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LOGARITHMIC-TYPE ESTIMATORS OF POPULATION COEFFICIENT OF VARIATION UNDER SUCCESSIVE SAMPLING IN THE PRESENCE OF NON-RESPONSE AND MEASUREMENT ERRORS

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ABSTRACT
This paper considered the effects of random non-response and measurement error on the estimation of the population coefficient of variation of the characteristic studied character in two-occasion successive sampling. To lessen the effects of non-response and measurement errors, logarithmic-type estimators for estimating the population coefficient of variation have been proposed. The theoretical properties of the estimators (biases and MSEs) were obtained using Taylor's series approximation method. Empirical studies have been conducted with simulated data and the results revealed that the proposed estimators demonstrated high level of efficiency, and appropriate recommendations have been made to survey statisticians for use in real-world scenarios.

KEYWORDS: Two-occasion successive sampling, Random non-response, Auxiliary variable, Coefficient of variation, Measurement error-

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RESUMEN
Este artículo considera los efectos de las no respuestas aleatorias y el error de medición en la estimación del coeficiente de variación poblacional de la característica bajo estudio en muestreos secuenciales en dos ocasiones. Para disminuir los efectos de la no respuesta y los errores de medición, se han propuesto estimadores de tipo logarítmico para estimar el coeficiente de variación poblacional. Las propiedades teóricas de los estimadores (sesgos y MSE) se obtuvieron utilizando el método de aproximación en Series de Taylor. Se realizaron estudios empíricos con datos simulados y los resultados revelaron que los estimadores propuestos demostraron un alto nivel de eficiencia. Se hicieron las recomendaciones apropiadas para encuestar a los estadísticos para su uso en escenarios del mundo real.

PALABRAS CLAVE: Muestreo secuencial en dos ocasiones, No respuesta aleatoria, Variable auxiliar, Coeficiente de variación, Error de medición,

1. INTRODUCTION

The coefficient of variation is a key parameter in population statistics that is frequently used to compare variability measured in different units. The coefficient of variation is a percentage that indicates how much variability there is in the data. For example, the coefficient of variation is a financial term that allows investors to assess how much volatility, or risk, is assumed concerning the projected return on investments, in the field of analytical chemistry, when doing quality assurance investigations in the domains of engineering or physics, it is utilized to express the precision and reproducibility of an assay. The coefficient of variation can also be used to compare the variabilities of two or more quantities that have different units or dimensions. for example, variabilities in weight and height of individuals, are in kilogram (kg) and centimeter (cm). The development and modification of estimators for estimating population coefficients of variation have recently gotten a lot of attention in sampling theory, and there have been a lot of efforts to enhance the precision of the estimate by using supplementary character information. Some authors that have worked in this direction include [23], [10], [15], [18], [14], [19], [25], [2]. However, the estimators proposed by these authors

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were based on the simple random sampling scheme which assumes that all the sampling units are accessible and availability of complete information about sampling units which are free from measurement errors. These assumptions are not often realistic in life situations. Incomplete sample information may occur as a result of non-availability of respondents, refusal to respond, non-understanding of a particular question etc while measurement errors may occur as a result of use of faulty equipment, inexperienced interviewers, etc. Also, in the presence surveys, the sampling scheme does not consider retention of some sampling units from the previous similar surveys, hence more expensive. Nevertheless, sometimes surveys do entail studying characteristics that tend to vary considerably over time. To address the limitations, we suggested new estimators based on successive sampling in the presence of non-response and measurement errors.

Successive sampling or successive occasions sampling is a procedure often adapted to monitor changes in a population over different points of time. Under successive sampling, some units drawn on the first occasion are preserved and used on the second occasion. The remaining sampling units are replaced with new units drawn on the current occasion. This partial replacement of units lowers survey expenses. Successive sampling is commonly employed in scientific and socioeconomic surveys to estimate the number of malaria deaths in a given year, for example, or the number of malaria deaths from year to year, or both at the same time, or to determine the average quantity of woollen clothing sold over the winter, or the difference in average woollen clothing sales between winter and summer, or both at the same time, etc.
[12] was the first to formulate the theory of successive sampling. [20], [21], and others have done recent work in this field. Additional information on a correlated variable may be used in many real-world circumstances to generate more exact estimations of population characteristics. For example, in health department surveys, the number of beds in the hospital, or the number of doctors employed may be treated as an auxiliary variable. Additional information on a correlated variable may be used in many real-world circumstances to generate more exact estimations of population characteristics. [6], and others have constructed estimators based on auxiliary data present on both occasions.
The influence of measurement errors on population mean estimators has been explored by [5], [13], and others. Also, simultaneous effects of non-response and measurement errors on estimators have been considered in the literature (See [22], [25]). Authors like [11], [4] and [16] study non-response using call-back approach while [1], [7], [8], [9], [17] and [24] utilized subsampling procedure. However, the majority of previous research has focused on the development of either estimators of population mean or variance in the presence of non-response or measurement error or both. Therefore, the current study considered the development of estimators of population coefficient of variation in the presence of random non-response and measurement errors under successive sampling using call-back approach.

2. NOTATIONS AND SAMPLE STRUCTURE

Let us consider a finite population $U=\{U_1,U_2,....U_N\}$ of size $N,$ which has been sampled over two occasions. Let the study character be denoted by X and Y on the first and second occasions respectively. We assume that information on an auxiliary variable Z is available on both occasions and its population mean is known. We further assume that response occurs on both occasions.
For the first occasion, Simple Random Sampling without Replacement (SRSWOR) is used to draw a preliminary sample of size n from the population where r_1 units do not respond. From the responding part of this sample, a second stage SRSWOR sample of size $m=n\lambda$ is drawn, where r_2 units do not respond. This sample is matched or retained for the second occasion, and information on the study variable Y is collected. In addition, a fresh sample of size $u=n-m=n\mu$ is drawn from the population using SRSWOR, and information on Y is collected again. Here, r_3 units do not respond. λ and $\mu(\lambda+\mu)=1$ are the fractions of matched and fresh samples respectively on the current (second) occasion.
Henceforth, the following notations have been used:

$\bar{X},\bar{Y}$: The population means of the study variables X and Y respectively.

$\bar{Z}$: The population mean of the auxiliary variable Z.

S_x^2,S_y^2 : The population variances of the study variables X and Y respectively.

S_z^2 : The population variance of the auxiliary variable Z.

$\bar{y}_m, \bar{y}_u, \bar{x}_n, \bar{x}_m, \bar{z}_n, \bar{z}_m, \bar{z}_u$: The sample means of the variables Y, X and Z respectively based on the respective sample sizes shown in suffice.

$\bar{z}_n = n^{-1}\sum_{i=1}^{n} z_i$, $\bar{z}_m = m^{-1}\sum_{i=1}^{m} z_i$, $\bar{z}_u = u^{-1}\sum_{i=1}^{u} z_i$: The sample means of auxiliary variable Z based on samples of size n, m and u respectively.

$s_{z_n}^2 = (n-1)^{-1}\sum_{i=1}^{n}\left(z_i - \bar{z}_n\right)^2$, $s_{z_m}^2 = (m-1)^{-1}\sum_{i=1}^{m}\left(z_i - \bar{z}_m\right)^2$, $s_{z_u}^2 = (u-1)^{-1}\sum_{i=1}^{u}\left(z_i - \bar{z}_u\right)^2$: The sample variances of auxiliary variable Z based on samples of size n, m and u respectively.

$\bar{y}_{m-r_2} = (m-r_2)^{-1}\sum_{i=1}^{m-r_2} y_i$, $\bar{y}_{u-r_3} = (u-r_3)^{-1}\sum_{i=1}^{u-r_3} y_i$: The sample means of Y based on the number of responding units in the samples of size m and u respectively on the first and second occasion.

$s_{y_{m-r_2}}^2 = (m-r_2-1)^{-1}\sum_{i=1}^{m-r_2}\left(y_i - \bar{y}_{m-r_2}\right)^2$, $s_{y_{u-r_3}}^2 = (u-r_3-1)^{-1}\sum_{i=1}^{u-r_3}\left(y_i - \bar{y}_{u-r_3}\right)^2$: The sample variances of Y based on the number of responding units in the samples of size m and u respectively on the first and second occasion.

$\bar{x}_{n-r_1} = (n-r_1)^{-1}\sum_{i=1}^{n-r_1} y_i$, $\bar{x}_{m-r_2} = (m-r_2)^{-1}\sum_{i=1}^{m-r_2} y_i$: The sample means of X based on the number of responding units in the samples of size n and m on the first and second occasion respectively.

$s_{x_{n-r_1}}^2 = (n-r_1-1)^{-1}\sum_{i=1}^{n-r_1}\left(x_i - \bar{x}_{n-r_1}\right)^2$, $s_{x_{m-r_2}}^2 = (m-r_2-1)^{-1}\sum_{i=1}^{m-r_2}\left(x_i - \bar{x}_{m-r_2}\right)^2$: The sample variances of X based on the number of responding units in the samples of size n and m respectively on the first and second occasion.

$c_{y_{m-r_2}} = \frac{s_{y_{m-r_2}}}{\bar{y}_{m-r_2}}, c_{x_{m-r_2}} = \frac{s_{x_{m-r_2}}}{\bar{x}_{m-r_2}}, c_{x_{n-r_1}} = \frac{s_{x_{n-r_1}}}{\bar{x}_{n-r_1}}, c_{z_m} = \frac{s_{z_m}}{\bar{z}_m}, c_{z_n} = \frac{s_{z_n}}{\bar{z}_n}, c_{z_u} = \frac{s_{z_u}}{\bar{z}_u}$: The sample coefficients of variation for the variables Y, X and Z respectively based on the respective sample sizes shown in suffice.

