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COMPUTING NAÏVE CLASS OF ESTIMATORS OF POPULATION VARIANCE USING ROBUST AUXILIARY PARAMETERS

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ABSTRACT
Variability is a natural phenomenon that can be found in nature and society, even in similar objects. Thus, it is imperative and exciting to know or estimate this variation. In the scripture, we suggest a naïve family of population variance estimators utilizing some robust auxiliary parameters, which also deal with the outliers and extreme values. The bias and the Mean Squared Error (MSE) of the introduced family of estimators are obtained up to an approximation of order one. Furthermore, the optimal values of the characterizing Searls constants are obtained. For these optimum values of the characterizing scalars, the least value of the MSE of the introduced estimator is also obtained. In comparison, the suggested estimator was compared to the competing estimators of the population variance. The efficiency criteria of the suggested estimators are also obtained over the competing estimators. These efficiency conditions have been validated using a real data set as well as a simulated data set. For practical utility in various areas of application, the estimator with the lowest MSE is recommended.

KEYWORDS: Primary Variable, Auxiliary Variable, Robust Parameter, Bias, MSE, Efficiency.

MSC: 62D05, 62D99, 94A20

RESUMEN
Variabilidad es un fenómeno natural que se haya en la naturaleza y la sociedad incluso entre similares objetos. Así que es imperativo y excitante el conocer o tener un estimado de la variación. En el trabajo sugerimos una familia naïve de estimadores de la varianza de la población usando algunos parámetros robustos auxiliares, los que también lo son para los outliers y valores extremos. El sesgo y el Error Cuadrático Medio (MSE) de la introducida familia de estimadores son obtenidos para el orden 1 de aproximación Además, los valores optimales para caracterizar las constantes de Searls son obtenidos . Para estos óptimos valores de los caracterizantes escalares, es introducido el estimador con el menor MSE el que es obtenido. El sugerido estimador de la varianza poblacional se comparó con sus competidores. Se derivó la eficiencia de los sugeridos estimadores respecto a sus competidores . Estas condiciones de eficiencia fueron validadas usando un conjunto real de datos y uno simulado. Para la utilidad práctica en varias áreas de aplicación, se recomienda el estimador con el menor MSE.

PALABRAS CLAVE: Variable Primaria, Variable Auxiliar, Parámetro Robusto , Sesgo, MSE, Eficiencia.

1. INTRODUCTION

It is always advisable to calculate the population parameters under consideration rather than to estimate them. However, due to financial and time constraints, sampling is the optimum method for large populations. The most natural and appropriate estimator for any population parameter is the corresponding statistic. Thus, for estimating the population variance (S_y^2) of the main variable (Y), sample variance (s_y^2) is the most suitable estimator. It is observed both theoretically and empirically in

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the literature that although s_y^2 is unbiased for S_y^2 but it has a reasonably large sampling variance. Thus one search for a more efficient estimator may be biased but have its sampling distribution closer to true S_y^2 that its MSE is less than the variance of sample variance. The search for a more efficient estimator is fulfilled through the use of auxiliary information provided by the auxiliary variable (X). The variable X has a high positive or negative correlation with the Y. Whenever the main variable and auxiliary variable have a positive correlation, then the ratio method is used for enhanced estimation of S_y^2 through ratio type estimators with one constraint that the regression line of Y on X goes through origin.
On the other hand product method is used for enhanced estimation of S_y^2 by product type estimation when Y and X are negatively correlated with the condition that the regression line is passing through origin. In either case regression method of estimation is used for improved estimation of S_y^2. In the present investigation, we are considering the case of positive correlation between Y and X, therefore, we are focusing on the ratio type estimators of S_y^2.
In search of more and more efficient estimators, various authors have given different ratio type estimators of S_y^2 utilizing the known X as its various conventional and non-conventional parameters. Isaki (1983) utilized the positively correlated X and suggested the conventional ratio estimator of S_y^2 while Prasad and Singh (1990) worked on some elevated ratio estimators of S_y^2 and shown improvement over usual ratio estimator. Upadhyaya and Singh (1999) utilized the known auxiliary parameter and proposed a new ratio type estimator of S_y^2 and shown theoretical and empirical improvement over existing estimators. Kadilar and Cingi (2006) introduced ratio type estimators of S_y^2 utilizing some known conventional auxiliary parameters and Singh *et al.* (2008) utilized the known parameter of auxiliary attribute and suggested improved ratio type estimators of S_y^2. Singh and Solanki (2010) worked on the enhanced estimation of S_y^2 using known auxiliary parameters in the presence of non-response on the main and auxiliary variables while Singh *et al.* (2011) introduced improved estimation of S_y^2 through exponential ratio type estimator utilizing the known two X.
Tailor and Sharma (2012) worked on some modified ratio estimators of S_y^2 utilizing known auxiliary parameters and Subramani and Kumarapandiyan (2012) proposed some improved ratio type estimators of S_y^2 using known quartiles of X and their functions. Solanki and Singh (2013) suggested an elevated class of estimators of S_y^2 using known auxiliary parameters and shown many existing estimators as the member of their family. They have also shown improvement over some existing estimators of S_y^2 making use of auxiliary parameters. Yadav and Kadilar (2013) proposed a generalized exponential ratio type class of estimators of S_y^2 and have shown that estimators by them performs superior to estimators of S_y^2 in competition, some of which are also members of the suggested family and Khan and Shabbir (2013) worked on a ratio estimator of S_y^2 utilizing known information on coefficient of correlation along with the third quartile of X.
Yadav and Kadilar (2014) suggested a two parameter class of estimators of S_y^2 utilizing known auxiliary parameters and shown improvement over existing estimators while Yadav *et al.* (2015) worked on an improved ratio estimator of S_y^2 utilizing known coefficient of correlation and quartiles of X. Adichwal *et al.* (2016) worked on a new generalized family of estimators of S_y^2 utilizing the known parameters of auxiliary attributes and shown improvement over existing estimators and Maqbool *et al.* (2016) proposed a general class of estimators of S_y^2 utilizing some known non-conventional auxiliary parameters and

shown that their estimators are more efficient than the estimators of S_y^2 in competing. Yadav *et al.* (2016) introduced a new family of estimators of S_y^2 using some known auxiliary parameters and shown improvement over competing estimators, many of which are also the members of the proposed family while Maqbool and Javaid (2017) proposed an elevated estimator of S_y^2 using known tri-mean and quartile average of X . Singh and Pal (2017) proposed a new ratio estimator of S_y^2 utilizing known coefficient of variation of X .

