entropies are derived. Kareema and [4] [9] and [5], presented some properties and called them exponential Pareto using an alternative frame work from a beta - generated distribution. A distribution is called exponential Pareto if it has cdf and pdf as follows:

$$G(x) = 1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}, \quad x > 0 \text{ and } \beta,\ \theta > 0 \tag{3}$$

and

$$g(x) = \frac{\beta\theta}{\rho}\left(\frac{x}{\rho}\right)^{\theta-1} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}, \quad x > 0 \text{ and } \beta,\ \theta,\ \rho > 0 \tag{4}$$

2. SUGGESTED KUMARASWAMY EXPONENTIAL PARETO DISTRIBUTION (K):

We established a cumulative density function (CDF) and probability density function (PDF) for the Kumaraswamy Exponential Pareto distribution (KEPD).

$$F(x) = 1 - \left\{1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{a}\right\}^{b} \tag{5}$$

$$f(x) = \frac{ab\beta\theta}{\rho}\left(\frac{x}{\rho}\right)^{\theta-1} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{a-1}\left(1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right)^{b-1},$$

$$x >, \beta, \theta, a, b, \rho > 0 \tag{6}$$

By applying the generalized binomial theorem

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{a-1}{i}\binom{b+b_i-1}{j}\frac{ab\beta\theta}{\rho}\left(\frac{x}{\rho}\right)^{\theta-1} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}j}, \quad x > 0, \beta, \theta, a, b, \rho > 0 \tag{7}$$

$$\text{let } w_i = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{a-1}{i}\binom{b+b_i-1}{j}$$

$$f(x) = w_i\frac{ab\beta\theta}{\rho}\left(\frac{x}{\rho}\right)^{\theta-1} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}j} \quad x > 0, j > 0 \text{ and } \beta, \theta, a, b, \rho > 0 \tag{8}$$

Moments and Properties of Suggested Modified Exponential Pareto Distribution (MEPD)

[14] and [15] studied A comprehensive class of statistical models is introduced for a univariate response variable, referred to as the generalized additive model for location, scale, and shape (GAMLSS). Also, the tail

shape parameter and the extremal index are the fundamental parameters governing the extreme behavior of the distribution

$$E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$E(x^r) = \int_0^{\infty} x^r w_i \frac{ab\beta\theta}{\rho}\left(\frac{x}{\rho}\right)^{\theta-1} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta} j}\, dx \quad (9)$$

r^th Moment

$$E(x^r) = w_i \frac{ab\rho^r}{(j+1)^{\frac{r}{\theta}+1}\beta^{\frac{r}{\theta}}}\Gamma\left(\frac{r}{\theta}+1\right)$$

$$E(x^r) = \mu = \frac{ab\rho^r}{\beta^{\frac{r}{\theta}}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma a\, \Gamma(b+bi)}{\Gamma(a-i)\Gamma(b+bi-j)i!j!}\Gamma\left(\frac{r}{\theta}+1\right) \quad (10)$$

1^st Moment i.e. Mean

$$E(x) = \mu = \frac{ab\rho}{\beta^{\frac{1}{\theta}}} w_i \Gamma\left(\frac{1}{\theta}+1\right) \quad (11)$$

2^nd Moment

$$E(x^2) = \mu_2' = \frac{ab\rho^2}{\beta^{\frac{2}{\theta}}} w_i \Gamma\left(\frac{2}{\theta}+1\right) \quad (12)$$

3^rd Moment

$$E(x^3) = \mu_3' = \frac{ab\rho^3}{\beta^{\frac{3}{\theta}}} w_i \Gamma\left(\frac{3}{\theta}+1\right) \quad (13)$$

4^th Moment

$$E(x^4) = \mu_4' = \frac{ab\rho^4}{\beta^{\frac{4}{\theta}}} w_i \Gamma\left(\frac{4}{\theta}+1\right) \quad (14)$$

Skewness

$$SK = \frac{E(x-\mu)^3}{\sigma^3} = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{(\mu_2' - \mu^2)^{\frac{3}{2}}}$$

$$SK = \frac{\frac{\lambda\alpha\rho^3}{\beta^{\frac{3}{\theta}}} w_i\Gamma(\frac{3}{\theta}+1) - 3*\frac{\lambda\alpha\rho^2}{\beta^{\frac{2}{\theta}}} w_i\Gamma(\frac{2}{\theta}+1)*\frac{\lambda\alpha\rho}{\beta^{\frac{1}{\theta}}} w_i\Gamma(\frac{1}{\theta}+1) + 2\left(\frac{\lambda\alpha\rho}{\beta^{\frac{1}{\theta}}} w_i\Gamma(\frac{1}{\theta}+1)\right)^3}{\left(\frac{\lambda\alpha\rho^2}{\beta^{\frac{2}{\theta}}} w_i\Gamma(\frac{2}{\theta}+1) - \left(\frac{\lambda\alpha\rho}{\beta^{\frac{1}{\theta}}} w_i\Gamma(\frac{1}{\theta}+1)\right)^2\right)^{\frac{3}{2}}} \quad (15)$$

Kurtosis

$$KS = \frac{E(x-\mu)^4}{\sigma^4} - 3 = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{(\mu_2' - \mu^2)^2}$$

$$KS = \frac{\frac{\lambda\alpha\rho^4}{\beta^{\frac{4}{\theta}}} w_i\Gamma(\frac{4}{\theta}+1) - 4*\frac{\lambda\alpha\rho^3}{\beta^{\frac{3}{\theta}}} w_i\Gamma(\frac{3}{\theta}+1)*\frac{\lambda\alpha\rho}{\beta^{\frac{1}{\theta}}} w_i\Gamma(\frac{1}{\theta}+1) + 6*\frac{\lambda\alpha\rho^2}{\beta^{\frac{2}{\theta}}} w_i\Gamma(\frac{2}{\theta}+1)*\left(\frac{\lambda\alpha\rho}{\beta^{\frac{1}{\theta}}} w_i\Gamma(\frac{1}{\theta}+1)\right)^2 - 3\left(\frac{\lambda\alpha\rho}{\beta^{\frac{1}{\theta}}} w_i\Gamma(\frac{1}{\theta}+1)\right)^4}{\left(\frac{\lambda\alpha\rho^2}{\beta^{\frac{2}{\theta}}} w_i\Gamma(\frac{2}{\theta}+1) - \left(\frac{\lambda\alpha\rho}{\beta^{\frac{1}{\theta}}} w_i\Gamma(\frac{1}{\theta}+1)\right)^2\right)^2} \quad (16) .$$