$\rho_{xy}, \rho_{yz}, \rho_{zx}$: The population correlation coefficients between the variables shown in suffice.

S_X^2, S_Y^2, S_Z^2 : The population mean squares of the variables X, Y and Z respectively.

$C_X, C_Y, C_Z, C_{XY}, C_{YZ}, C_{ZX}$: The coefficient of variation based on the variables in the suffices.

2.1 Non-Response Probability Functions

Let $r_1\{r_1 = 0,1,2,....,n-2\}$ be the number of units in the sample S_n of size n on which information on X could not be recorded due to random non-response. Let $r_2\{r_2 = 0,1,2,....,m-2\}$ be the number of units in the sample S_m of size m on which information on Y could not be recorded due to random non-response on the second occasion. Finally, let $r_3\{r_3 = 0,1,2,....,u-2\}$ be the number of units in the sample of size u on which information on Y could not be recorded due to random non-response on the second occasion. We assume that $0 \le r_1 \le (n-2)$, $0 \le r_2 \le (m-2)$, and $0 \le r_3 \le (u-2)$. If p_1, p_2, p_3 are the probabilities of non-responses among the $(n-2)$, $(m-2)$ and $(u-2)$ possible values of non-responses respectively and $q_1 = 1-p_1,\ q_2 = 1-p_2,\ q_3 = 1-p_3$ are the probabilities of responses among the n, m and u possible values of sampling units respectively, then r_1, r_2 and r_3 have the following discrete probability distributions respectively.

$$P(r_1) = \frac{n-r_1}{nq_1 + 2p_1}\binom{n-2}{r_1} p_1^{r_1} q_1^{n-r_1-2};\ r_1 = 0,1,2,........,n-2 \quad (1)$$

$$P(r_2)=\frac{m-r_2}{nq_2+2p_2}\binom{m-2}{r_2}p_2^{r_2}q_2^{m-r_2-2};\ r_2=0,1,2,........,m-2 \tag{2}$$

$$P(r_3)=\frac{u-r_3}{nq_3+2p_3}\binom{u-2}{r_3}p_3^{r_3}q_3^{u-r_3-2};\ r_3=0,1,2,........,u-2 \tag{3}$$

where $q_1=1-p_1$, $q_2=1-p_2$ and $q_3=1-p_3$

Here, $\binom{n-2}{r_1}$, $\binom{m-2}{r_2}$ and $\binom{u-2}{r_3}$ are the total numbers of ways of obtaining $r_i\ (i=1,2,3)$ non-response out of $(n-2)$, $(m-2)$ and $(u-2)$ possible values of non-responses respectively.

3. PROPOSED ESTIMATORS

Motivated by the above discussions and following the estimation strategies of population coefficient of variation as developed by [2] and [25], we suggest the following estimators T_1,T_2,T_3 of population coefficient of variation based on the sample S_m of size m common to both the occasions as

$$T_1=c_{y_m}^{*}+\log\left(1+k_1\left(\bar{x}_n^{*}-\bar{x}_m^{*}\right)\right) \tag{4}$$

$$T_2=c_{y_m}^{*}+\log\left(1+k_2\left(s_{x_n}^{*2}-s_{x_m}^{*2}\right)\right) \tag{5}$$

$$T_3=c_{y_m}^{*}+\log\left(1+k_3\left(c_{x_n}^{*}-c_{x_m}^{*}\right)\right) \tag{6}$$

where $k_j,\ j=1,2,3$ are suitable scalar quantities to be determined by minimizing the mean square error of the estimators $t_j,\ j=1,2,3$, $c_{y_m}^{*}=c_{y_{m-r_2}}+\log\left(1+\frac{C_z-c_{z_m}}{C_z+c_{z_m}}\right)$, $\bar{x}_m^{*}=\bar{x}_{m-r_2}+\log\left(1+\frac{\bar{Z}-\bar{z}_m}{\bar{Z}+\bar{z}_m}\right)$, $s_{x_m}^{*2}=s_{x_{m-r_2}}^{2}+\log\left(1+\frac{S_z^2-s_{z_m}^2}{S_z^2+s_{z_m}^2}\right)$, $c_{x_m}^{*}=c_{x_{m-r_2}}+\log\left(1+\frac{C_z-c_{z_m}}{C_z+c_{z_m}}\right)$, $c_{x_n}^{*}=c_{x_{n-r_1}}+\log\left(1+\frac{C_z-c_{z_n}}{C_z+c_{z_n}}\right)$, $\bar{x}_n^{*}=\bar{x}_{n-r_1}+\log\left(1+\frac{\bar{Z}-\bar{z}_n}{\bar{Z}+\bar{z}_n}\right)$, $s_{x_n}^{*2}=s_{x_{n-r_1}}^{2}+\log\left(1+\frac{S_z^2-s_{z_n}^2}{S_z^2+s_{z_n}^2}\right)$.

Similarly, we suggest the following estimators t_1,t_2,t_3 of population coefficient of variation based on the fresh sample S_u of size u drawn on the current occasion as

$$t_1=c_{y_{u-r_3}}+\log\left(1+a_1\frac{\bar{Z}-\bar{z}_u}{\bar{Z}+\bar{z}_u}\right) \tag{7}$$

$$t_2=c_{y_{u-r_3}}+\log\left(1+a_2\frac{S_z^2-s_{z_u}^2}{S_z^2+s_{z_u}^2}\right) \tag{8}$$

$$t_3=c_{y_{u-r_3}}+\log\left(1+a_3\frac{C_z-c_{z_u}}{C_z+c_{z_u}}\right) \tag{9}$$

where $a_i, i=1,2,3$ are suitable scalar quantities to be determined by minimizing the mean square error of the estimators $t_j, j=1,2,3$

Now, we define the following convex linear combination of pairs of the proposed logarithmic type estimators $T_j, t_j, \; j=1,2,3$ denoted by $\Phi_i, i=1,2,3...,9$ for population coefficient of variation C_y of the study variable Y on the current occasion in two-occasion successive sampling as

$$\Phi_i = \theta_i T_j + \left(1-\theta_i\right) t_j, \quad i=1,2,...,9, \; j=1,2,3. \tag{10}$$

where $\theta_i \left(0 \le \theta_i \le 1\right)$ are suitable scalar quantities to be determined by minimizing the mean square error of the estimators $\Phi_i, i=1,2,..,9$ in order to deal with the problems of each occasion and change from one occasion to the next.

3.1 Properties of the proposed estimators

The biases and mean square errors of proposed estimators are derived up to the first order of approximations under large sample assumptions and using the following transformations: $\bar{y}_{m-r_2} = \bar{Y}\left(1+e_0\right)$, $\bar{x}_{m-r_2} = \bar{X}\left(1+e_1\right)$, $\bar{x}_{n-r_1} = \bar{X}\left(1+e_2\right)$, $\bar{y}_{u-r_3} = \bar{Y}\left(1+e_3\right)$, $\bar{z}_m = \bar{Z}\left(1+e_4\right)$, $\bar{z}_n = \bar{Z}\left(1+e_5\right)$, $\bar{z}_u = \bar{Z}\left(1+e_6\right)$, $s^2_{y_{m-r_2}} = S^2_y\left(1+e_7\right)$, $s^2_{z_m} = S^2_z\left(1+e_8\right)$, $s^2_{z_n} = S^2_z\left(1+e_9\right)$, $s^2_{x_{n-r_1}} = S^2_x\left(1+e_{10}\right)$, $s^2_{y_{u-r_3}} = S^2_y\left(1+e_{11}\right)$, $s^2_{z_u} = S^2_z\left(1+e_{12}\right)$, $s^2_{x_{m-r_2}} = S^2_x\left(1+e_{13}\right)$.

Such that $E\left(e_i\right)=0$ and $\left|e_i\right| \le 1, \; \forall \; i=0,1,2,......,13$ .Then, we obtain the following expectations:

$E\left(e_0^2\right)=\eta_2 C_y^2$, $E\left(e_1^2\right)=\eta_2 C_x^2$, $E\left(e_2^2\right)=\eta_1 C_x^2$, $E\left(e_3^2\right)=\eta_3 C_y^2$, $E\left(e_4^2\right)=\eta_5 C_z^2$, $E\left(e_5^2\right)=\eta_4 C_z^2$, $E\left(e_6^2\right)=\eta_6 C_z^2$, $E\left(e_7^2\right)=\eta_2\left(\lambda_{400}-1\right)$, $E\left(e_8^2\right)=\eta_5\left(\lambda_{004}-1\right)$, $E\left(e_9^2\right)=\eta_4\left(\lambda_{004}-1\right)$, $E\left(e_0 e_1\right)=\eta_2 C_{xy}$
$E\left(e_{10}^2\right)=\eta_1\left(\lambda_{040}-1\right)$, $E\left(e_{11}^2\right)=\eta_3\left(\lambda_{400}-1\right)$, $E\left(e_{12}^2\right)=\eta_6\left(\lambda_{004}-1\right)$, $E\left(e_{13}^2\right)=\eta_2\left(\lambda_{040}-1\right)$,
$E\left(e_0 e_2\right)=\eta_1 C_{xy}$, $E\left(e_0 e_4\right)=\eta_5 C_{yz}$, $E\left(e_0 e_5\right)=\eta_4 C_{yz}$, $E\left(e_0 e_7\right)=\eta_2 C_y \lambda_{300}$, $E\left(e_0 e_8\right)=\eta_5 C_y \lambda_{003}$,
$E\left(e_0 e_9\right)=\eta_4 C_y \lambda_{003}$, $E\left(e_0 e_{10}\right)=\eta_1 C_y \lambda_{030}$, $E\left(e_0 e_{13}\right)=\eta_2 C_y \lambda_{030}$, $E\left(e_1 e_2\right)=\eta_1 C_x^2$ $E\left(e_1 e_4\right)=\eta_5 C_{xz}$,
$E\left(e_1 e_5\right)=\eta_4 C_{xz}$, $E\left(e_1 e_7\right)=\eta_2 C_x \lambda_{210}$, $E\left(e_1 e_8\right)=\eta_5 C_x \lambda_{012}$, $E\left(e_1 e_9\right)=\eta_4 C_x \lambda_{012}$, $E\left(e_1 e_{10}\right)=\eta_1 C_x \lambda_{030}$,
$E\left(e_1 e_{13}\right)=\eta_2 C_x \lambda_{030}$, $E\left(e_2 e_4\right)=\eta_1 C_{xz}$, $E\left(e_2 e_5\right)=\eta_4 C_{xz}$, $E\left(e_2 e_7\right)=\eta_1 C_x \lambda_{210}$, $E\left(e_2 e_8\right)=\eta_1 C_x \lambda_{012}$,
$E\left(e_2 e_9\right)=\eta_4 C_x \lambda_{012}$, $E\left(e_2 e_{10}\right)=\eta_1 C_x \lambda_{030}$, $E\left(e_2 e_{13}\right)=\eta_1 C_x \lambda_{030}$, $E\left(e_3 e_6\right)=\eta_6 C_{yz}$ $E\left(e_3 e_{11}\right)=\eta_3 C_y \lambda_{300}$,
$E\left(e_3 e_{12}\right)=\eta_6 C_y \lambda_{102}$, $E\left(e_4 e_5\right)=\eta_4 C_z^2$, $E\left(e_4 e_7\right)=\eta_4 C_z \lambda_{201}$, $E\left(e_4 e_8\right)=\eta_5 C_z \lambda_{003}$, $E\left(e_4 e_9\right)=\eta_4 C_z \lambda_{003}$
$E\left(e_4 e_{10}\right)=\eta_1 C_z \lambda_{021}$, $E\left(e_4 e_{13}\right)=\eta_4 C_z \lambda_{021}$, $E\left(e_5 e_7\right)=\eta_4 C_z \lambda_{201}$, $E\left(e_5 e_8\right)=\eta_4 C_z \lambda_{003}$ $E\left(e_5 e_9\right)=\eta_4 C_z \lambda_{003}$
$E\left(e_5 e_{10}\right)=\eta_4 C_z \lambda_{021}$, $E\left(e_5 e_{13}\right)=\eta_4 C_z \lambda_{021}$, $E\left(e_6 e_{11}\right)=\eta_6 C_z \lambda_{201}$, $E\left(e_6 e_{12}\right)=\eta_6 C_z \lambda_{003}$,
$E\left(e_7 e_8\right)=\eta_5\left(\lambda_{202}-1\right)$, $E\left(e_7 e_9\right)=\eta_4\left(\lambda_{202}-1\right)$, $E\left(e_7 e_{10}\right)=\eta_1\left(\lambda_{220}-1\right)$, $E\left(e_7 e_{13}\right)=\eta_2\left(\lambda_{220}-1\right)$,
$E\left(e_8 e_9\right)=\eta_4\left(\lambda_{004}-1\right)$, $E\left(e_8 e_{10}\right)=\eta_1\left(\lambda_{022}-1\right)$, $E\left(e_8 e_{13}\right)=\eta_5\left(\lambda_{022}-1\right)$, $E\left(e_9 e_{10}\right)=\eta_4\left(\lambda_{022}-1\right)$,
$E\left(e_9 e_{13}\right)=\eta_4\left(\lambda_{022}-1\right)$, $E\left(e_{10} e_{13}\right)=\eta_1\left(\lambda_{040}-1\right)$, $E\left(e_{11} e_{12}\right)=\eta_6\left(\lambda_{202}-1\right)$.

where $\eta_1 = \frac{1}{nq_1 + 2p_1} - \frac{1}{N}, \eta_2 = \frac{1}{mq_2 + 2p_2} - \frac{1}{N}, \eta_3 = \frac{1}{uq_3 + 2p_3} - \frac{1}{N}, \eta_4 = \frac{1}{n} - \frac{1}{N}, \eta_5 = \frac{1}{m} - \frac{1}{N},$

$\eta_6 = \frac{1}{u} - \frac{1}{N}$

$\lambda_{pqs} = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^p\left(x_i - \bar{X}\right)^q\left(z_i - \bar{Z}\right)^s$

3.1.1 Biases and MSEs of Estimators $T_i,\ i = 1,2,3$

Using transformations above, equation (4)-(6), estimators $T_i,\ i = 1,2,3$ becomes,

$$T_1 - C_y = C_y\left(e_0^2 - e_0 + \frac{e_7}{2} - \frac{e_0e_7}{2} - \frac{e_7^2}{8}\right) + \frac{e_4}{2} - \frac{3e_4^2}{8} - \frac{3e_8^2}{32} - \frac{e_8}{4} + k_1\left\{\bar{X}\left(e_2 - e_1\right) - \frac{\left(e_5 - e_4\right)}{2} + \frac{3\left(e_5^2 - e_4^2\right)}{8}\right\}$$
$$-\frac{k_1^2}{2}\left\{\bar{X}^2\left(e_2^2 + e_1^2 - 2e_1e_2\right) + \frac{\left(e_5^2 + e_4^2 - 2e_4e_5\right)}{4} - \bar{X}\left(e_2e_5 - e_2e_4 - e_1e_5 + e_1e_4\right)\right\} \quad (11)$$

$$T_2 - C_y = C_y\begin{pmatrix} e_0^2 - e_0 + \frac{e_7}{2} \\ -\frac{e_0e_7}{2} - \frac{e_7^2}{8}\end{pmatrix} + \frac{e_4}{2} - \frac{3e_4^2}{8} - \frac{3e_8^2}{32} - \frac{e_8}{4} + k_2\left\{S_x^2\left(e_{10} - e_{13}\right) - \frac{\left(e_9 - e_8\right)}{2} + \frac{3\left(e_9^2 - e_8^2\right)}{8}\right\}$$
$$-\frac{k_2^2}{2}\left\{S_x^4\left(e_{10}^2 + e_{13}^2 - 2e_{10}e_{13}\right) + \frac{\left(e_9^2 + e_8^2 - 2e_8e_9\right)}{4} - S_x^2\left(e_9e_{10} - e_8e_{10} - e_9e_{13} + e_8e_{13}\right)\right\} \quad (12)$$

$$T_3 - C_y = C_y\left(e_0^2 - e_0 + \frac{e_7}{2} - \frac{e_0e_7}{2} - \frac{e_7^2}{8}\right) + \frac{e_4}{2} - \frac{3e_4^2}{8} - \frac{3e_8^2}{32} - \frac{e_8}{4} + k_3\left\{\frac{\left(e_5 - e_4\right)}{2} + \frac{3\left(e_5^2 - e_4^2\right)}{8} - \frac{3\left(e_5^2 - e_4^2\right)}{32}\right.$$
$$\left. - \frac{\left(e_9 - e_8\right)}{4} + C_x\left(e_2^2 - e_2 + \frac{e_{10}}{2} - \frac{e_2e_{10}}{2} - \frac{e_{10}^2}{8} - e_1^2 + e_1 - \frac{e_{13}}{2} + \frac{e_1e_{13}}{2} + \frac{e_{13}^2}{8}\right)\right\}$$
$$+\frac{k_3^2}{2}\left\{C_x^2\left(e_2^2 - e_2e_{10} + e_{13}^2 - 2e_{10}e_{13} + e_1e_{13} + \frac{e_{10}^2}{2} + e_1e_{10} - \frac{e_{10}e_{13}}{2} + e_1^2 - e_1e_{13} + \frac{e_{13}^2}{4}\right)\right. \quad (13)$$
$$+C_x\left(e_2e_4 - e_2e_5 + \frac{e_5e_{10}}{2} - \frac{e_4e_{10}}{2} + e_1e_5 - e_1e_4 - \frac{e_2e_8 - e_2e_9 + e_1e_9 - e_1e_8}{2}\right.$$
$$\left.\left. - \frac{e_9e_{10} - e_8e_{10}}{4} - \frac{\left(e_5^2 + e_5^2 - 2e_4e_5\right) - \left(e_5e_9 - e_5e_8 - e_4e_9 + e_4e_8\right)}{8} - \frac{\left(e_9^2 + e_8^2 - 2e_8e_9\right)}{32}\right)\right\}$$