Recently Muneer *et al.* (2018) proposed a naive ratio-product exponential estimator of S_y^2 using known auxiliary parameters. Yasmeen *et al.* (2018) worked on a new exponential ratio estimator of S_y^2 using transformed X while Zahid and Shabbir (2019) proposed an elevated estimator of S_y^2 using dual auxiliary information under sensitive issue. Yadav *et al.* (2019) worked on enhanced estimation of S_y^2 utilizing tri-mean and third quartile of X . Cekim and Kadilar (2020) proposed an Ln-Type estimator for elevated estimation of S_y^2 utilizing known auxiliary parameters and Sanaullah *et al.* (2020) proposed a novel hybrid type family of estimators of S_y^2 utilizing transformed auxiliary information. Muhammad and Amjad (2021) worked on the enhanced estimation of S_y^2 through transformed ratio type estimator of S_y^2 utilizing known auxiliary information while Daraz and Khan (2021) suggested a difference-cum-ratio-type exponential estimator of S_y^2 under *SRSWOR* scheme.

In this investigation, we propose a naïve family of ratio type estimators of S_y^2 using known auxiliary parameters at the estimation stage as robust non-conventional parameters which also deals with the outliers and extreme values. The sampling properties, bias and MSE of introduced family of estimators is derived theoretically till the first order approximation. The whole paper has been presented in various sections. Section 2 represents the review of some existing competing estimators of S_y^2 . In Section 3, a naive family of estimators of S_y^2 has been introduced.

The bias and MSE expressions for the introduced family of estimators were derived up to the first order o f approximation in section 4. Section 5 depicts the comparison of efficiency of the introduced class with the estimators in competition. In Section 6, a numerical study has been executed to judge the candidature of different estimators, and section 7 represents the results and findings of the present investigation. Finally, the article ends at Section 8 with the conclusions.

2. REVIEW OF EXISTING ESTIMATORS

Under this section, we have represented different estimators of S_y^2 using and without using auxiliary parameters. Till now, different authors have given different estimators of S_y^2 and studied their sampling properties mainly biases and MSEs till first order approximation.

Sample variance is one of the most suitable estimators of S_y^2 as it possesses almost all desirable properties of a good estimator with only one but major drawback that it is not the most efficient estimator of S_y^2 . The estimator and its variance up to approximation of order one is given by,

$$t_0 = s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$V(t_0) = \gamma S_y^4 \lambda_{40}^* \qquad (1)$$

Where,

$\lambda_{rs} = \dfrac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{r/2}}$, $\lambda_{rs}^* = (\lambda_{rs} - 1)$, $\mu_{rs} = \dfrac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^r (X_i - \bar{X})^s$, $\gamma = \dfrac{1-f}{n}$ and $f = \dfrac{n}{N}$.

Isaki (1983) proposed the conventional ratio estimator of S_y^2 by making use of known $\bar{X}$. The estimator and the MSE of his estimator is given by,

$$t_R = s_y^2\left[\frac{S_x^2}{s_x^2}\right]$$

$$MSE(t_R) = \gamma S_y^4[\lambda_{40}^* + \lambda_{04}^* - 2\lambda_{22}^*] \quad (2)$$

Singh *et al.* (2013) introduced the given below ratio estimators of S_y^2 utilizing the known quartiles and their functions of X. The estimators and the MSEs of their estimators up to approximation of order one is given by,

$$t_1 = s_y^2\left[\frac{S_x^2+\alpha Q_1^2}{s_x^2+\alpha Q_1^2}\right],\ t_2 = s_y^2\left[\frac{S_x^2+\alpha Q_2^2}{s_x^2+\alpha Q_2^2}\right],\ t_3 = s_y^2\left[\frac{S_x^2+\alpha Q_3^2}{s_x^2+\alpha Q_3^2}\right],\ t_4 = s_y^2\left[\frac{S_x^2+\alpha Q_r^2}{s_x^2+\alpha Q_r^2}\right],$$

$$t_5 = s_y^2\left[\frac{S_x^2+\alpha Q_d^2}{s_x^2+\alpha Q_d^2}\right],\ t_6 = s_y^2\left[\frac{S_x^2+\alpha Q_a^2}{s_x^2+\alpha Q_a^2}\right]$$

$$MSE(t_i) = \gamma S_y^4[\lambda_{40}^* + R_i^2\lambda_{04}^* - 2R_i\lambda_{22}^*],\ i = 1,2,...,6 \quad (3)$$

Where,

$0 \le \alpha \le 1$, $Q_i\,(i = 1,2,3)$ are the quartiles of auxiliary variable, $Q_r = Q_3 - Q_1$ is inter-quartile range, $Q_d = \dfrac{Q_3 - Q_1}{2}$ is semi-quartile range and $Q_a = \dfrac{Q_3 + Q_1}{2}$ is semi-quartile average and

$$R_1 = \frac{S_x^2}{S_x^2+\alpha Q_1^2},\ R_2 = \frac{S_x^2}{S_x^2+\alpha Q_2^2},\ R_3 = \frac{S_x^2}{S_x^2+\alpha Q_3^2},\ R_4 = \frac{S_x^2}{S_x^2+\alpha Q_r^2},\ R_5 = \frac{S_x^2}{S_x^2+\alpha Q_d^2},$$

$$R_6 = \frac{S_x^2}{S_x^2+\alpha Q_a^2}.$$

Solanki *et al.* (2015) suggested some modified ratio estimators of S_y^2 using the known quartiles and the functions of the quartiles of X. The estimators and the MSEs of their estimators is given by,