Quantile

$$x = p\left(\frac{-1}{\beta}\log\left\{1 - \left[1 - (1-q)^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right\}\right)^{\frac{1}{\theta}} \quad (17)$$

$$Median = p\left(\frac{-1}{\beta}\log\left\{1 - \left[1 - (1-0.5)^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right\}\right)^{\frac{1}{\theta}} \quad (18)$$

Hazard function

The function that measures the lowest or highest chance of an event surviving a certain time based on its past survival time t is called the hazard function. By definition, F(x) is given by:

$$h(x) = \frac{f(x)}{1 - F(x)}$$

$$h(x) = \frac{ab\beta\theta}{\rho}\left(\frac{x}{\rho}\right)^{\theta-1} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{a-1}\left(1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right)^{b-1}$$

$$\left(1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right)^{b} \quad (19)$$

Survival function

The survival function quantifies the probability that a device, patient, or any other objects will continue to exist beyond a specific time 't,' and it is expressed as follows:

$$s(x) = 1 - F(x)$$

It implies that $s(x)$ is

$$S(x) = \left(1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right)^{b} \qquad (20)$$

Maximum Likelihood Estimation

In this section, we perform calculations to determine the maximum likelihood estimates (MLEs) of the parameters of the KEP distribution.

If $x_1, x_2, \dots, x_n$ is a random sample of size n observations from KEPD $(a, b, \beta, \theta, \rho)$, then the log likelihood function is given by:

From the equation 6, we have

$$f(x: a; b; \beta; \theta; \rho) = \frac{ab\beta\theta}{\rho^{\theta}} (x)^{\theta-1} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}} \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{a-1} \left(1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right)^{b-1}$$

$$Lf(x: a; b; \beta; \theta; \rho) = n\ln a + n\ln\beta + n\ln b + n\ln\theta - n\theta\ln\rho + (\theta - 1)\sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} \left(\frac{x_i}{\rho}\right)^{\theta}$$
$$+(a-1)\sum_{i=1}^{n} \ln\left(1 - e^{-\beta\left(\frac{x}{\rho}\right)}\right) + (b-1)\sum_{i=1}^{n} \ln\left\{1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)}\right)^{a}\right\}$$

$$\partial Lf(x: a; b; \beta; \theta; \rho) = n\ln a + n\ln\beta + n\ln b + n\ln\theta - n\theta\ln\rho + (\theta - 1)\sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} \left(\frac{x_i}{\rho}\right)^{\theta}$$
$$+(a-1)\sum_{i=1}^{n} \ln\left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right) + (b-1)\sum_{i=1}^{n} \ln\left\{1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right\}$$

$$\frac{\partial Lf(x: a; b; \beta; \theta)}{\partial\rho} = -\frac{n\theta}{\rho} - \frac{\beta\theta}{\rho^{\theta+1}}\sum_{i=1}^{n}(x_i)^{\theta} + \frac{\beta\theta(a-1)}{\rho^{\theta+1}}\sum_{i=1}^{n}\frac{(x_i)^{\theta} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}}{1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}} -$$
$$\frac{\beta\theta a(b-1)}{\rho^{\theta+1}}\sum_{i=1}^{n}\frac{(x_i)^{\theta} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)}{\left\{1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)}\right)^{a}\right\}}$$

After equating the nonlinear equations to zero, the maximum likelihood estimators of parameters and can be obtained by simultaneously solving the equations using the Newton-Raphson iteration process.

$$\frac{\partial Lf(x: a; b; \beta; \rho)}{\partial\theta} = \frac{n}{\theta} - n\ln\rho + \sum_{i=1}^{n}\ln x_i - \frac{\beta}{\rho^{\theta}}\sum_{i=1}^{n}(x_i)^{\theta}\left(\frac{x_i}{\rho}\right) - \frac{\beta(a-1)}{\rho^{\theta}}\sum_{i=1}^{n}\frac{(x_i)^{\theta}\ln\left(\frac{x_i}{\rho}\right) e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}}{1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}}$$
$$-\frac{\beta\alpha(b-1)}{\rho^{\theta}}\sum_{i=1}^{n}\frac{(x_i)^{\theta}\ln\left(\frac{x_i}{\rho}\right) e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)}{\left\{1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right\}}$$

$$\frac{\partial Lf(x: a; b; \theta; \rho)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln\left(\frac{x_i}{\rho}\right)^{\theta} + \frac{(\alpha-1)}{\rho^{\theta}}\sum_{i=1}^{n}\frac{(x_i)^{\theta} e^{-\beta\left(\frac{x}{p}\right)^{\theta}}}{1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}} - \frac{\beta\alpha(\lambda-1)}{\rho^{\theta}} *$$
$$\sum_{i=1}^{n}\frac{(x_i)^{\theta} e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)}{\left\{1 - \left(1 - e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right\}}$$

$$\frac{\partial Lf(x;b;\beta;\theta;\rho)}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n} ln\left(1-e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)+\frac{\alpha(\lambda-1)}{\rho^{\theta}}\sum_{i=1}^{n}\frac{\left(1-e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha-1}}{\left\{1-\left(1-e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right\}}$$

$$\frac{\partial Lf(x:a;\beta;\theta;\rho)}{\partial b}=\frac{n}{\lambda}+\sum_{i=1}^{n} ln\left\{1-\left(1-e^{-\beta\left(\frac{x}{\rho}\right)^{\theta}}\right)^{\alpha}\right\}$$

Figure 1 shows a graph of the probability density function of KEPD, which exhibits various shapes depending on the combination of scale and shape parameters. This shows that it could be high or low, depending on the values of the parameters. An increase in the shape parameter θ tends to change the skewness of the distribution from a right to left skew. As the shape of the parameter increases, the tail becomes heavier.