Taking expectations of (11), (12), (13) and apply the results of expected values of the error terms in section 5, we obtained the biases of estimator $T_i,\ i = 1,2,3$ as

$$Bias\left(T_1\right) = f_1 + k_1f_2 + k_1^2f_3 \quad (14)$$

$$Bias\left(T_2\right) = g_1 + k_2g_2 + k_2^2g_3 \quad (15)$$

$$Bias\left(T_3\right) = h_1 + k_3h_2 + k_3^2h_3 \quad (16)$$

where $f_1 = C_y\eta_2\left(C_y^2 - \frac{C_y\lambda_{300}}{2} - \frac{(\lambda_{400}-1)}{8}\right) + \frac{3}{8}\eta_5\left(C_z^2 + \frac{(\lambda_{004}-1)}{4}\right)$, $f_2 = \frac{3}{8}\eta_9 C_z^2$, $g_2 = \frac{3}{8}\eta_9(\lambda_{004}-1)$, ,

$$f_3 = \frac{1}{2}\left(\eta_7 \bar{X}^2 C_x^2 - \frac{1}{4}\eta_9 C_z^2 - \bar{X}\eta_8 C_{xz}\right), g_1 = C_y\eta_2\left(C_y^2 - \frac{C_y\lambda_{300}}{2} - \frac{(\lambda_{400}-1)}{8}\right) + \frac{3}{8}\eta_5\left(C_z^2 + \frac{(\lambda_{004}-1)}{4}\right)$$

$$g_3 = \frac{\eta_7 S_x^4(\lambda_{040}-1)}{2} - \frac{\eta_9(\lambda_{004}-1)}{8} - \frac{\eta_8 S_x^2(\lambda_{022}-1)}{2}$$

$$h_1 = C_y\eta_2\left(C_y^2 - \frac{C_y\lambda_{300}}{2} - \frac{(\lambda_{400}-1)}{8}\right) + \frac{3\eta_5}{8}\left(C_z^2 + \frac{(\lambda_{004}-1)}{4}\right),$$

$$h_2 = C_x\left(\eta_7\left(\frac{C_x\lambda_{030}}{2} + \frac{(\lambda_{040}-1)}{8} - C_x^2\right) - \frac{3}{8}\eta_5\left(C_z^2 + \frac{(\lambda_{004}-1)}{4}\right)\right),$$

$$h_3 = \frac{1}{2}\left\{C_x^2\left(-\eta_7 C_x^2 + (\lambda_{004}-1)\left(\eta_2 - \frac{1}{4}\eta_1\right) + \eta_1\left(C_{xz} - \frac{C_x\lambda_{021}}{2}\right) + \eta_4 C_x\lambda_{021} - \eta_5 C_{xz}\right)\right.$$
$$\left. - \frac{C_x}{2}\left(-\eta_8 C_x\lambda_{021} + \frac{1}{2}\eta_{10}(\lambda_{022}-1) - \frac{1}{4}\eta_9\left(\left(C_z^2 - C_z\lambda_{003}\right) + \frac{1}{4}(\lambda_{004}-1)\right)\right)\right\}$$

Squaring both sides of (17), (18) and (19) and take expectation of the resulting equations to obtain the MSE of estimator $T_i,\ i = 1,2,3$ as

$$MSE(T_1) = A_0 + k_1 A_1 + k_1^2 A_2 \tag{17}$$

$$MSE(T_2) = B_0 + k_2 B_1 + k_2^2 B_2 \tag{18}$$

$$MSE(T_3) = D_0 + k_3 D_1 + k_3^2 D_2 \tag{19}$$

where, $\eta_7 = \eta_2 - \eta_1$, $\eta_8 = \eta_5 - \eta_1$, $\eta_9 = \eta_4 - \eta_5$, $\eta_{10} = \eta_4 - \eta_1$,

$$A_0 = C_y^2\eta_2\left(C_y^2 + \frac{(\lambda_{400}-1)}{4} - C_y\lambda_{300}\right) + C_y\eta_5\left(\frac{C_y\lambda_{201}}{2} - C_{yz}\right) - \frac{C_y}{2}\eta_5\left(\frac{(\lambda_{202}-1)}{2} - C_y\lambda_{003}\right) + \frac{1}{4}\eta_5\left(C_z^2 - C_z\lambda_{003}\right)$$

$$, A_1 = -2\bar{X}C_y\left(\frac{C_x\lambda_{210}}{2} - C_{xy}\right)\eta_7 + C_y\eta_9\left(C_{xy} - \frac{C_x\lambda_{210}}{2}\right) - \bar{X}\eta_8 C_{xz} - \frac{1}{2}\eta_9 C_z^2 + \frac{1}{2}\bar{X}\eta_8 C_x\lambda_{012} + \eta_9 C_z\lambda_{003},$$

$$A_2 = \bar{X}^2\eta_7 C_x^2 - \frac{1}{4}\eta_9 C_z^2 - \bar{X}\eta_{10}C_{xz}.$$

$$B_0 = C_y^2\eta_2\left(C_y^2 + \frac{(\lambda_{400}-1)}{4} - C_y\lambda_{300}\right) + \eta_5\left(C_y\left(\frac{C_y\lambda_{201}}{2} - C_{yz}\right) - \frac{C_y}{2}\left(\frac{(\lambda_{202}-1)}{2} - C_y\lambda_{003}\right)\right.$$
$$\left. + \frac{\left(C_z^2 - C_z\lambda_{003}\right)}{4} + \frac{(\lambda_{004}-1)}{16}\right)$$

$$B_1 = -S_x^2\left(\frac{\lambda_{220}-1}{2} - C_y\lambda_{030}\right)\eta_7 + \frac{1}{2}\eta_9\left(C_y\lambda_{003} - \frac{\lambda_{202}-1}{2} - C_z\lambda_{003} + \frac{1}{2}(\lambda_{004}-1)\right) - S_x^2\eta_8\left(C_z\lambda_{021} - \frac{1}{2}(\lambda_{022}-1)\right)$$

$$B_2 = S_x^4\eta_7(\lambda_{040}-1) - \frac{1}{4}\eta_9(\lambda_{004}-1) - S_x^2\eta_{10}(\lambda_{022}-1).$$

$$D_0 = C_y^2\eta_2\left(C_y^2+\frac{(\lambda_{400}-1)}{4}-C_y\lambda_{300}\right)+\eta_5\left(C_y\left(\frac{C_y\lambda_{201}}{2}-C_{yz}\right)-\frac{C_y}{2}\left(\frac{(\lambda_{202}-1)}{2}-C_y\lambda_{003}\right)\right.$$
$$\left.+\frac{C_z^2}{2}-C_z\lambda_{003}+\frac{1}{16}(\lambda_{004}-1)\right)$$
$$C_1 = -C_x\eta_7\left(\frac{\lambda_{202}-1}{4}-\frac{C_x\lambda_{210}}{2}-\frac{C_y\lambda_{030}}{2}+C_{xy}\right)+\frac{1}{2}\eta_9\left(\frac{C_z\lambda_{210}}{2}-C_{yz}-\frac{\lambda_{202}-1}{4}\right.$$
$$\left.+\frac{C_y\lambda_{030}}{2}-C_z^2+\frac{C_z\lambda_{030}}{2}+C_z\lambda_{003}-\frac{(\lambda_{004}-1)}{2}\right)+C_x\eta_8\left(\frac{C_z\lambda_{021}}{2}-C_{xz}+\frac{1}{2}\left((\lambda_{021}-1)-C_x\lambda_{012}\right)\right)$$
$$C_2 = C_x^2\left(-\eta_7\left(C_x\lambda_{012}-C_x^2\right)+\eta_2\frac{(\lambda_{040}-1)}{2}\right)+\frac{1}{16}\eta_5(\lambda_{004}-1)+C_x\eta_1\begin{pmatrix}C_{xz}-\frac{C_z\lambda_{021}}{2}\\ -\frac{C_x\lambda_{012}}{2}+\frac{(\lambda_{022}-1)}{4}\end{pmatrix}-\frac{1}{4}\eta_9\left(C_z^2-C_z\lambda_{003}\right)$$

Differentiating (20), (21) and (22) with respect to k_1, k_2, k_3 respectively, equate the results to zeros and solve for k_1, k_2, k_3, we obtain,

$$k_{1(opt)} = -2^{-1}A_1A_2^{-1} \tag{20}$$
$$k_{2(opt)} = -2^{-1}B_1B_2^{-1} \tag{21}$$
$$k_{3(opt)} = -2^{-1}D_1D_2^{-1} \tag{22}$$

Substituting $k_{1(opt)}$ $k_{1(opt)}$ and $k_{1(opt)}$ into (17), (18) and (19) respectively, minimum MSE of $T_i,\ i=1,2,3$ is obtained as in (23), (24) and (25) respectively

$$MSE(T_1)_{min} = A_0-\frac{A_1^2}{4A_2} \tag{23}$$
$$MSE(T_2)_{min} = B_0-\frac{B_1^2}{4B_2} \tag{24}$$
$$MSE(T_3)_{min} = D_0-\frac{D_1^2}{4D_2} \tag{25}$$

3.1.2 Biases and MSEs of Estimators $t_i,\ i=1,2,3$

Expressing $t_i,\ i=1,2,3$ in terms of error terms defined above, we have,

$$t_1-C_y = C_y\left(e_3^2-e_3+\frac{e_{11}}{2}-\frac{e_{11}e_3}{8}-\frac{e_{11}^2}{8}\right)-a_1\left(\frac{e_6}{2}-\frac{e_6^2}{4}\right)-\frac{a_1^2e_6^2}{4} \tag{26}$$
$$t_2-C_y = C_y\left(e_3^2-e_3+\frac{e_{11}}{2}-\frac{e_{11}e_3}{8}-\frac{e_{11}^2}{8}\right)-a_2\left(\frac{e_{12}}{2}-\frac{e_{12}^2}{4}\right)-\frac{a_2^2e_{12}^2}{4} \tag{27}$$
$$t_3-C_y = C_y\left(e_3^2-e_3+\frac{e_{11}}{2}-\frac{e_{11}e_3}{8}-\frac{e_{11}^2}{8}\right)-a_3\left(\frac{e_6^2}{4}-\frac{e_6}{2}+\frac{e_{12}}{2}+\frac{e_{12}e_6}{8}-\frac{e_{12}^2}{8}\right)-a_3^2\left(\frac{e_6^2}{4}+\frac{e_{12}^2}{16}-\frac{e_{12}e_6}{8}\right) \tag{28}$$

Take expectation from both sides to obtain the bias of t_1 as

$$Bias\left(t_1\right)=C_y\eta_3\left(C_y^2-\frac{C_y\lambda_{300}}{8}-\frac{\left(\lambda_{400}-1\right)}{8}\right)+\frac{1}{4}\eta_6\left(a_1-a_1^2\right)C_z^2 \tag{29}$$

$$Bias\left(t_2\right)=C_y\eta_3\left(C_y^2-\frac{C_y\lambda_{300}}{2}-\frac{\left(\lambda_{400}-1\right)}{8}\right)+\frac{1}{4}\eta_6\left(a_2-a_2^2\right)\left(\lambda_{004}-1\right) \tag{30}$$

$$Bias\left(t_3\right)=C_y\eta_3\left(C_y^2-\frac{C_y\lambda_{300}}{2}-\left(\lambda_{400}-1\right)\right)-\eta_6\left(a_3\left(\frac{C_z^2}{4}-\frac{\lambda_{004}-1}{8}\right)-a_3^2\left(\frac{C_z^2}{4}+\frac{\lambda_{004}-1}{16}\right)\right) \tag{31}$$