$$t_7 = s_y^2\left[\frac{C_xS_x^2+\alpha Q_1^2}{C_xs_x^2+\alpha Q_1^2}\right],t_8 = s_y^2\left[\frac{\beta_2(x)S_x^2+\alpha Q_2^2}{\beta_2(x)s_x^2+\alpha Q_2^2}\right],t_9 = s_y^2\left[\frac{\bar{X}S_x^2+\alpha Q_3^2}{\bar{X}s_x^2+\alpha Q_3^2}\right],$$

$$t_{10} = s_y^2\left[\frac{\beta_2(x)S_x^2+\alpha Q_r^2}{\beta_2(x)s_x^2+\alpha Q_r^2}\right],\ t_{11} = s_y^2\left[\frac{\rho S_x^2+\alpha Q_d^2}{\rho s_x^2+\alpha Q_d^2}\right],\ t_{12} = s_y^2\left[\frac{\rho S_x^2+\alpha Q_a^2}{\rho s_x^2+\alpha Q_a^2}\right]$$

$$MSE(t_i) = \gamma S_y^4[\lambda_{40}^* + R_i^2\lambda_{04}^* - 2R_i\lambda_{22}^*],\ i = 7,8,...,12 \quad (4)$$

Where,

$$0 \le \alpha \le 1 \text{ and } R_7 = \frac{C_xS_x^2}{C_xS_x^2+\alpha Q_1^2},\ R_8 = \frac{\beta_2(x)S_x^2}{\beta_2(x)S_x^2+\alpha Q_2^2},\ R_9 = \frac{\bar{X}S_x^2}{\bar{X}S_x^2+\alpha Q_3^2},$$

$$R_{10} = \frac{\beta_2(x)S_x^2}{\beta_2(x)S_x^2+\alpha Q_r^2},\ R_{11} = \frac{\rho S_x^2}{\rho S_x^2+\alpha Q_d^2},\ R_{12} = \frac{\rho S_x^2}{\rho S_x^2+\alpha Q_a^2}.$$

Lone *et al.* (2021) worked on modified estimators of S_y^2 using some known non-conventional parameters of X. The estimators and MSEs of their estimators are given by,

$$t_{13}=[\delta_1 s_y^2+\delta_2(S_x^2-s_x^2)]\left[\frac{C_x S_x^2+\alpha TM^2}{C_x s_x^2+\alpha TM^2}\right], t_{14}=[\delta_1 s_y^2+\delta_2(S_x^2-s_x^2)]\left[\frac{C_x S_x^2+\alpha MR^2}{C_x s_x^2+\alpha MR^2}\right],$$

$$t_{15}=[\delta_1 s_y^2+\delta_2(S_x^2-s_x^2)]\left[\frac{C_x S_x^2+\alpha HL^2}{C_x s_x^2+\alpha HL^2}\right], t_{16}=[\delta_1 s_y^2+\delta_2(S_x^2-s_x^2)]\left[\frac{C_x S_x^2+\alpha G^2}{C_x s_x^2+\alpha G^2}\right],$$

$$t_{17}=[\delta_1 s_y^2+\delta_2(S_x^2-s_x^2)]\left[\frac{C_x S_x^2+\alpha D^2}{C_x s_x^2+\alpha D^2}\right], t_{18}=[\delta_1 s_y^2+\delta_2(S_x^2-s_x^2)]\left[\frac{C_x S_x^2+\alpha S_{pw}^2}{C_x s_x^2+\alpha S_{pw}^2}\right]$$

$$MSE(t_i)=S_y^4[1+\delta_1^2 d_1+\delta_2^2 d_2+2\delta_1\delta_2 d_3-2\delta_1 d_4-2\delta_2 d_5]\ ,\ i=13,14,...,18 \quad (5)$$

Where, δ_1 and δ_2 are characterizing constants to be determined such that the $MSE(t_i)$ ($i=13,14,...,18$) are minimum, $|\alpha|\leq 1$ and $d_1=[1+\gamma(\mu_{40}^*+3\theta^2\mu_{04}^*-4\theta\mu_{22}^*)]$,

$d_2=\gamma R^2\mu_{04}^*$, $d_3=\gamma R(2\theta\mu_{04}^*-\mu_{22}^*)$, $d_4=[1+\gamma\theta(\theta\mu_{04}^*-\mu_{22}^*)]$, $d_5=\gamma R\theta\mu_{04}^*$, $R=\frac{S_x^2}{S_y^2}$,

$R_{13}=\frac{C_x S_x^2}{C_x S_x^2+\alpha TM^2}$, $R_{14}=\frac{C_x S_x^2}{C_x S_x^2+\alpha MR^2}$, $R_{15}=\frac{C_x S_x^2}{C_x S_x^2+\alpha HL^2}$, $R_{16}=\frac{C_x S_x^2}{C_x S_x^2+\alpha G^2}$,

$R_{17}=\frac{C_x S_x^2}{C_x S_x^2+\alpha D^2}$, $R_{18}=\frac{C_x S_x^2}{C_x S_x^2+\alpha S_{pw}^2}$, $TM=(Q_1+2Q_2+Q_3)/4$ is the tri-mean of X,

$HL=\text{Median }[X_j+X_k]/2, 1\leq j\leq k\leq N$ is the Hodges-Lehmann estimator of X,

$MR=[X_{(1)}+X_{(N)}]/2$ is the mid-range of X, $G=(4/N-1)\sum_{i=1}^{N}[(2i-N-1)/2N]X_{(i)}$ is the Gini's mean difference of X, $D=[2\sqrt{\pi}/N(N-1)]\sum_{i=1}^{N}[i-(N+1)/2]X_{(i)}$ is the Downton's method of X and $S_{pw}=(\sqrt{\pi}/N^2)\sum_{i=1}^{N}(2i-N-1)X_{(i)}$ is the probability-weighted moments of X.