Figure 1: Graph of Probability Density Function (KEPD)

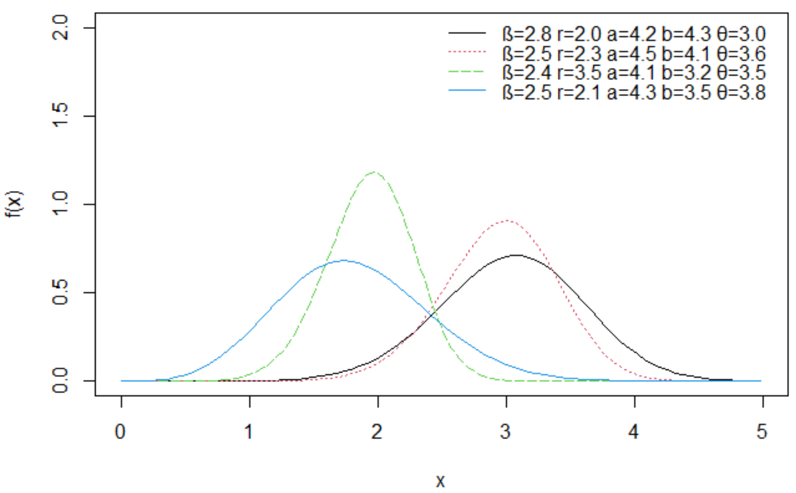
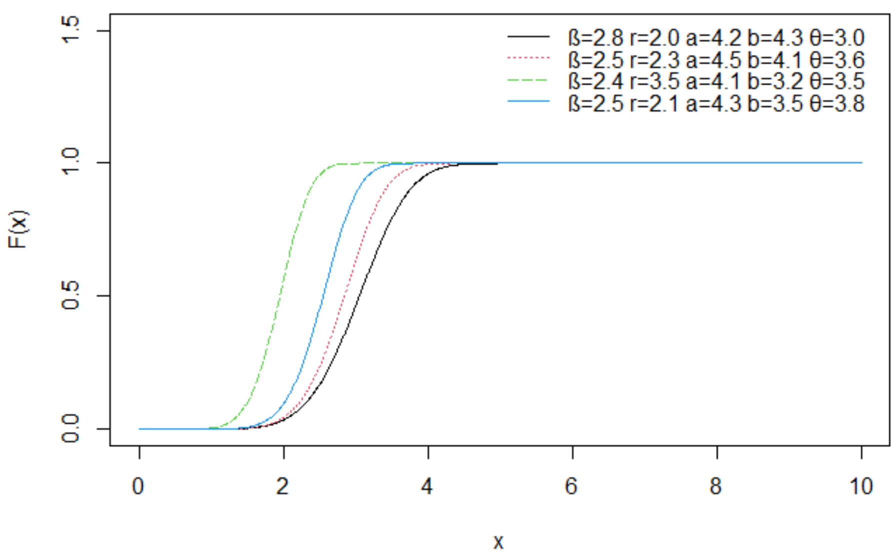


Figure 2 depicts a graph of the

Figure 2: Graph of Cummulative Density Function (KEPD)



cumulative distribution function of the KEPD. It shows a satisfactory cumulative plot level because the plot does not exceed 1, which shows that our distribution is a true probability distribution function.

Figure 3 represents the graph of the hazard function of KEPD. The hazard function of KEPD is an increase function of the shape and scale parameters.

Figure 3: Graph of Harzard Function (KEPD)

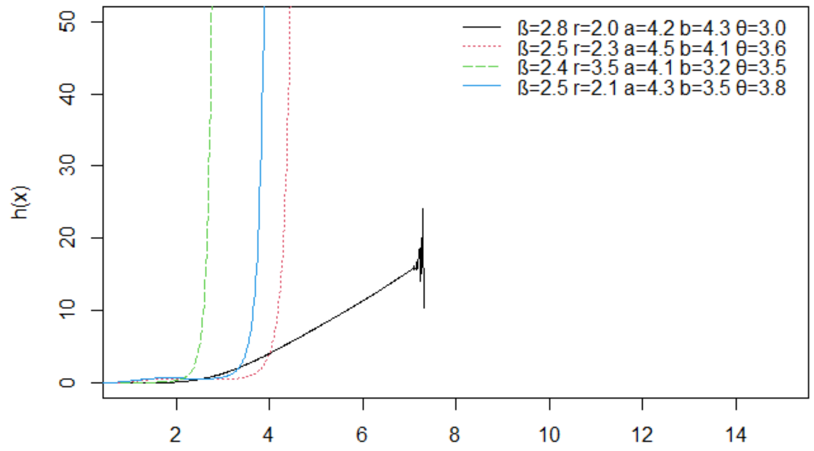
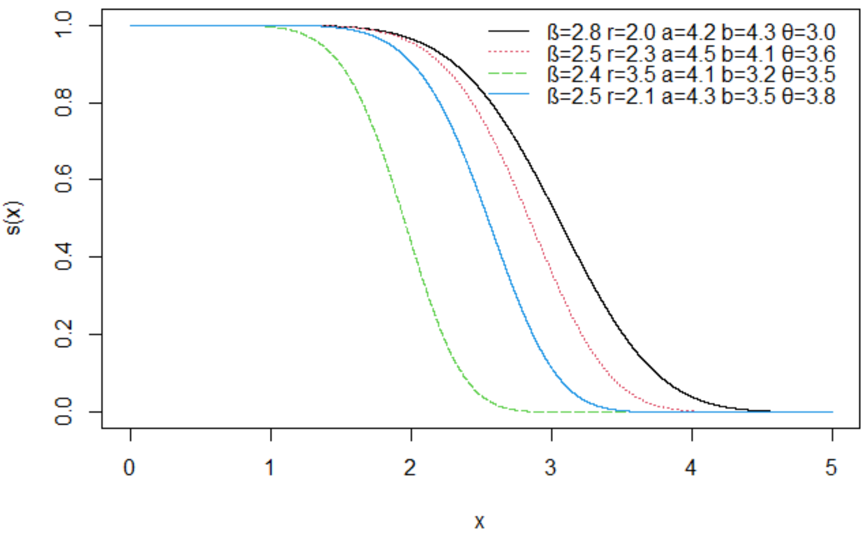


Figure 4 illustrates a graph of the survival function of KEPD, which delayed its decline. The survival function of KEPD is very high because it delays its drop. The survival function of KEPD drops across the x-axis as the shape and scale parameters increase.