Squaring both sides of (26)-(28) and take expectation to obtain the MSE of $t_i,\ i=1,2,3$ as in

$$MSE\left(t_1\right)=C_y^2\eta_3\left(C_y^2+\frac{\left(\lambda_{400}-1\right)}{4}-C_y\lambda_{300}\right)+\eta_6\left(\frac{a_1^2C_z^2}{4}-a_1C_y\left(C_z\lambda_{201}-C_{yz}\right)\right) \tag{32}$$

$$MSE\left(t_2\right)=C_y^2\eta_3\left(C_y^2+\frac{\left(\lambda_{400}-1\right)}{4}-C_y\lambda_{300}\right)+\eta_6\left(\frac{a_2^2C_z^2}{4}-a_2C_y\left(\frac{\lambda_{002}-1}{2}-C_y\lambda_{102}\right)\right) \tag{33}$$

$$MSE\left(t_3\right)=C_y^2\eta_3\left(C_y^2+\frac{\left(\lambda_{400}-1\right)}{4}-C_y\lambda_{300}\right)+\eta_6\left(a_3^2\begin{pmatrix}\left(\frac{\left(\lambda_{400}-1\right)}{16}+\frac{C_z^2}{4}-\frac{C_z\lambda_{003}}{4}\right)\\ -aC_y\left(\frac{\left(\lambda_{202}-1\right)}{4}-\frac{C_y\lambda_{102}}{2}+2C_{yz}\right)\end{pmatrix}\right) \tag{34}$$

Differentiating (32)-(34) with respect to a_1, equate to zero and solve for a_1, we obtain,

$$a_{1(opt)}=\frac{2C_y\left(C_z\lambda_{201}-C_{yz}\right)}{C_z^2} \tag{35}$$

$$a_{2(opt)}=\frac{C_y\left(\lambda_{002}-1-2C_y\lambda_{102}\right)}{C_z^2} \tag{36}$$

$$a_{3(opt)}=\frac{2C_y\left(\left(\lambda_{202}-1\right)-2C_y\lambda_{102}+8C_{yz}\right)}{\left(\lambda_{400}-1\right)+4C_z^2-4C_z\lambda_{003}} \tag{37}$$

Substituting $a_{2(opt)}$ into (32)-(34), minimum MSE of $t_i,\ i=1,2,3$ is obtained as

$$MSE\left(t_1\right)_{\min}=C_y^2\eta_3\left(C_y^2+\frac{\left(\lambda_{400}-1\right)}{4}-C_y\lambda_{300}\right)-\eta_6\frac{C_y^2\left(C_z\lambda_{201}-C_{yz}\right)^2}{C_z^2} \tag{38}$$

$$MSE\left(t_2\right)_{\min}=C_y^2\eta_3\left(C_y^2+\frac{\left(\lambda_{400}-1\right)}{4}-C_y\lambda_{300}\right)-\frac{1}{4C_z^2}\eta_6C_y^2\left(\lambda_{002}-1-2C_y\lambda_{102}\right)^2 \tag{39}$$

$$MSE\left(t_3\right)_{\min}=C_y^2\eta_3\left(C_y^2+\frac{\left(\lambda_{400}-1\right)}{4}-C_y\lambda_{300}\right)-\frac{1}{4}\eta_6C_y^2\frac{\left(\left(\lambda_{202}-1\right)-2C_y\lambda_{102}+8C_{yz}\right)^2}{\left(\lambda_{400}-1\right)+4C_z^2-4C_z\lambda_{003}} \tag{40}$$

3.1.3 Biases and MSEs of Estimators $\Phi_i,\ i=1,2,...,9$

Using the results of subsections 3.1.1 and 3.2.2, the biases and MSEs of the estimators $\Phi_i,\ i=1,2,...,9$ are obtained as

$$Bias\left(\Phi_i\right)=\theta_i Bias\left(T_j\right)+\left(1-\theta_i\right)Bias\left(t_j\right) \tag{41}$$

$$MSE\left(\Phi_i\right)=\theta_i^2 MSE\left(T_j\right)+\left(1-\theta_i\right)^2 MSE\left(t_j\right) \tag{42}$$

Differentiating (42) with respect to $\theta_i,\ i=1,2,...,9$ respectively, equate results to zero and solve for $\theta_i,\ i=1,2,...,9$, we obtained,

$$\theta_{i(opt)}=\frac{MSE\left(t_j\right)_{\min}}{MSE\left(T_j\right)_{min}+MSE\left(t_j\right)_{\min}} \tag{43}$$

Substituting (43) in (42) respectively, we obtain minimum MSE of $\Phi_i,\ i=1,2,...,9$ denoted by $MSE\left(\Phi_i\right)_{\min},\ i=1,2,...,9$ as

$$MSE\left(\Phi_i\right)_{\min}=\frac{MSE\left(T_j\right)_{min}MSE\left(t_j\right)_{\min}}{MSE\left(T_j\right)_{min}+MSE\left(t_j\right)_{\min}} \tag{44}$$

3.2. Effects of Measurement Error on the Efficiency of $\Phi_i,\ i=1,2,...,9$

Let the true values of X, Y and Z respectively be x_i, y_i and z_i while the observed values be x_i^*, y_i^* and z_i^*. The measurement errors on X, Y and Z are defined as $u_i=x_i-x_i^*$, $v_i=y_i-y_i^*$ and $w_i=z_i-z_i^*$ respectively such that $u_i\in U\sim N\left(0,S_u^2\right), v_i\in V\sim N\left(0,S_v^2\right), w_i\in W\sim N\left(0,S_w^2\right)$ and pairs of X, Y, Z, U, V and W are uncorrelated. Here we considered two cases.

Case I: Auxiliary variable Z assumed to be free measurement errors, that is, $w_i=0$. Therefore, the joint moment about that mean is given as

$$\tau_{pqr}=\frac{1}{N-1}\sum_{i=1}^{N}v_i^p u_i^q\left(z_i-\bar{Z}\right)^s \tag{45}$$

Take into consideration the effects of the measurement errors U and V, the expression for the minimum MSE of the proposed estimators $\Phi_i,\ i=1,2,...,9$ are obtained as

$$MSE\left(\Phi_i\right)_{\min.m(I)}=\frac{MSE\left(T_j\right)_{min.m(I)}MSE\left(t_j\right)_{\min.m(I)}}{MSE\left(T_j\right)_{min.m(I)}+MSE\left(t_j\right)_{\min.(I)}} \tag{46}$$

where $MSE\left(T_1\right)_{min.m(I)}=A_{0.m(I)}-\frac{A_{1.m(I)}^2}{4A_{2.m(I)}}$, $MSE\left(T_2\right)_{min.m(I)}=B_{0.m(I)}-\frac{B_{1.m(I)}^2}{4B_{2,m(I)}}$,

$MSE\left(T_3\right)_{min.m(I)}=D_{0.m(I)}-\frac{D_{1.m(I)}^2}{4D_{2.m(I)}}$, $A_{2.m(I)}=\eta_7\left(S_x^2+S_u^2\right)-\frac{\eta_9}{4}C_z^2-\eta_{10}\bar{X}C_{xz}$

$$A_{0.m(I)}=\eta_2\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}+\frac{\left(\tau_{400}-1\right)}{4}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{300}\right)+\frac{1}{4}\eta_5\left(C_z^2-C_z\tau_{003}\right)$$

$$+\eta_5\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{\tau_{201}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}-C_{yz}-\frac{1}{2}\left(\frac{\left(\tau_{202}-1\right)}{2}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{003}\right)\right)$$

$$A_{1.m(I)}=-2\eta_7\bar{X}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{C_x\tau_{210}}{2}-C_{xy}\right)+\eta_9\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(C_{xy}-\frac{C_x\tau_{210}}{2}\right),$$

$$-\eta_8\bar{X}C_{xz}-\frac{\eta_9}{2}C_z^2+\frac{1}{2}\eta_8\sqrt{\left(S_x^2+S_u^2\right)}\tau_{012}+\eta_9C_z\tau_{003}$$

$$B_{0.m(I)}=\eta_2\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}+\frac{(\tau_{400}-1)}{4}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{300}\right)+\eta_5\left(\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{\tau_{201}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}-C_{yz}\right)\right.$$

$$\left.-\frac{1}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{(\tau_{202}-1)}{2}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{003}\right)+\frac{1}{4}\left(C_z^2-C_z\tau_{003}\right)+\frac{1}{16}(\tau_{004}-1)\right)$$

$$B_{1m(I)}=-\eta_7\left(S_x^2+S_u^2\right)\left(\frac{\tau_{220}-1}{2}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{030}\right)+\frac{\eta_9}{2}\left(\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{003}-\frac{\tau_{202}-1}{2}-C_z\tau_{003}+\frac{1}{2}(\tau_{004}-1)\right)$$

$$+\left(S_x^2+S_u^2\right)\left(\frac{1}{nq_1+2p_1}-\frac{1}{m}\right)\left(C_z\tau_{021}-\frac{1}{2}(\tau_{022}-1)\right)$$

$$,\ B_{2.m(I)}=\left(S_x^4+S_u^4\right)\left(\eta_7(\tau_{040}-1)-\eta_{10}(\tau_{022}-1)\right)-\frac{\eta_9}{4}(\tau_{004}-1)$$