The optimum values of δ_1 and δ_2 respectively are,

$$\delta_{1(opt)}=\frac{(d_2 d_4-d_3 d_5)}{(d_1 d_2-d_3^2)},\ \delta_{2(opt)}=\frac{(d_1 d_5-d_3 d_4)}{(d_1 d_2-d_3^2)}$$

The least values of $MSE(t_i)$ for these optimum values of δ_1 and δ_2 is,

$$MSE_{\min}(t_i)=S_y^4\left[1-\frac{(d_2 d_4^2-2d_3 d_4 d_5+d_1 d_5^2)}{(d_1 d_2-d_3^2)}\right] \quad (6)$$

3. INTRODUCED CLASS OF ESTIMATORS

It is well established from the literature that exponential ratio estimators are always more efficient than the ratio type estimators of any parameter of the main variable under consideration utilizing the known auxiliary parameters. Motivated by the above result and applying it to Lone *et al.* (2021), we introduce a naive class of exponential ratio type estimators of S_y^2 as,

$$t_{pj}=[\kappa_1 s_y^2+\kappa_2(S_x^2-s_x^2)]\exp\left[\frac{(aS_x^2+\alpha L_j^2)-(as_x^2+\alpha L_j^2)}{(aS_x^2+\alpha L_j^2)+(as_x^2+\alpha L_j^2)}\right] \quad (7)$$

Where κ_1 and κ_2 are characterizing scalars to be determined such that the $MSE(t_{pj})$ is minimum, $|\alpha|\leq 1$ and a and L_j are conventional and non-conventional auxiliary parameters respectively.

Remark 1: If we put $\kappa_1 = 1$, the introduced family of estimators takes the form of the following new family of estimators as,

$$t_{p1j} = [s_y^2 + \kappa_2 (S_x^2 - s_x^2)] \exp\left[\frac{(aS_x^2 + \alpha L_j^2) - (as_x^2 + \alpha L_j^2)}{(aS_x^2 + \alpha L_j^2) + (as_x^2 + \alpha L_j^2)}\right]$$

Remark 2: If we put $\kappa_2 = 1$, the proposed family of estimators becomes the following new family of estimators as,

$$t_{p2j} = [\kappa_1 s_y^2 + (S_x^2 - s_x^2)] \exp\left[\frac{(aS_x^2 + \alpha L_j^2) - (as_x^2 + \alpha L_j^2)}{(aS_x^2 + \alpha L_j^2) + (as_x^2 + \alpha L_j^2)}\right]$$

Remark 3: If we put $\kappa_1 = 1$ and $\kappa_1 = 1$, the introduced family of estimators reduces to the following family of estimators as,

$$t_{p3j} = [s_y^2 + (S_x^2 - s_x^2)] \exp\left[\frac{(aS_x^2 + \alpha L_j^2) - (as_x^2 + \alpha L_j^2)}{(aS_x^2 + \alpha L_j^2) + (as_x^2 + \alpha L_j^2)}\right]$$

Remark 4: If we put $\kappa_2 = 0$, the suggested family of estimators takes the form to the following class of estimators as,

$$t_{p4j} = \kappa_1 s_y^2 \exp\left[\frac{(aS_x^2 + \alpha L_j^2) - (as_x^2 + \alpha L_j^2)}{(aS_x^2 + \alpha L_j^2) + (as_x^2 + \alpha L_j^2)}\right]$$

Now different members of the introduced family of estimators may be obtained for different values of a and L_j. Some of the members of the suggested class of estimators using non-conventional auxiliary parameters are presented in the following Table1, given below as,

Table 1: Some members of the suggested class of estimators of S_y^2

S.No.	Estimator
1.	$t_{p1} = [\kappa_1 s_y^2 + \kappa_2 (S_x^2 - s_x^2)] \exp\left[\frac{(C_x S_x^2 + \alpha TM^2) - (C_x s_x^2 + \alpha TM^2)}{(C_x S_x^2 + \alpha TM^2) + (C_x s_x^2 + \alpha TM^2)}\right]$
2.	$t_{p2} = [\kappa_1 s_y^2 + \kappa_2 (S_x^2 - s_x^2)] \exp\left[\frac{(C_x S_x^2 + \alpha MR^2) - (C_x s_x^2 + \alpha MR^2)}{(C_x S_x^2 + \alpha MR^2) + (C_x s_x^2 + \alpha MR^2)}\right]$
3.	$t_{p3} = [\kappa_1 s_y^2 + \kappa_2 (S_x^2 - s_x^2)] \exp\left[\frac{(C_x S_x^2 + \alpha HL^2) - (C_x s_x^2 + \alpha HL^2)}{(C_x S_x^2 + \alpha HL^2) + (C_x s_x^2 + \alpha HL^2)}\right]$
4.	$t_{p4} = [\kappa_1 s_y^2 + \kappa_2 (S_x^2 - s_x^2)] \exp\left[\frac{(C_x S_x^2 + \alpha G^2) - (C_x s_x^2 + \alpha G^2)}{(C_x S_x^2 + \alpha G^2) + (C_x s_x^2 + \alpha G^2)}\right]$
5.	$t_{p5} = [\kappa_1 s_y^2 + \kappa_2 (S_x^2 - s_x^2)] \exp\left[\frac{(C_x S_x^2 + \alpha D^2) - (C_x s_x^2 + \alpha D^2)}{(C_x S_x^2 + \alpha D^2) + (C_x s_x^2 + \alpha D^2)}\right]$
6.	$t_{p6} = [\kappa_1 s_y^2 + \kappa_2 (S_x^2 - s_x^2)] \exp\left[\frac{(C_x S_x^2 + \alpha S_{pw}^2) - (C_x s_x^2 + \alpha S_{pw}^2)}{(C_x S_x^2 + \alpha S_{pw}^2) + (C_x s_x^2 + \alpha S_{pw}^2)}\right]$

4. BIAS AND MSE OF SUGGESTED ESTIMATOR

In this section, the expressions for the bias and MSE of the introduced family of estimators have been derived up to the first order of approximation. For the bias and MSE of the introduced estimator, we have used the following standard results regarding the errors given as:

$e_0 = \frac{s_y^2}{S_y^2} - 1$, $e_1 = \frac{s_x^2}{S_x^2} - 1$, $E(e_0) = E(e_1) = 0$ and $E(e_0^2) = \gamma \lambda_{40}^*$, $E(e_1^2) = \gamma \lambda_{04}^*$,, $E(e_0 e_1) = \gamma \lambda_{22}^*$

Expressing t_{pj} in terms of $e_i's$ ($i=0,1$), we have

$$t_{pj}=[\kappa_1S_y^2(1+e_0)-\kappa_2S_x^2e_1]\exp\left[\frac{-aS_x^2e_1}{2(aS_x^2+\alpha L_j^2)+aS_x^2e_1}\right]=[\kappa_1S_y^2(1+e_0)-\kappa_2S_x^2e_1]\exp\left[\frac{-\frac{R_j}{2}e_1}{1+\frac{R_j}{2}e_1}\right]$$

$$=[\kappa_1S_y^2(1+e_0)-\kappa_2S_x^2e_1]\exp\left[-\frac{R_j}{2}e_1\left(1+\frac{R_j}{2}e_1\right)^{-1}\right]=[\kappa_1S_y^2(1+e_0)-\kappa_2S_x^2e_1]\exp\left(-\frac{R_j}{2}e_1+\frac{R_j^2}{4}e_1^2\right)$$

Expanding the expression on right hand side, simplifying and taking the terms up to the first order of approximation, we have,

$$t_{pj}=\kappa_1S_y^2(1+e_0-\frac{R_j}{2}e_1-\frac{R_j}{2}e_0e_1+\frac{3}{8}R_j^2e_1^2)-\kappa_2S_x^2(e_1-\frac{R_j}{2}e_1^2)$$

Subtracting S_y^2 on both sides of the above equation, we have

$$t_{pj}-S_y^2=\kappa_1S_y^2(1+e_0-\frac{R_j}{2}e_1-\frac{R_j}{2}e_0e_1+\frac{3}{8}R_j^2e_1^2)-\kappa_2S_x^2(e_1-\frac{R_j}{2}e_1^2)-S_y^2$$

$$t_{pj}-S_y^2=S_y^2[\kappa_1(1+e_0-\frac{R_j}{2}e_1-\frac{R_j}{2}e_0e_1+\frac{3}{8}R_j^2e_1^2)-\kappa_2R(e_1-\frac{R_j}{2}e_1^2)-1]$$

(8)

Squaring on both sides of (8) and simplifying, taking expectations on both sides and putting values of different expectations we get the MSE of t_{pj} as,

$$MSE(t_{pj})=S_y^4\begin{bmatrix}1+\kappa_1^2(1+\gamma\lambda_{40}^*+R_j^2\gamma\lambda_{04}^*-2R_j\gamma\lambda_{22}^*)+\kappa_2^2R^2\gamma\lambda_{04}^*-2\kappa_1\kappa_2R\gamma(\lambda_{22}^*-R_j\lambda_{04}^*)\\-2\kappa_1(1-\frac{R_j}{2}\gamma\lambda_{22}^*+\frac{3}{8}R_j^2\gamma\lambda_{04}^*)-2\kappa_2R\frac{R_j}{2}\gamma\lambda_{04}^*\end{bmatrix}$$

$$MSE(t_{pj})=S_y^4[1+\kappa_1^2A+\kappa_2^2B-2\kappa_1\kappa_2C-2\kappa_1D-2\kappa_2F] \tag{9}$$

Where,

$A=(1+\gamma\lambda_{40}^*+R_j^2\gamma\lambda_{04}^*-2R_j\gamma\lambda_{22}^*)$, $B=R^2\gamma\lambda_{04}^*$, $C=R\gamma(\lambda_{22}^*-R_j\lambda_{04}^*)$, $D=(1-\frac{R_j}{2}\gamma\lambda_{22}^*+\frac{3}{8}R_j^2\gamma\lambda_{04}^*)$, $F=R\frac{R_j}{2}\gamma\lambda_{04}^*$.

The optimum values of κ_1 and κ_2 which minimizes the MSE of t_{pj} respectively are,

$$\kappa_{1(opt)}=\frac{(CF+BD)}{(AB-C^2)},\ \kappa_{2(opt)}=\frac{(AF+CD)}{(AB-C^2)}$$

The minimum value of the MSE of the proposed estimator t_{pj} for the above optimal values of κ_1 and κ_2 is,

$$MSE_{\min}(t_{pj})=S_y^4\left[1-\frac{\begin{Bmatrix}2(CF+BD)(AF+CD)C+2(CF+BD)(AB-C^2)D\\+2(AF+CD)(AB-C^2)F-(CF+BD)^2A-(AF+CD)B\end{Bmatrix}}{(AB-C^2)^2}\right]$$

$$MSE_{\min}(t_{pj})=S_y^4\left[1-\frac{P}{Q}\right] \tag{10}$$

Where,

$$P=\begin{Bmatrix}2(CF+BD)(AF+CD)C+2(CF+BD)(AB-C^2)D\\+2(AF+CD)(AB-C^2)F-(CF+BD)^2A-(AF+CD)B\end{Bmatrix},\ Q=(AB-C^2)^2$$

5. EFFICIENCY COMPARISON

Under this section, the introduced family of estimators has been compared with the mentioned competing estimators of S_y^2 and the efficiency conditions of the proposed estimator over competing estimators are obtained.