Figure 4: Graph of Survival Function (KEPD)



Simulation Study:

Performing a simulation study in R involves of Mean, Standard Deviation, Median Skewness and Kurtosis from Kumaraswamy Exponential Pareto Distribution (KEPD) of different Sample Sizes: [8].

Table 1: Simulation Study of Mean, Standard Deviation and Median from Kumaraswamy Exponential Pareto Distribution (KEPD) of Sample Size n=50

			β =2 $\rho = 2$			β =2 $\rho = 4$			β =4 $\rho = 2$		
a	b	θ	Mean	SD	MD	Mean	SD	MD	Mean	Sd	MD
2	2	2	0.2445	0.0549	0.2477	0.4821	0.1254	0.4682	0.1205	0.0313	0.1170
		4	0.1223	0.0275	0.1239	0.2411	0.0627	0.2341	0.0603	0.0157	0.0585
		6	0.0815	0.0183	0.0825	0.1607	0.0418	0.1561	0.0402	0.0104	0.0390
	4	2	0.1067	0.0210	1.2906	0.2105	0.0475	0.2069	0.0526	0.0119	0.0517
		4	0.0534	0.0105	0.0543	0.1052	0.0238	0.1035	0.0263	0.0059	0.0258

		6	0.0355	0.0069	0.0362	0.0702	0.0158	0.0670	0.0175	0.0040	0.0172
		2	0.0685	0.0130	0.0698	0.1350	0.0294	0.1331	0.0338	0.0073	0.0333
		4	0.0342	0.0065	0.0349	0.0675	0.0147	0.0665	0.0168	0.0037	0.0166
	6	6	0.0228	0.0043	0.0233	0.0450	0.0097	0.0444	0.0113	0.0024	0.0111
		2	0.2901	0.0344	0.2872	0.5801	0.0689	0.5743	0.1450	0.0172	0.1436
		4	0.1450	0.0172	0.1436	0.2901	0.0344	0.2872	0.0725	0.0086	0.0718
	2	6	0.0967	0.0115	0.0957	0.1934	0.0230	0.1914	0.0483	0.0057	0.0479
		2	0.1240	0.0123	0.1232	0.2481	0.0246	0.2465	0.0620	0.0062	0.0616
		4	0.0620	0.0062	0.0616	0.1240	0.0123	0.1232	0.0310	0.0031	0.0308
	4	6	0.0413	0.0041	0.0411	0.0827	0.0082	0.0822	0.0207	0.0021	0.0205
		2	0.0791	0.0075	0.0787	0.1583	0.0150	0.1574	0.0396	0.0038	0.0394
		4	0.0396	0.0038	0.0394	0.0791	0.0075	0.0787	0.0198	0.0019	0.0197
4	6	6	0.0264	0.0025	0.0262	0.0528	0.0050	0.0525	0.0132	0.0013	0.01311
		2	0.3080	0.0238	0.3062	0.6159	0.0475	0.6124	0.1540	0.01188	0.1531
		4	0.1540	0.0119	0.1530	0.3080	0.0238	0.3062	0.0770	0.0059	0.0765
	2	6	0.1027	0.0079	0.1021	0.2053	0.0158	0.2041	0.0513	0.0040	0.0510
		2	0.1305	0.0083	0.1300	0.2610	0.0166	0.2600	0.0653	0.0042	0.0650
		4	0.0653	0.0042	0.0650	0.1305	0.0083	0.1300	0.0326	0.0021	0.0325
	4	6	0.0435	0.0028	0.0433	0.0870	0.0055	0.0867	0.0218	0.0014	0.0217
		2	0.0831	0.0050	0.0828	0.1662	0.0101	0.1656	0.0416	0.0025	0.0414
		4	0.0416	0.0025	0.0414	0.0831	0.0050	0.0828	0.0208	0.0013	0.0207
6	6	6	0.0277	0.0017	0.0276	0.0554	0.0034	0.0552	0.0139	0.0008	0.0138

Source: Computed from the simulated data on KEPD

Table 2: Simulation study of Skewness and Kurtosis from Kumaraswamy Exponential Pareto Distribution (KEPD) of sample $size\ n = 50$

			β =2	$\rho = 2$	β =2	$\rho = 4$	β =4	$\rho = 2$
a	b	θ	Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
		2	-0.0933	-1.0692	-0.0933	-1.0692	-0.0933	-1.0692
		4	0.2091	-1.3695	0.2091	-1.3695	0.2091	-1.3695
	2	6	0.3544	-1.0574	0.3544	-1.0574	0.3544	-1.0574
		2	-0.1857	-1.0556	-0.1857	-1.0556	-0.1857	-1.0556
		4	0.1414	-1.3933	0.1414	-1.3933	0.1414	-1.3933
	4	6	0.2769	-1.1104	0.2769	-1.1104	0.2769	-1.1104
		2	-0.2073	-1.0486	-0.2073	-1.0486	-0.2073	-1.0486
		4	0.1255	-1.3970	0.1255	-1.3970	0.1255	-1.3970
2	6	6	0.2585	-1.1205	0.2585	-1.1205	0.2585	-1.1205
		2	-0.1604	-1.0620	-0.1604	-1.0620	-0.1604	-1.0620
		4	0.1599	-1.3881	0.1599	-1.3881	0.1599	-1.3881
	2	6	0.2984	-1.0974	0.2984	-1.0974	0.2984	-1.0974
		2	-0.2130	-1.0465	-0.2130	-1.0465	-0.2130	-1.0465
		4	0.1214	-1.3979	0.1214	-1.3979	0.1214	-1.3979
	4	6	0.2538	-1.1230	0.2538	-1.1230	0.2538	-1.1230
4	6	2	-0.2244	-1.0420	-0.2244	-1.0420	-0.2244	-1.0420

		4	0.1130	-1.3994	0.1130	-1.3994	0.1130	-1.3994
		6	0.2440	-1.1279	0.2440	-1.1279	0.2440	-1.1279
		2	-0.1857	-1.0556	-0.1857	-1.0556	-0.1857	-1.0556
		4	0.1414	-1.3933	0.1414	-1.3933	0.1414	-1.3933
	2	6	0.2769	-1.1104	0.2769	-1.1104	0.2769	-1.1104
		2	-0.2225	-1.0428	-0.2225	-1.0428	-0.2225	-1.0428
		4	0.1144	-1.3991	0.1144	-1.3991	0.1144	-1.3991
	4	6	0.2456	-1.1271	0.2456	-1.1271	0.2456	-1.1271
		2	-0.2303	-1.0395	-0.2303	-1.0395	-0.2303	-1.0395
		4	0.1087	-1.4002	0.1087	-1.4002	0.1087	-1.4002
6	6	6	0.2389	-1.1303	0.2389	-1.1303	0.2389	-1.1303