$$D_{0.m(I)}=\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\eta_2\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}+\frac{(\tau_{400}-1)}{4}-C_y\tau_{300}\right)+\eta_5\left(\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{\tau_{201}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}-C_{yz}\right)\right.$$

$$\left.-\frac{1}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{(\tau_{202}-1)}{2}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{003}\right)+\frac{C_z^2}{2}-C_z\tau_{003}+\frac{1}{16}(\tau_{004}-1)\right)$$

$$D_{1.m(I)}=-\eta_7\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\left(\frac{\tau_{202}-1}{4}-\frac{\tau_{210}}{2}\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}-\frac{\tau_{030}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}+C_{xy}\right)$$

$$+\frac{\eta_9}{2}\left(\frac{C_z\tau_{210}}{2}-C_{yz}-\frac{\tau_{202}-1}{4}+\frac{\tau_{030}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}-C_z^2+\frac{C_z\tau_{030}}{2}+C_z\tau_{003}-\frac{(\tau_{004}-1)}{2}\right)$$

$$-\eta_1\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\left(\frac{C_z\tau_{021}}{2}-C_{xz}+\frac{1}{2}\left((\tau_{021}-1)-\tau_{012}\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\right)\right)$$

$$D_{2.m(I)}=\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)\left(-\eta_7\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\left(\tau_{012}-\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\right)+\eta_2\frac{(\tau_{040}-1)}{2}\right)+\eta_1\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}$$

$$\left(C_{xz}-\frac{C_z\tau_{021}}{2}-\frac{\tau_{012}}{2}\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}+\frac{(\tau_{022}-1)}{4}\right)+\frac{1}{16}\left(\frac{1}{m}-\frac{1}{N}\right)\left((\tau_{004}-1)+\frac{1}{4}\left(C_z^2-C_z\tau_{003}\right)\right)$$

$$MSE(t_1)_{\min.m(I)}=\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\eta_3\left(\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)+\frac{(\tau_{400}-1)}{4}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\tau_{300}\right)-\eta_6\frac{\left(C_z\tau_{201}-C_{yz}\right)^2}{C_z^2}\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)$$

$$MSE(t_2)_{\min.m(I)} = \left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)\eta_3\left(\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right) + \frac{(\tau_{400}-1)}{4} - \sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\tau_{300}\right)$$

$$-\frac{1}{4C_z^2}\eta_6\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)\left(\tau_{002} - 1 - 2\sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\tau_{102}\right)^2$$

$$MSE(t_3)_{\min} = \left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)\eta_3\left(\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right) + \frac{(\tau_{400}-1)}{4} - \sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\tau_{300}\right)$$

$$-\frac{1}{4}\eta_6\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)\frac{\left((\tau_{202}-1) - 2\sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\tau_{102} + 8C_{yz}\right)^2}{(\tau_{400}-1) + 4C_z^2 - 4C_z\tau_{003}}$$

Case II: Auxiliary variable Z assumed to be characterized by measurement errors. Therefore, the joint moment about that mean is given as

$$\varpi_{pqs} = \frac{1}{N-1}\sum_{i=1}^{N} v_i^p u_i^q w_i^s \tag{47}$$

Take into consideration the effects of the measurement errors U, V and W, the expression for the minimum MSE of the proposed estimators $\Phi_j,\ j = 1,2,\ldots,9$ are obtained as

$$MSE(\Phi_i)_{\min.m(II)} = \frac{MSE(T_j)_{min.m(II)}\, MSE(t_j)_{\min.m(II)}}{MSE(T_j)_{min.m(II)} + MSE(t_j)_{\min.(II)}} \tag{48}$$

where $MSE(T_1)_{min.m(II)} = A_{0.m(II)} - \dfrac{A_{1.m(II)}^2}{4A_{2.m(II)}}$, $MSE(T_2)_{min.m(II)} = B_{0.m(II)} - \dfrac{B_{1.m(II)}^2}{4B_{2,m(II)}}$,

$$MSE(T_3)_{min.m(II)} = D_{0.m(II)} - \frac{D_{1.m(II)}^2}{4D_{2.m(II)}}\ ,\ A_{2.m(II)} = \eta_7\left(S_x^2 + S_u^2\right) - \frac{1}{4}\eta_9\left(C_z^2 + \frac{S_w^2}{\bar{Z}^2}\right) - \bar{X}\eta_{10}C_{xz}$$

$$A_{0.m(II)} = \eta_2\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2} + \frac{(\varpi_{400}-1)}{4} - \sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{300}\right) + \frac{1}{4}\eta_5\left(C_z^2 + \frac{S_w^2}{\bar{Z}^2} - \sqrt{C_z^2 + \frac{S_w^2}{\bar{Z}^2}}\varpi_{003}\right)$$

$$+\eta_5\sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{\varpi_{201}}{2}\sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)} - C_{yz} - \frac{1}{2}\left(\frac{(\varpi_{202}-1)}{2} - \sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{003}\right)\right)$$

$$A_{1.m(I)} = -2\eta_7\bar{X}\sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{C_x\varpi_{210}}{2} - C_{xy}\right) - \eta_8\left(\bar{X}C_{xz} - \frac{1}{2}\sqrt{\left(S_x^2 + S_u^2\right)}\varpi_{012}\right)$$

$$+\eta_9\left(\sqrt{\left(C_y^2 + \frac{S_v^2}{\bar{Y}^2}\right)}\left(C_{xy} - \frac{C_x\varpi_{210}}{2}\right) + \sqrt{C_z^2 + \frac{S_w^2}{\bar{Z}^2}}\varpi_{003} - \frac{1}{2}\left(C_z^2 + \frac{S_w^2}{\bar{Z}^2}\right)\right),.$$

$$B_{2.m(II)} = \left(S_x^4 + S_u^4\right)\left(\eta_7\left(\varpi_{040}-1\right) - \eta_{10}\left(\varpi_{022}-1\right)\right) - \frac{1}{4}\eta_9\left(\varpi_{004}-1\right)$$

$$B_{0.m(II)}=\eta_5\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}+\frac{(\varpi_{400}-1)}{4}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{300}\right)+\eta_5\left(\frac{1}{4}\left(\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)-\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\varpi_{003}\right)\right.$$

$$\left.+\frac{1}{16}(\varpi_{004}-1)+\left(\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{\varpi_{201}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}-C_{yz}\right)-\frac{1}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{(\varpi_{202}-1)}{2}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{003}\right)\right)\right)$$

$$B_{1m(II)}=-\eta_7\left(S_x^2+S_u^2\right)\left(\frac{\varpi_{220}-1}{2}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{030}\right)-\eta_8\left(S_x^2+S_u^2\right)\left(\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\varpi_{021}-\frac{1}{2}(\varpi_{022}-1)\right)$$

$$+\frac{\eta_9}{2}\left(\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{003}-\frac{\varpi_{202}-1}{2}-C_z\varpi_{003}+\frac{1}{2}(\varpi_{004}-1)\right)$$

$$D_{0.m(II)}=\eta_2\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}+\frac{(\varpi_{400}-1)}{4}-C_y\varpi_{300}\right)+\eta_5\left(\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}+\frac{1}{2}\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)-\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\varpi_{003}\right.$$

$$\left.+\frac{1}{16}(\varpi_{004}-1)\left(\frac{\varpi_{201}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}-C_{yz}\right)-\frac{1}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\left(\frac{(\varpi_{202}-1)}{2}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{003}\right)\right)$$

$$D_{1.m(II)}=-\eta_7\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\left(\frac{\varpi_{202}-1}{4}-\frac{\varpi_{210}}{2}\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}-\frac{\varpi_{030}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}+C_{xy}\right)$$

$$+\frac{1}{2}\eta_5\left(\frac{\varpi_{210}}{2}\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}-C_{yz}-\frac{\varpi_{202}-1}{4}+\frac{\varpi_{030}}{2}\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}-\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)+\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\left(\frac{\varpi_{030}}{2}+\varpi_{003}\right)\right.$$

$$\left.-\frac{(\varpi_{004}-1)}{2}\right)-\eta_1\left(C_x+\frac{S_u}{\bar{X}}\right)\left(\frac{C_z\varpi_{021}}{2}-C_{xz}+\frac{1}{2}\left((\varpi_{021}-1)-\varpi_{012}\left(C_x+\frac{S_u}{\bar{X}}\right)\right)\right)$$

$$D_{2.m(I)}=\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)\left(-\eta_7\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\left(\varpi_{012}-\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\right)+\frac{\eta_2(\varpi_{040}-1)}{2}\right)+\frac{1}{16}\eta_5\left((\varpi_{004}-1)\right.$$

$$\left.+\frac{1}{4}\left(\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)-\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\varpi_{003}\right)\right)+\eta_1\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}\left(C_{xz}-\frac{C_z\varpi_{021}}{2}-\frac{\varpi_{012}}{2}\sqrt{\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)}+\frac{(\varpi_{022}-1)}{4}\right)$$

$$MSE(t_1)_{\min.m(II)}=\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\left(\frac{1}{uq_3+2p_3}-\frac{1}{N}\right)\left(\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)+\frac{(\varpi_{400}-1)}{4}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{300}\right)$$

$$-\left(\frac{1}{u}-\frac{1}{N}\right)\frac{\left(\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\varpi_{201}-C_{yz}\right)^2}{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)$$

$$MSE\left(t_2\right)_{\min.m(II)}=\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\eta_3\left(\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)+\frac{\left(\varpi_{400}-1\right)}{4}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{300}\right)$$

$$-\frac{1}{4\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\eta_6\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\left(\varpi_{002}-1-2\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{102}\right)^2$$

$$MSE\left(t_3\right)_{\min.m(II)}=\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\eta_3\left(\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)+\frac{\left(\varpi_{400}-1\right)}{4}-\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{300}\right)$$

$$-\frac{1}{4}\eta_6\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\frac{\left(\left(\varpi_{202}-1\right)-2\sqrt{\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)}\varpi_{102}+8C_{yz}\right)^2}{\left(\varpi_{400}-1\right)+4\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)-4\sqrt{\left(C_z^2+\frac{S_w^2}{\bar{Z}^2}\right)}\varpi_{003}}$$

4. EFFICIENCY COMPARISONS THROUGH EMPIRICAL STUDY

To evaluate the efficiency of the proposed estimators $\Phi_i,\ i=1,2,3,...,9$, the MSEs of $\Phi_i,\ i=1,2,3,...,9$ were compare to that of estimator Φ_0 which is defined as follows:

$$\Phi_0=\left(1-\theta_0\right)c_{y_{u-r_3}}+\theta_0 c_{y_{m-r_2}}c_{x_{n-r_1}}/c_{x_{m-r_2}} \tag{49}$$

where $\theta_0\left(0\le\theta_0\le1\right)$ is an unknown quantity to be obtained by minimizing the MSE of Φ_0.