The introduced estimator t_{pj} performs better than the sample variance t_0 under the condition if,

$V(t_0) - MSE_{\min}(t_{pj}) > 0$, or $\gamma\lambda_{40}^* - \left[1 - \frac{P}{Q}\right] > 0$, or $\gamma\lambda_{40}^* + \frac{P}{Q} > 1$

The proposed estimator t_{pj} performs more efficiently than the estimator t_R of Isaki (1983) if the following condition is fulfilled as,

$MSE(t_R) - MSE_{\min}(t_{pj}) > 0$, or $\gamma[\lambda_{40}^* + \lambda_{04}^* - 2\lambda_{22}^*] - \left[1 - \frac{P}{Q}\right] > 0$, or $\gamma[\lambda_{40}^* + \lambda_{04}^* - 2\lambda_{22}^*] + \frac{P}{Q} > 1$

The suggested estimator t_{pj} is better than the Singh *et al.* (2013) estimators t_i, $i = 1, 2, ..., 6$ for the condition if,

$MSE(t_i) - MSE_{\min}(t_{pj}) > 0$, or $\gamma[\lambda_{40}^* + R_i^2\lambda_{04}^* - 2R_i\lambda_{22}^*] - \left[1 - \frac{P}{Q}\right] > 0$, or $\gamma[\lambda_{40}^* + R_i^2\lambda_{04}^* - 2R_i\lambda_{22}^*] + \frac{P}{Q} > 1$

The introduced estimator t_{pj} is better than the estimator t_i, $i = 7, 8, ..., 12$ of Solanki *et al.* (2015) if the following condition is fulfilled as,

$MSE(t_i) - MSE_{\min}(t_{pj}) > 0$, or $\gamma[\lambda_{40}^* + R_i^2\lambda_{04}^* - 2R_i\lambda_{22}^*] - \left[1 - \frac{P}{Q}\right] > 0$, or $\gamma[\lambda_{40}^* + R_i^2\lambda_{04}^* - 2R_i\lambda_{22}^*] + \frac{P}{Q} > 1$

The introduced estimator t_{pj} has lesser MSE than the Lone *et al.* (2021) estimators t_i, $i = 13, 14, ..., 18$ under the condition if,

$MSE_{\min}(t_i) - MSE_{\min}(t_{pj}) > 0$, or $\frac{P}{Q} - \frac{(d_2 d_4^2 - 2d_3 d_4 d_5 + d_1 d_5^2)}{(d_1 d_2 - d_3^2)} > 0$

6. EMPIRICAL STUDY

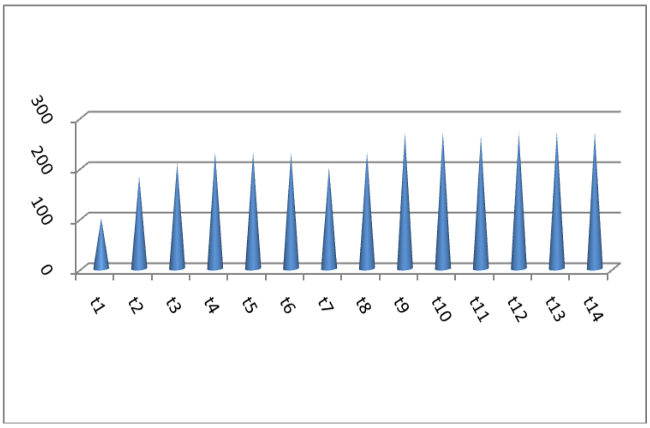
To compare the performances of introduced and the competing estimators of S_y^2 and to verify the efficiency conditions of the suggested estimator to be more efficient than the competing estimators, we have considered a natural population from Murthy (1967). The parameters of the above population are presented in Table 2.

Table 2: Parameters of the considered population

$N = 80$, $n = 20$, $\bar{Y} = 51.8264$, $\bar{X} = 11.2646$, $\rho = 0.9413$,
$S_y = 18.3569$, $C_y = 0.3542$, $S_x = 8.4563$, $C_x = 0.7507$, $\mu_{40} = 2.2667$,
$\mu_{04} = 2.8664$, $\mu_{22} = 2.2209$, $Q_1 = 5.1500$, $Q_2 = 10.3000$,
$Q_3 = 16.9750$, $Q_r = 11.8250$, $Q_d = 5.9125$, $Q_a = 11.0625$,
$TM = 9.3180$, $MR = 10.4050$, $HL = 17.9550$, $G = 9.0408$,
$D = 8.0138$, $S_{pw} = 7.9136$

The Percentage Relative Efficiency (PRE) of different competing estimators with respect to t_0 are presented in Table 3 for different values of $0 \le \alpha \le 1$. Table 4 represents the PRE of estimators of Lone *et al.* (2021) with respect to t_0 for different values of $|\alpha| \le 1$ and the PRE of the suggested estimators with respect to t_0 for different values of $|\alpha| \le 1$ are presented in Table 5.

Table 3: PRE of different estimator of S_y^2 with respect to t_0



S.No.	Estimator	Value of α	PRE
1.	t_0	-	100.00
2.	t_R	-	183.23
3.	t_1	1.00	209.67
4.	t_2	0.50	230.60
5.	t_3	0.30	230.28
6.	t_4	0.70	230.57
7.	t_5	1.00	200.87
8.	t_6	0.60	230.80
9.	t_7	1.00	270.38
10.	t_8	1.00	270.61
11.	t_9	1.00	263.50
12.	t_{10}	0.80	270.58
13.	t_{11}	1.00	270.61
14.	t_{12}	0.30	270.58

Table 4: PRE of Lone *et al.* (2021) estimators with respect to t_0 for different values of $|\alpha| \leq 1$

S. No.	Value of α	PRE of Estimator					
		t_{13}	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}
1.	-1.00	257.0391	259.3591	291.8213	258.2640	256.4163	253.0021
2.	-0.75	255.2481	257.2658	275.3213	256.0011	253.9286	252.8256
3.	-0.50	254.2359	256.9329	273.3492	255.3651	253.6165	252.4152
4.	-0.25	253.9139	256.1231	273.1029	254.9606	253.0259	252.0081
5.	0.00	253.1029	255.8569	273.0051	254.0436	252.8169	251.5621
6.	0.25	252.9320	255.0091	272.9321	253.8186	252.0411	251.0561
7.	0.50	253.1001	254.8563	272.3649	253.5646	251.9103	250.3961
8.	0.75	254.2569	254.6561	272.1536	253.3963	251.3851	250.0011
9.	1.00	254.2031	254.5341	271.9806	252.8646	251.0091	249.5651