Source: Computed from the simulated data on KEPD

Table 3: Simulation Study of Mean, Standard Deviation and Median from Kumaraswamy Exponential Pareto Distribution (KEPD) of Sample Size n=100

			β =2 $\rho = 2$			β =2 $\rho = 4$			β =4 $\rho = 2$		
λ	α	θ	Mean	SD	MD	Mean	SD	MD	Mean	Sd	MD
		2	0.2369	0.0606	0.2236	0.4829	0.1082	0.4651	0.1207	0.0271	0.1163
		4	0.1184	0.0303	0.1118	0.2415	0.0541	0.2326	0.0604	0.0135	0.0581
	2	6	0.0790	0.0202	0.0745	0.1610	0.0361	0.1550	0.0402	0.0090	0.0388
		2	0.1037	0.0230	0.0994	0.2112	0.0413	0.2058	0.0528	0.0103	0.0514
		4	0.0518	0.0115	0.0497	0.1056	0.0206	0.1028	0.0264	0.0052	0.0257
	4	6	0.0346	0.0077	0.2008	0.0704	0.0138	0.0686	0.0176	0.0034	0.0171
		2	0.0666	0.0142	0.0640	0.1356	0.0255	0.1324	0.0339	0.0064	0.0331
		4	0.0333	0.0071	0.0320	0.0678	0.0128	0.0662	0.0169	0.0032	0.0165
2	6	6	0.0222	0.0047	0.0213	0.0452	0.0085	0.0441	0.0113	0.0021	0.0110
		2	0.2878	0.0334	0.2813	0.5811	0.0597	0.5727	0.1453	0.0149	0.1432
		4	0.1439	0.0167	0.1407	0.2906	0.0299	0.2863	0.0720	0.0083	0.0703
	2	6	0.0959	0.0111	0.0938	0.1937	0.0199	0.1909	0.0478	0.0053	0.0476
		2	0.1232	0.0120	0.1211	0.2485	0.0214	0.2459	0.0621	0.0054	0.0615
		4	0.0616	0.0060	0.0606	0.1242	0.0107	0.1229	0.0308	0.0030	0.0303
	4	6	0.0411	0.0040	0.0404	0.0828	0.0071	0.0820	0.0205	0.0019	0.0205
		2	0.0787	0.0073	0.0774	0.1586	0.0131	0.1570	0.0396	0.0033	0.0393
		4	0.0393	0.0036	0.0387	0.0793	0.0065	0.0785	0.0197	0.0018	0.0194
4	6	6	0.0262	0.0024	0.0258	0.0529	0.0044	0.0523	0.0131	0.0012	0.0130
		2	0.3064	0.0230	0.3021	0.6167	0.0413	0.6112	0.1542	0.0103	0.1528
		4	0.1532	0.0115	0.1510	0.3083	0.0206	0.3056	0.0766	0.0058	0.0755
	2	6	0.1021	0.0077	0.1007	0.2056	0.0138	0.2037	0.0510	0.0037	0.0509
		2	0.1300	0.0081	0.1286	0.2613	0.0145	0.2596	0.0653	0.0036	0.0649
		4	0.0650	0.0040	0.0643	0.1307	0.0072	0.1298	0.0325	0.0020	0.0321
	4	6	0.0433	0.0027	0.0429	0.0871	0.0048	0.0865	0.0216	0.0013	0.0216
		2	0.0828	0.0049	0.0820	0.1664	0.0088	0.1654	0.0305	0.0016	0.0303
6	6	4	0.0414	0.0024	0.0409	0.0832	0.0044	0.0827	0.0207	0.0012	0.0205

		6	0.0276	0.0016	0.0273	0.0555	0.0029	0.0551	0.0137	0.0008	0.0138

Source: Computed from the simulated data on KEPD

When we keep the other parameters constant, an examination of Tables 1 and 3 reveals that the mean, standard deviation, and median exhibit an upward trend as the scale parameter increases. Similarly, for the fixed shape parameters a, b, and θ, the mean, standard deviation, and median increase as the scale parameters increase. Conversely, the shape parameters a, b, and θ displayed a decrease in response to increases in the mean, standard deviation, and median. Furthermore, when one of the shape parameters is increased while the others remain constant, the mean, standard deviation, and median values of the KEPD distribution decrease. The skewness and kurtosis in Tables 2 and 4 increase when the shape parameters β and ρ are held constant while decreasing as the scale parameter increases, and certain trends emerge. However, when the scale parameters were kept constant and the shape parameters increased, different patterns emerge in the data. Specifically, an increase in shape parameters tends to lead to higher skewness and lower kurtosis values in the distribution.

4. REAL LIFE APPLICATION

In this section, the Kumaraswamy Exponential Power Distribution (KEPD) is applied to actual datasets, and the values obtained from the KEPD model are compared with those from its sub-models for evaluation and contrast. The data comprise the Cumulative Grade Point Average (CGPA) of Auchi Polytechnic, Auchi.2019/2020 first semester results in the Department of Statistics.

Table 4: CGPA of 103 Students in the Department of statistics Auchi Polytechnic

3.47	3.17	2.91	2.42	3.09	2.59	3.12	2.30	2.66	2.83	3.01	3.46	2.44
3.19	2.38	2.36	2.59	2.86	2.87	2.53	2.81	3.01	3.10	2.85	2.96	2.74
3.08	3.11	3.11	3.66	2.95	2.71	3.22	2.88	2.37	2.87	2.46	2.96	2.42
2.55	3.09	3.36	2.60	3.03	2.86	3.00	2.99	2.66	3.18	2.75	3.04	2.64
2.59	3.49	2.26	2.74	2.55	2.79	2.43	2.67	2.83	3.08	2.84	2.40	3.14
2.50	2.72	2.87	3.02	3.13	2.58	2.79	3.28	2.80	2.83	2.41	2.06	2.81
2.59	2.62	3.14	2.50	2.97	2.97	2.51	2.78	2.26	3.08	2.90	2.46	3.03
2.81	2.75	2.56	2.35	2.50	2.32	3.01	3.23	2.76	3.06	2.92	2.45	

Source: Department of Statistics Auchi Polytechnic, 2018/2019 first semester result

Figure 5 Displays histograms and Cumulative Distribution Function (CDF) plots representing an empirical distribution for the Cumulative Grade Point Average (CGPA) of 103 students in the Department of Statistics at Auchi Polytechnic, Auchi.