The optimum expression for θ_0 and minimum MSE of Φ_0 up to second degree approximation in the absence of measurement errors are obtained as in (50) and (51) respectively

$$\theta_{0(opt)}=\frac{\operatorname{var}\left(t_0\right)}{MSE\left(T_0\right)+\operatorname{var}\left(t_0\right)} \tag{50}$$

$$MSE\left(\Phi_0\right)_{\min}=\frac{MSE\left(T_0\right)\operatorname{var}\left(t_0\right)}{MSE\left(T_0\right)+\operatorname{var}\left(t_0\right)} \tag{51}$$

where

$$MSE\left(T_0\right)=C_y^2\left(\eta_2\left(\frac{\lambda_{400}-1}{4}+C_y^2-\lambda_{300}C_y\right)+\eta_7\left(C_x^2+\frac{\lambda_{040}-1}{4}-\lambda_{030}C_y+2C_{yx}+\lambda_{210}C_x-\lambda_{030}C_x-\frac{\lambda_{220}-1}{2}\right)\right)$$

$$\operatorname{var}\left(t_0\right)=C_y^2\eta_3\left(\frac{\lambda_{400}-1}{4}+C_y^2-\lambda_{300}C_y\right)$$

The optimum expression for θ_0 and minimum MSE of Φ_0 up to second degree approximation in the presence of measurement errors are obtained as in (52) and (53) respectively

$$\theta_{0(opt)}=\frac{\operatorname{var}\left(t_0\right)_m}{MSE\left(T_0\right)_m+\operatorname{var}\left(t_0\right)_m} \tag{52}$$

$$MSE\left(\Phi_0\right)_{\min}=\frac{MSE\left(T_0\right)_m\operatorname{var}\left(t_0\right)_m}{MSE\left(T_0\right)_m+\operatorname{var}\left(t_0\right)_m} \tag{53}$$

where

$$MSE\left(T_0\right)_m = C_y^2\left(\eta_2\left(\frac{\lambda_{400}-1}{4}+\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)-\lambda_{300}\sqrt{C_y^2+\frac{S_v^2}{\bar{Y}^2}}\right)+\eta_7\right.$$

$$\left.\left(\left(C_x^2+\frac{S_u^2}{\bar{X}^2}\right)+\frac{\lambda_{040}-1}{4}-\lambda_{030}\sqrt{C_y^2+\frac{S_v^2}{\bar{Y}^2}}+2C_{yx}+\sqrt{C_x^2+\frac{S_u^2}{\bar{X}^2}}\left(\lambda_{210}-\lambda_{030}\right)-\frac{\lambda_{220}-1}{2}\right)\right),$$

$$\operatorname{var}\left(t_0\right)_m = \left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)\eta_3\left(\frac{\lambda_{400}-1}{4}+\left(C_y^2+\frac{S_v^2}{\bar{Y}^2}\right)-\lambda_{300}\sqrt{C_y^2+\frac{S_v^2}{\bar{Y}^2}}\right)$$

To assess the performance of the proposed estimators $\Phi_i,\ i=1,2,3,...,9$ with respect to Φ_0, absolute relative bias (ARB) and percentage relative efficiency (PRE) of the estimators were computed using (54) and (55) respectively.

$$ARB\left(\Phi_i\right)=\frac{\left|E\left(\Phi_i-C_y\right)\right|}{C_y},\quad i=0,1,2,...,9 \tag{54}$$

$$PRE\left(\Phi_i\right)=\frac{E\left(\Phi_0-C_y\right)^2}{E\left(\Phi_i-C_y\right)^2}\times 100,\quad i=0,1,2,...,9 \tag{55}$$

where $E\left(\Phi_i-C_y\right)=\frac{1}{M}\sum_{j=1}^{M}\left(\Phi_{ij}-C_y\right),\ E\left(\Phi_i-C_y\right)^2=\frac{1}{M}\sum_{j=1}^{M}\left(\Phi_{ij}-C_y\right)^2$

Artificial populations of size $N=10,000$ from which SRSWOR of size $n=m+u=1000$ were drawn $M=1000$ times are simulated from multivariate normal distribution with parameters $\mu_X=20,\mu_Y=30,\mu_Z=50,\sigma_X^2=10,\sigma_Y^2=9,\sigma_Z^2=10,\mu_U=\mu_V=\mu_W=0,\sigma_U^2=\sigma_V^2=\sigma_W^2=1$ for different probabilities $p_1,\ p_2,\ p_3$ of non-response in the presence of measurement errors and ABR and PRE of were computed for each estimators using (54)-(55) as presented in Tables 1,2,3.

Table 1: Absolute Relative Biases and PREs of $\Phi_i, i = 0,1,2,...,9$ for $\rho_{xy} = \rho_{xz} = \rho_{yz} = 0.8$

Est.	$m = 900, u = 100$						$m = 800, u = 200$					
	$p_1 = p_2 = p_3 = 0.05$		$p_1 = p_2 = p_3 = 0.1$		$p_1 = p_2 = p_3 = 0.15$		$p_1 = p_2 = p_3 = 0.1$		$p_1 = p_2 = p_3 = 0.15$		$p_1 = p_2 = p_3 = 0.2$	
	ARB	PRE	ARB	PRE	ARB	PRE	ARB	PRE	ARB	PRE	ARB	PRE
Proposed Estimators under Case I												
Φ_0	0.00243	100	0.00279	100	0.00331	100	0.00211	100	0.00139	100	0.00045	100
Φ_1	0.08009	0.76	0.1375	12.28	0.1621	2.94	0.01763	19.27	0.04066	449.91	0.04329	2473.9
Φ_2	1.97775	0	2.02897	0.12	2.16363	0.01	0.29126	6.8	0.59078	7.76	0.54355	139.72
Φ_3	0.08681	0.35	0.1406	36.19	0.16789	1.79	0.01789	1018.56	0.04308	110.92	0.04498	76.28
Φ_4	0.00201	108.76	0.00605	224.73	0.00442	9.9	0.00431	177.68	0.00718	100.96	0.00069	45.86
Φ_5	0.00193	104.86	0.00615	142.99	0.00482	9.29	0.00436	185.08	0.00717	101.22	7e-04	47.35
Φ_6	0.00148	50.63	0.00626	166.25	0.00484	10.82	0.00431	189.22	0.00718	101.24	0.00067	47.03
Φ_7	0.00216	295.76	0.00625	146.79	0.00459	7.95	0.00307	625.09	0.00694	138.94	0.00157	105.23
Φ_8	0.00209	318.29	0.00638	95.98	0.00504	7.31	0.0037	356679.9	0.00839	77.81	0.00152	62.4
Φ_9	0.00155	82.99	0.00647	113.44	0.00505	8.64	0.0031	4415214	0.00738	78.38	0.00185	65.42
	4.7e-06		2.83e-05		6.2e-06		2.15e-05		5.38e-05		0.0001718	
Case II												
Φ_0	0.0025	100	0.0028	100	0.00331	100	0.00211	100	0.00139	100	0.00045	100
Φ_1	0.13882	0.06	0.23444	0.06	0.35544	0.01	0.01728	2.16	0.1093	1.68	0.1705	12.07
Φ_2	0.16672	0.05	0.20039	0.08	0.23329	0.03	0.01586	0.82	0.07597	4.23	0.07897	35.96
Φ_3	0.15276	0.06	0.24355	0.07	0.37503	0.01	0.01753	0.81	0.11606	5.38	0.1775	2.06
Φ_4	0.00323	22.06	0.0063	450.59	0.00468	12.79	0.00365	2551.99	0.0071	125.8	0.00066	32.41
Φ_5	0.0039	18.66	0.00641	257.28	0.00514	11.93	0.00377	6301.8	0.00722	118.51	0.00065	31.95
Φ_6	0.00387	21.21	0.00657	270.44	0.0052	13.97	0.00365	6201.4	0.00715	117.76	0.00067	32.11
Φ_7	0.00313	29.56	0.00642	248.61	0.0048	9.92	0.00263	2514.24	0.00684	143.57	0.00183	61.59
Φ_8	0.00379	25.14	0.00654	152.95	0.0053	9.07	0.0032	837.99	0.00826	73.77	0.00186	36.11
Φ_9	0.00377	28.24	0.00669	166.73	0.00534	10.75	0.00263	1745.17	0.00727	76.53	0.00205	42.01
	5.7e-06		2.82e-05		6.2e-06		2.15e-05		5.38e-05		0.0001715	

Table 2: Absolute Relative Biases and PREs of $\Phi_i, i = 0,1,2,...,9$ for $\rho_{xy} = \rho_{xz} = \rho_{yz} = 0.6$