Table 5: PRE of suggested class of estimators with respect to t_0 for different values of $|\alpha| \leq 1$

S. No.	Value of α	PRE of Estimator					
		t_{p1}	t_{p2}	t_{p3}	t_{p4}	t_{p5}	t_{p6}
1.	-1.00	296.5953	297.2631	336.5945	296.0036	296.8787	292.9524
2.	-0.75	293.5351	294.8557	316.6194	294.4013	293.0179	292.7493
3.	-0.50	291.3713	295.4728	314.3516	293.6699	292.6592	291.2775
4.	-0.25	291.0012	294.5416	314.0683	294.2047	291.9798	290.8093
5.	0.00	291.0683	294.2354	315.9559	292.1501	291.7394	290.2964
6.	0.25	290.8718	294.2605	315.8718	291.8914	289.8473	289.7145
7.	0.50	291.0653	294.0837	315.2196	292.5993	290.6968	288.9556
8.	0.75	293.3954	293.8545	314.9765	292.4057	290.0929	288.5013
9.	1.00	293.3336	293.7142	314.7777	291.7943	289.6605	287.9999

Figure 1 shows the PRE of sample variance, Isaki (1983) ratio estimator, Singh *et al.* (2013) estimators and Solanki *et al.* (2015) estimators of S_y^2 with respect to sample variance. Figure 2 depicts the PRE of Lone *et al.* (2021) estimators of S_y^2 with respect to sample variance and Figure 3 represents the PRE of suggested estimators of S_y^2 with respect to sample variance. Series 1 to Series 6 represents six Lone *et al.* (2021) and suggested estimators for the same non-conventional **Figure 1: PRE of different competing estimators with respect to t_0**
auxiliary parameters.

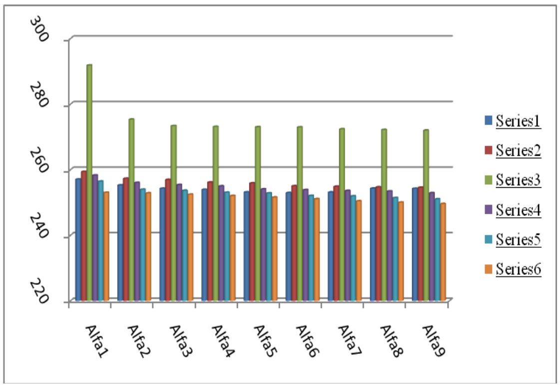


Figure 2: PRE of Lone *et al.* (2021) estimators w.r.t. t_0

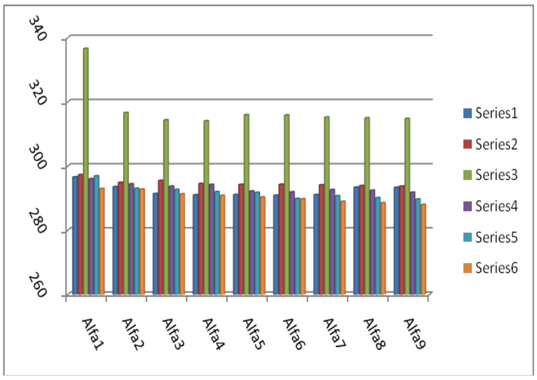


Figure 3: PRE of suggested estimators w.r.t. t_0

7. RESULTS AND DISCUSSION

It may be observed from table 3 that the PREs of the competing Isaki (1983) ratio estimator, Singh *et al.* (2013) estimators, and Solanki *et al.* (2015) estimators of S_y^2 with respect to sample variance lie between [183.23 270.61] and among these estimators, the estimator t_{11} is having the highest PRE over t_0 as 270.61, which can also be verified from Figure 1. From table 4, it is evident that the PRE of Lone *et al.* (2021) family of estimators with respect to t_0 ranges from [249.5651 291.8213] for different values of $|\alpha|\leq 1$ and further among six suggested estimators t_{15} is most efficient as it has the highest PRE over t_0 as 291.8213 among this class. The PREs of the introduced family of estimators for different $|\alpha|\leq 1$ ranges values [287.9999 336.5945] and the among the proposed family of estimators, the estimator t_{p3} is most efficient as it has the highest PRE as 336.5945 over t_0 among the class of all competing estimators.

8. CONCLUSION

In this article, we introduced a naive family of estimators of S_y^2 for enhanced estimation of S_y^2. We have derived the expressions for the bias and MSE of the proposed class of estimators up to the approximation of order one. The theoretical efficiency conditions for the introduced estimator to be better than the competing estimators are obtained, and these efficiency conditions are verified through the real natural population. It is investigated that the suggested class of estimators has the highest PRE with respect to sample variance in comparison to Isaki (1983) ratio estimator, Singh *et al.* (2013) family of estimators, Solanki *et al.* (2015) class of estimators, and Lone *et al.* (2021) family of estimators of S_y^2. The introduced family of estimators utilized the known information on some non-conventional robust auxiliary parameters. The proposed family of estimators is also good for estimating S_y^2 while dealing with populations with outliers and extreme values X . Thus the introduced class of estimators could be utilized for elevated estimation of S_y^2 in different areas of applications like Agricultural Sciences, Biological Sciences, Business, Economics, Mathematical Sciences, Medical Sciences, Social Sciences, etc.

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REFERENCES

[1] ADICHWAL, N.K., SHARMA, P., VERMA, H.K and SINGH, R. (2016): Generalized Class of Estimators for Population Variance Using Auxiliary Attribute, **Int. J. Appl. Comput. Math**, 2, 499-508.

[2] CEKIM, H.O. and KADILAR, C. (2020): Ln-Type Variance Estimators in Simple Random Sampling, **Pakistan Journal of Statistics and Operation Research**,16, 689-696.

[3] DARAZ, U. and KHAN, M. (2021): Estimation of variance of the difference-cum-ratio-type exponential estimator in simple random sampling, **RMS: Research in Mathematics & Statistics**, 8, 1-8.

[4] ISAKI, C, T. (1983): Variance estimation using auxiliary information, **Journal of American Statistical Association,** 78, 117- 123.