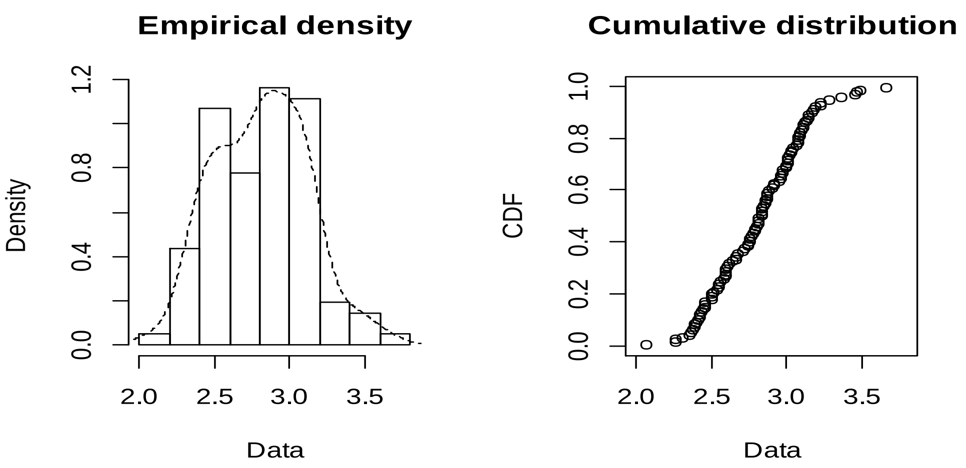
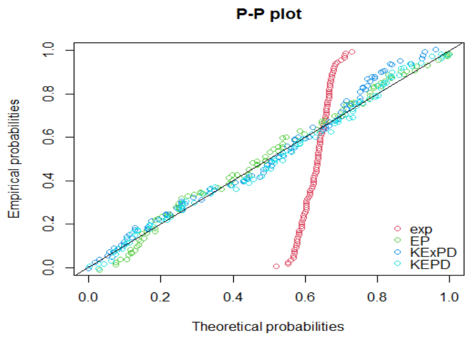
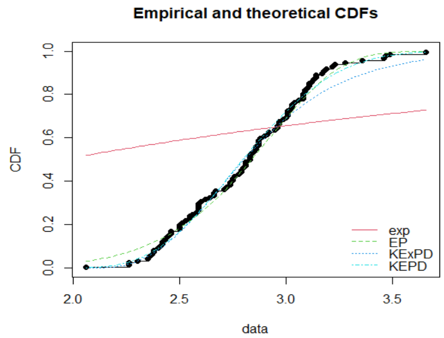


Fig. 6: Graph showing theoretical quantities





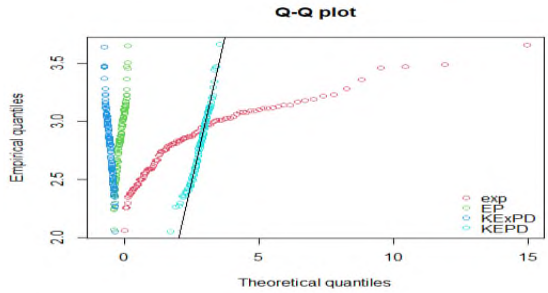


Table5: Summary Statistics CGPA Data

Min.	Max.	Mean	Median	1st Quart.	3rd Quart.	S.D.	Skewness	Kurtosis
2.06	3.66	2.81	2.83	2.57	3.03	0.3099681	0.1292561	-0.3230131

The statistical summary table in Table 5, as presented above, indicates that the CGPA (Cumulative Grade Point Average) data follows a right-skewed distribution and exhibits dispersion when modeled under the KEPD (Kumaraswamy Exponential Power Distribution).

Correlation matrix KEPD =

$$\begin{pmatrix} 1.000000 & -0.011566 & -0.1591165 & 0.2480000 & -0.1869646 \\ -0.011566 & 1.000000 & 0.4706987 & -0.038965 & 0.5566150 \\ -0.1591165 & 0.4706987 & 1.0000000 & 0.1786846 & 0.8490320 \\ 0.2480000 & -0.038965 & 0.1786846 & 1.0000000 & 0.2426484 \\ -0.1869646 & 0.5566150 & 0.8490320 & 0.2426484 & 1.0000000 \end{pmatrix}$$

The correlation between pairs of parameters varies, and some combinations may show positive correlations, whereas others may show negative correlations. This enabled us to identify pairs that have both positive and negative correlation.

The estimated parameters asymptotic variance covariance matrix is

$$I_{ij}^{-1} = \begin{pmatrix} 3.94156 & -1.846e-04 & -4.862e-03 & 9.255e-03 & -4.938e-03 \\ -1.846e-04 & 6.454e-05 & 5.824e-05 & -5.888e-06 & 5.953e-05 \\ -4.862e-03 & 5.824e-05 & 2.369e-04 & 5.169e-05 & 1.738e-04 \\ 9.255e-03 & -5.888e-06 & 5.169e-05 & 3.533e-04 & 6.068e-05 \\ 1.770e-04 & 5.953e-05 & 1.738e-04 & 6.068e-05 & 1.770e-04 \end{pmatrix}$$

Table 6 This information is derived from the CGPA (Cumulative Grade Point Average) data.