Est.	$m=900, u=100$						$m=800, u=200$					
	$p_1=p_2=p_3=0.05$		$p_1=p_2=p_3=0.1$		$p_1=p_2=p_3=0.15$		$p_1=p_2=p_3=0.1$		$p_1=p_2=p_3=0.15$		$p_1=p_2=p_3=0.2$	
	ARB	PRE	ARB	PRE	ARB	PRE	ARB	PRE	ARB	PRE	ARB	PRE
Proposed Estimators under Case I												
Φ_0	0.00954	100	0.00416	100	0.00891	100	0.0038	100	0.00145	100	0.0007	100
Φ_1	0.08354	13.2	0.12105	383.14	0.15509	0.05	0.03272	21.61	0.04189	15.94	0.0608	11.14
Φ_2	1.86345	0.04	2.22326	0.51	2.52762	0	0.59677	0.05	0.6328	0.08	0.8219	0.06
Φ_3	0.09153	7.57	0.12615	147.78	0.16073	0.29	0.03254	14.21	0.04152	5.83	0.0596	1826.91
Φ_4	0.0039	6817.2	0.00182	150.82	0.00189	3.64	0.00163	2358.93	0.00506	31.41	0.0004	482.04
Φ_5	0.00385	3671.15	0.00219	130.55	0.00208	6.39	0.00207	983.54	0.0067	22.37	0.0003	322.8
Φ_6	0.00329	42547.36	0.00147	161.15	0.00228	13.9	0.00173	1160.14	0.0052	23.42	0.0002	1622175
Φ_7	0.00401	12750.43	0.00142	237.62	0.00198	88.24	0.00161	208966	0.00445	34.83	0.0006	545.26
Φ_8	0.00396	5485.21	0.00177	206.37	0.00218	10.29	0.0021	24184.6	0.00625	23.44	4e-04	333.01
Φ_9	0.00332	2239085	0.00104	260.09	0.00238	7.19	0.00171	12167.9	0.00461	24.49	0.00022	6888.82
	0.0001219		9.16e-05		1e-07		5.78e-05		9.99e-05		1.08e-05	
Case II												
Φ_0	0.00718	100	0.00322	100	0.00664	100	100	0.00358	0.00135	100	0.00076	100
Φ_1	0.08441	4.14	0.15407	3.21	0.20682	263.47	3.63	0.03167	0.04679	2.66	0.06209	19.41
Φ_2	0.82226	0.06	1.22472	0.04	1.44118	0.28	0.06	0.24617	0.29917	0.07	0.34606	0.58
Φ_3	0.09265	2.71	0.16073	2.95	0.21441	3.17	3.17	0.03137	0.0466	1.46	0.06053	3.91
Φ_4	0.00372	519.42	0.00172	107.13	0.00196	458.04	114660.4	0.0019	0.00469	20.49	0.0015	3651.86
Φ_5	0.00365	356.18	0.00204	95.35	0.00218	148.58	22617.41	0.00241	0.00626	14.12	0.00156	2569.83
Φ_6	0.00316	795.91	0.00131	110.28	0.00242	103.76	14795.41	0.00202	0.00481	15.34	0.00116	1474.15
Φ_7	0.0038	1073.76	0.00112	280.24	0.00199	25.78	1645.22	0.00187	0.00396	21.13	0.0014	2983.94
Φ_8	0.00373	625.73	0.00142	259.26	0.00223	15.79	1261.47	0.00242	0.00559	13.83	0.0014	1952.23
Φ_9	0.00314	2372.64	0.00067	295.69	0.00248	15.77	2776.94	0.00199	0.0041	15.24	0.001	1015.42
	7.45e-05		4.83e-05		1.3e-06		4.28e-05		6.99e-05		7.6e-06	

Table 3: Absolute Relative Biases and PREs of $\Phi_i, i = 0,1,2,...,9$ for $\rho_{xy} = \rho_{xz} = \rho_{yz} = 0.3$

Est.	$m = 900, u = 100$						$m = 800, u = 200$					
	$p_1 = p_2 = p_3 = 0.05$		$p_1 = p_2 = p_3 = 0.1$		$p_1 = p_2 = p_3 = 0.15$		$p_1 = p_2 = p_3 = 0.1$		$p_1 = p_2 = p_3 = 0.15$		$p_1 = p_2 = p_3 = 0.2$	
	RB	PRE	RB	PRE	RB	PRE	RB	PRE	RB	PRE	RB	PRE
Proposed Estimators under Case I												
Φ_0	0.00116	100	0.00048	100	0.0011	100	0.00015	100	0.00068	100	0.00149	100
Φ_1	0.03309	177.6	0.02292	61.39	0.0141	1.43	0.03065	450.46	0.03425	41.81	0.0474	268.1
Φ_2	2.18035	0.2	1.50213	0.02	0.74093	0	2.02806	0.02	1.91418	0.01	2.41747	0.05
Φ_3	0.03302	16073.07	0.02694	65.68	0.017	1.43	0.03217	554.66	0.03572	345.37	0.04871	1030.68
Φ_4	0.00282	81.84	0.00347	1288.86	0.00198	234.91	0.00026	161.02	0.004	105.65	7e-05	175.17
Φ_5	0.00309	78.44	0.00369	1110.16	0.002	162.73	0.00046	142.11	0.00562	63.05	0.00021	91.38
Φ_6	0.00283	85.39	0.00328	1267.81	0.00215	233.8	4e-05	156.58	0.00371	70.38	0.00017	209.07
Φ_7	0.00262	112.17	0.00267	638.95	0.00205	128.13	0.00013	124.14	0.00399	111.5	6e-05	173.04
Φ_8	0.00342	100.14	0.00306	481	0.00209	146.33	0.00029	104.39	0.00559	66.93	2e-04	91.25
Φ_9	0.00264	129.86	0.00228	623.71	0.00233	128.74	0.00017	121.11	0.00371	73.84	0.00018	205.76
	0.0001585		0.0001253		5.9e-06		3.63e-05		9.75e-05		2.16e-05	
Case II												
Φ_0	0.00115	100	0.00048	100	0.0011	100	0.00015	100	0.00068	100	0.00149	100
Φ_1	0.05351	157.62	0.03977	21.17	0.04621	0.33	0.03008	1034.3	0.03997	30.46	0.0597	82.99
Φ_2	2.48842	0.28	1.78626	0.01	1.79632	0	1.34255	0.04	1.51642	0.02	2.03889	0.16
Φ_3	0.05352	2126.3	0.04385	19.53	0.04927	0.36	0.03201	626.81	0.04196	215.19	0.06147	35.87
Φ_4	0.00301	122.89	0.00379	593.17	0.00191	29.3	7e-05	72.5	0.00433	177.2	0.00114	181.94
Φ_5	0.00315	120.59	0.00395	540.42	0.00193	23.98	0.00013	61.26	0.00592	101.91	0.00151	91.5
Φ_6	0.00302	125.6	0.00366	600.64	0.00204	28.23	3e-04	74.64	0.00395	101.36	0.00083	258.24
Φ_7	0.00253	167.31	0.00293	311.97	0.00216	5614.97	0.00012	57.3	0.00427	186.95	0.0011	158.24
Φ_8	0.00335	151.27	0.00332	240.04	0.00224	20179.42	0.00012	46.47	0.00584	107.51	0.00145	81.24
Φ_9	0.00256	195.14	0.00255	320.35	0.00244	25835.38	5e-04	58.8	0.00389	105.67	0.00079	218.02
	0.0001586		0.0001253		5.8e-06		3.63e-05		9.73e-05		2.16e-05	

5. DISCUSSIONS AND CONCLUSIONS

From the empirical results, it is observed that the results in Tables 1-3

i. showed that the efficiency of the estimators considered in the study increases as thr correlation between the variables increases.

ii. showed that the proposed estimators $\Phi_4, \Phi_5, \Phi_6, \Phi_7, \Phi_8$ and Φ_9 have smaller ARBs compared to Φ_0 in all different probabilities of non-response under cases I and II with exception of view cases.

iii. revealed that under case I when $m = 900,\ u = 100$ the estimators $\Phi_4, \Phi_5, \Phi_6, \Phi_7, \Phi_8$ and Φ_9 have higher PREs compared to Φ_0 in almost different probabilities p_1, p_2, p_3 of non-response except when $p_1, = p_2 = p_3 = 0.15$ for $\rho_{XY} = \rho_{XZ} = p_{YZ} = 0.8$ and $\rho_{XY} = \rho_{XZ} = p_{YZ} = 0.6$.

iv. revealed that under case I when the probability of non-response is 0.15 for m=900, u=100 and when the probability of non-response is 0.2 for m=800, u=200, the proposed estimators $\Phi_i,\ i = 1, 2, 3, ..., 9$ have lower PREs than Φ_0.

v. showed that, under case II when the probability of non-response is 0.05 for m=900, u=100 and when the probability of non-response is 0.2 for m=800, u=200, the proposed estimators $\Phi_i,\ i = 1, 2, 3, ..., 9$ have lower PREs than Φ_0.

vi. revealed that except for situations mentioned (iv) and (v), the proposed estimators $\Phi_i,\ i = 1, 2, 3, ..., 9$ have higher PREs than Φ_0. This implies that the proposed estimators $\Phi_i,\ i = 1, 2, 3, ..., 9$ have a higher tendency to produce closer estimates (less variability) to population means than Φ_0 in the presence of non-response and measurement errors when sampling is done via successive sampling techniques.

From the discussion above, it can be concluded that the proposed estimators $\Phi_i,\ i = 1, 2, 3, ..., 9$ demonstrate a high level of efficiency in estimating the population mean when the variable of interest and auxiliary variable are characterized by non-response and measurement errors. We may therefore recommend the estimators to statisticians for use in practical situations.

Conflict of Interest. No conflict of interest associated with this manuscript.

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