[5] KADILAR, C. and CINGI, H. (2006): Improvement in variance estimation using auxiliary information, **Hacettepe Journal of Mathematics and Statistics**, 35, 111-115.

[6] KHAN, M., and SHABBIR, J. (2013): A Ratio Type Estimator for the Estimation of Population Variance using Quartiles of an Auxiliary Variable, **Journal of Statistics Applications & Probability,** 2, 319-325.

[7] LONE, S.A., SUBZAR, M. and SHARMA, A. (2021): Enhanced Estimators of Population Variance with the Use of Supplementary Information in Survey Sampling, **Mathematical Problems in Engineering,** DOI: 10.1155/2021/9931217.

[8] MAQBOOL, S. and JAVAID, S. (2017): Variance Estimation Using Linear Combination of Tri-mean and Quartile Average, **American Journal of Biological and Environmental Statistics**, 3, 5-9.

[9] MAQBOOL, S., RAJA, T. A., and SHAKEEL J. (2016): Generalized modified ratio estimator using non-conventional location parameter, **Int. J. Agricult. Stat. Sci**, 12, 95-97.

[10] MUHAMMAD, I. and AMJAD, S. (2021): Efficient transformed ratio-type estimator using single auxiliary information, **Kuwait Journal of Science**, 48,1-10.

[11] MUNEER, S., KHALIL, A., SHABBIR, J. and NARJIS, G. (2018): A new improved ratio-product type exponential estimator of finite population variance using auxiliary information, **Journal of Statistical Computation and Simulation**, 88,.3179-3192.

[12] MURTHY, M. N. (1967): **Sampling: Theory and Methods**, Statistical Publishing Society, Calcutta, India.

[13] PRASAD, B. and SINGH, H. P. (1990): Some improved ratio type estimators of finite population variance in sample surveys, **Communication in Statistics: Theory and Methods**,. 19, 1127-1139.

[14] SANAULLAH, A. NIAZ, I. SHABBIR, J. and EHSAN, I. (2020): A class of hybrid type estimators for variance of a finite population in simple random sampling, **DOI:10.1080/03610918.2020.1776873**

[15] SINGH, H.P. and PAL, S.K. (2017): Estimation of population variance using known coefficient of variation of an auxiliary variable in sample surveys, **Journal of Statistics and Management Systems,** 20, 91-111.

[16] SINGH, H.P. and SOLANKI, R.S. (2010): Estimation of finite population variance using auxiliary information in presence of random non-response, **Gujarat Statistical Review**, 37,.46-58.

[17] SINGH, R., CHAUHAN, P., SAWAN, N. and SMARANDACHE, F. (2008): Ratio estimators in simple random sampling using information on auxiliary attribute, **Pakistan Journal of Statistics and Operation Research**,. 4, 47–53.

[18] SINGH, R., CHAUHAN, P., SAWAN, N. and SMARANDACHE, F (2011): Improved exponential estimator for population variance using two auxiliary variables, **Italian Journal of Pure and Applied Mathematics,** 28, 101-108.

[19] SINGH, R., PAL. S.K. and SOLANKI, R.S.(2013): Improved estimation of finite population variance using quartiles, **STAT˙IST˙IK: Journal of the Turkish Statistical Association**, l. 6,116–121.

[20] SOLANKI, R. S. SINGH, H. P. and PAL, S.K. (2015):. Improved ratio-type estimators of finite population variance using quartiles, **Hacettepe Journal of Mathematics and Statistics**, 44, 747–754.

[21] SOLANKI, R.S. and SINGH, H.P. (2013): An improved class of estimators for the population variance, **Model Assisted Statistics and Applications**, 8, 229-238.

[22] SUBRAMANI, J. and KUMARAPANDIYAN, G. (2012): Variance estimation using quartiles and their functions of an auxiliary variable, **International Journal of Statistics and Applications**, 2, 67-72.

[23] SUBRAMANI, J., and KUMARAPANDIYAN, G. (2013): A new modified ratio estimator of population mean when median of the auxiliary variable is known, **Pakistan Journal of Statistics and Operation Research**, 9,.137–145.
[24] TAILOR, R. and SHARMA, B. (2012): Modified estimators of population variance in presence of auxiliary information, **Statistics in Transition-new series**, 13, 37-46.
[25] UPADHYAYA, L. N. and SINGH, H. P. (1999): Use of auxiliary information in the estimation of population Variance, **Mathematical Forum**, 4, 33-36.
[26] YADAV S.K. and KADILAR, C. (2013): Improved Class of Ratio and Product Estimators, **Applied Mathematics and Computation**, 219, 10726-10731.
[27] YADAV S.K. and KADILAR, C.. (2014): A two-parameter variance estimator using auxiliary information, **Applied Mathematics and Computation**, 226, 117–122.
[28] YADAV, S. K., MISHRA, S. S., KUMAR, S. and KADILAR, C. (2016): A New Improved Class of Estimators For The Population Variance, **Journal of Statistics Applications & Probability,** 5, 1-7.
[29] YADAV, S. K., MISHRA, S. S., SHUKLA, A. K. and TIWARI, V. (2015): Improvement of Estimator for Population Variance using Correlation Coefficient and Quartiles of the Auxiliary Variable, **Journal of Statistics Applications & Probability**, 4, 259-263.
[30] YADAV, S.K., SHARMA, D.K. and MISHRA, S.S. (2019): Searching efficient estimator of population variance using tri-mean and third quartile of auxiliary variable, **Int. J. Business and Data Analytics,** 1, 30-40.
[31] YASMEEN, Y., AMIN, M. and HANIF, M. (2018): Exponential Estimators of Finite Population Variance Using Transformed Auxiliary Variables, **Proceedings of the National Academy of Sciences, India Section A: Physical Sciences**, DOI 10.1007/s40010-017-0410-5.
[32] ZAHID, E. and SHABBIR, J. (2019): Estimation of finite population mean for a sensitive variable using dual auxiliary information in the presence of measurement errors, **PLoS ONE** , 14,.e0212111.