Model	a	b	θ	β	ρ
KEPD	8.1567164	0.3121007	6.3562919	0.1694785	1.7486917
	(1.9853355)	**(**0.0080398**)**	(0.0153909)	**(**0.0187969**)**	**(**0.01330313**)**
KExPD	2.765001	5.826792	1.854824	1.986728	3.068769
	(0.80304451)	(1.06319192)	(0.53869802)	(0.04503899)	(0.40886975)
EP			9.6074434	0.9550032	2.9354928
			(0.6935308)	(17.8366244)	(5.7067006)
Exp				0.3558227	
				(0.03505998)	

Figure 7: Parameters' Likelihood Estimate Standard Errors for CGPA Data in Parenthesis

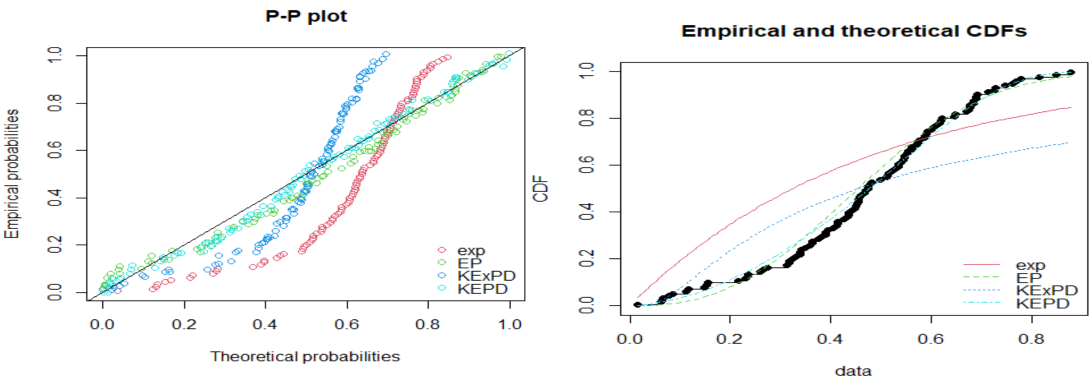


Table 7 provides information on the log-likelihood, goodness-of-fit statistics based on information criteria, and p-values for the CGPA (Cumulative Grade Point Average) of 103 students in the Statistics Department.

Model	**LL**	**AIC**	**BIC**	**K-S**	**C-M**	**AD**	**P-value**
KEPD	-24.43644	58.87288	72.04652	0.0545	0.06557	0.42655	0.9026
KExPD	-45.42353	100.8471	114.0207	0.1021	0.16168	1.32225	0.2180
EP	-29.60274	65.20549	73.10968	0.0677	0.08110	0.804289	0.7064
Exp	-209.4322	420.8645	423.4992	0.5428	8.12744	37.63606	1.332e-15

Figure 8 depicts cows that are owned by the Carnaúba farm. This data was originally displayed Cordeiro and Brito in 2012.

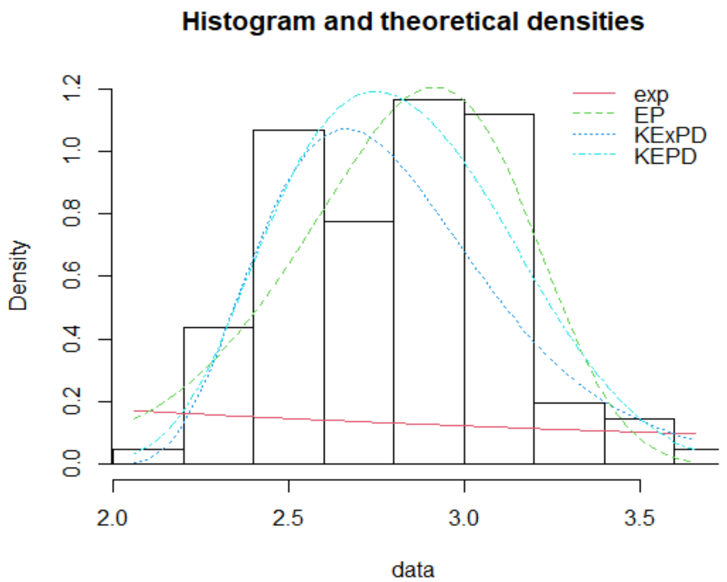


Table 8 shows the total amount of milk produced by 107 cows of the SINDI race during their first lactation.

0.4353	0.7617	0.7249	0.5338	0.6102	0.1119	0.1148	0.4541	0.3894	0.4663	0.5337
0.4248	0.6879	0.5929	0.4139	0.3468	0.7278	0.6738	0.4458	0.4426	0.6832	0.3739
0.5128	0.5758	0.3311	0.6162	0.4552	0.0156	0.5101	0.5273	0.46	0.3401	0.1534
0.6895	0.5382	0.0659	0.6777	0.6848	0.5517	0.5435	0.522	0.3176	0.432	0.4505
0.7459	0.1467	0.2348	0.4564	0.7792	0.0597	0.4131	0.6453	0.2148	0.0842	0.2669
0.2593	0.2344	0.4788	0.3247	0.3394	0.4518	0.6476	0.0638	0.6695	0.3809	0.4037
0.6184	0.6	0.5695	0.2291	0.4811	0.3879	0.5615	0.2735	0.6208	0.4682	0.5541
0.8769	0.1513	0.7119	0.7675	0.59	0.474	0.5138	0.848	0.4729	0.3623	0.5866
0.4978	0.5471	0.5841	0.4359	0.5732	0.3122	0.0764	0.8135	0.5617	0.4099	
0.6046	0.6915	0.6756	0.3371	0.5469	0.3163	0.3933	0.3615	0.5617	0.3586	

Table 9: displays descriptive statistics

Min	Max	Median	Mean	S.D.	Skewness	Kurtosis
0.0167	0.8780	0.4741	0.4688	0.1919961	-0.3352894	-0.3138844

$$\begin{pmatrix} 1.000000 & 0.001394 & -0.455460 & 0.846081 & 0.758577 \\ 0.001394 & 1.000000 & -0.025949 & -0.001782 & 0.034359 \\ -0.455460 & -0.025949 & 1.0000000 & -0.475426 & -0.500516 \\ 0.846081 & -0.001782 & -0.475426 & 1.0000000 & 0.9626509 \\ 0.758577 & 0.034359 & -0.500516 & 0.9626509 & 1.0000000 \end{pmatrix}$$

The specific correlations between pairs of parameters can vary depending on the combinations considered. This enabled us to identify pairs that have both positive and negative correlations.

$$I_{ij}^{-1} = \begin{pmatrix} 1.9702e-03 & 6.8313e-06 & -2.789e-03 & 1.9999e-02 & 3.164e-03 \\ 6.8313e-06 & 1.2199e-02 & -3.954e-04 & -1.048e-04 & 3.567e-04 \\ -2.7890e-03 & -3.954e-04 & 1.903e-02 & -3.493e-02 & 6.490e-03 \\ 1.9999e-02 & -1.048e-04 & -3.493e-02 & 0.2836065 & 4.819e-02 \\ 3.1647e-03 & 3.567e-04 & 6.491e-03 & 4.819\,e-02 & 8.835e-03 \end{pmatrix}$$

Fig 9: Graphical presentation of Empirical density and cumulative distribution

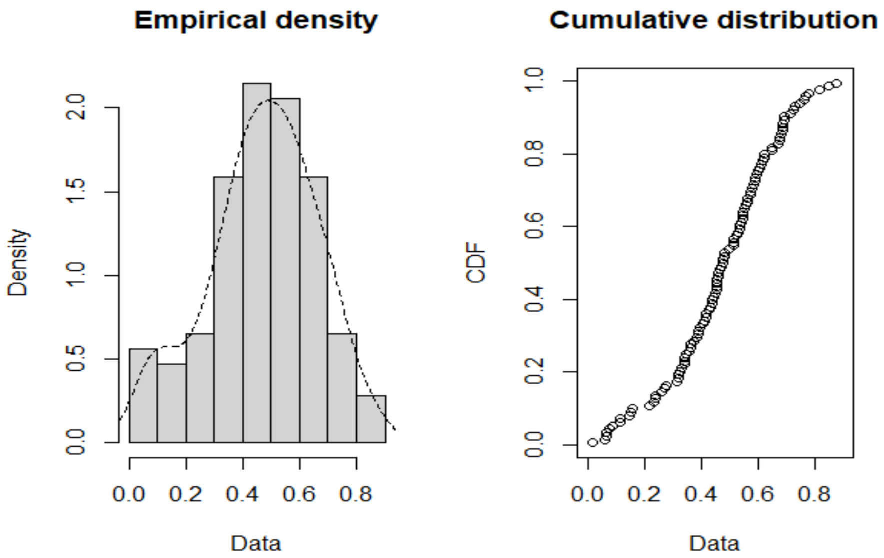


Table10:Parameter-Maximum Likelihood Estimates Cow dada's Standard Errors in Parenthesis

Model	a	b	θ	β	ρ
KEPD	8.1567164	0.3121007	6.3562919	0.1694785	1.7486917
	(1.98533547)	(0.0080398)	(0.0153909)	(0.01879693)	(0.01330312)
KExPD	2.90414268	3.79432964	0.42482771	0.01273337	0.01273337

	(0.515278)	(0.527318641)	(0.0414983)	(0.0015369)	0.001536852
EP			2.6011655	2.9750497	0.7962341
			(0.2098341)	(49.355305)	(5.078457)
Exp				2.132872	
				(0.206193)	

Table 11 presents associated data such as log-likelihood, information criteria, goodness-of-fit statistics, and p-values.

$Model$	LL	AIC	BIC	K-S	C-M	AD	P-value
$KEPD$	27.46764	−44.93528	−31.571	0.0743	0.09019	0.570830	0.7904
$KExPD$	−65.6009	141.2018	154.5659	0.30690	2.642099	13.83951	0.0312
EP	21.34751	−36.69502	−28.67654	0.0832	0.189395	1.483814	0.4258
Exp	−297.0648	53.90155	56.57438	0.3193	3.28607	16.18175	3.357e-10

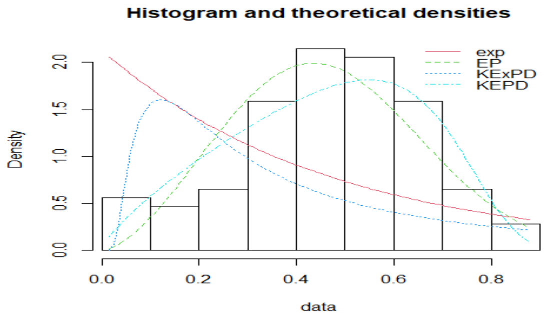


Fig 10: Histogram and theoretical densities

5. CONCLUSION

Using the Kumaraswamy family as the generator and Exponential Pareto Distribution as the reference distribution, we developed and derived the Kumaraswamy Exponential Pareto (KEPD) distribution. Figure 1 shows a plot of the (KEPD) density function with several sub models. The graph illustrates that the KExPD (Kumaraswamy Exponential Power Distribution) can exhibit left, right, or unimodal shapes depending on the values of its shape parameters. In Figure 2, examining the y-axis reveals that the cumulative distribution function (CDF) of KEPD consistently remains at or below 1. The shapes of these graphs are strongly influenced by the shape parameter values.

In Figure 3, the hazard function graph displays increasing and constant patterns for various parameter values, owing the closed-form nature of the quantile function of the suggested distribution, which was used for the simulation.

Tables 1 and 3 demonstrate that when other parameters are held constant, the mean, standard deviation, and median of KEPD increase as the scale parameter (ρ) increases. Similarly, for fixed shape parameters (a, b, and θ), as the scale parameter increases, the mean, standard deviation, and median also increase, and vice versa.

The skewness and kurtosis trends in Tables 2 and 4 show an upward trajectory when the shape parameters (β and ρ) are kept constant, but decrease when the scale parameter increases, and vice versa. It is observed that the Kumaraswamy Exponential Pareto distribution (KEPD) provides a better fit in terms of the Kolmogorov-Smirnov distance, Anderson-Darling and Cramer-von, BIC, AIC statistic, or higher log-likelihood values than those of Kumaraswamy Exponentiated Pareto Distribution (KEPD), Exponential Pareto distribution (EP) and Exponential Distribution, see details in table 8 and 11. The analysis shows that the proposed model is more flexible compared to existing models, as it stated in the introduction that an additional shape or scale parameter makes the new distribution more flexible than the existing distribution. This newly created distribution was demonstrated to be adaptable and capable of fitting a variety of data types. Based on the aging characteristics of the newly created distribution, the results of this study will help choose the distribution to be employed. They will also help identify the best distribution in terms of different stochastic orders.

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