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AN APPLICATION OF DISCRETE FRACTIONAL CALCULUS IN STATISTICS

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ABSTRACT
We introduced generalized random variables of discrete type, studied some of their properties and then related these to continuous random variable which has been studied by Ganji and Gharari [8]. With this introduction, we obtained a new relationship between discrete fractional calculus and statistics. Also, the fractional versions of the discrete uniform distribution are developed and their statistical properties are discussed.

KEY WORDS: Fractional sum, Fractional difference, Discrete uniform distribution, Generalized random variable.

MSC: 60E05, 39A12.

RESUMEN
Hemos introducido variables aleatorias generalizadas de tipo discreto, estudiamos algunas de sus propiedades y, a continuación relacionamos éstos a la variable aleatoria continua que ha sido estudiado por Ganji y Gharari [8]. Con esta introducción, se obtuvo una nueva relación entre el cálculo fraccional discreto y estadísticas. Además, las versiones fraccionarias de la distribución uniforme discreta se desarrollan y sus propiedades estadísticas se discuten.

1. INTRODUCTION

The discrete fractional calculus deals with the study of fractional order sums and differences and their diverse applications ([1], [2], [3], [4], [5], [10]).
As continuous fractional calculus that has widespread applications in different fields of science and engineering, applications of discrete fractional calculus will be ideal, too.
In this work, we follow our previous works, [6], [7] and [8], about applications of fractional calculus in statistics, we present an application of discrete fractional calculus in statistics. Having delta and nabla fractional sum and difference operators, we introduce two types of generalized random variables with their properties and we obtained a new relationship between discrete fractional calculus and statistics. By considering this relationship, we represent the discrete uniform distributions in term of fractional versions, which are called fractional discrete uniform distributions. The parameter space of these distributions, in the analogy with the ordinary discrete uniform, $DU\{0,1,\dots,N\}$, is extended from $\mathbb{N}_0$ to $\mathbb{N}_{-1}$. This is exactly like to our recent work, [8], in the generalized continuous random variable (GCRV), $\Phi_\alpha(x)$, in which the parameter space has been extended from $(0,\infty)$ to $(-1,\infty)$.
In such fractional distributions, our work is forming subclasses of family of distributions, in which we get different results by considering statistical properties of one of these subclasses.

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The article is organized as follows: The rest of this section contains summary of some notations and definitions in delta and nabla calculus. The second section contains the definitions of delta and nabla Riemann left fractional sums and differences. The third section contains the generalized discrete random variable type I and fractional discrete uniform distributions of type I. The fourth section contains the generalized discrete random variable type II and fractional discrete uniform distribution type II.
For a natural number n, the fractional polynomial is defined by,

$$t^{\underline{n}} = \prod_{j=0}^{n-1}(t-j) = \frac{\Gamma(t+1)}{\Gamma(t+1-n)}, \tag{1}$$

where Γ denotes the special gamma function and the product is zero when $j = t+1$ for some j. More generally, for arbitrary α, define

$$t^{\underline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \tag{2}$$

where the convention is that, division at pole is zero. The forward and backward difference operators are defined by

$$\Delta f(t) = f(t+1) - f(t), \qquad \nabla f(t) = f(t) - f(t-1), \tag{3}$$

respectively and we define iteratively the operators $\Delta^m = \Delta(\Delta^{m-1})$ and $\nabla^m = \nabla(\nabla^{m-1})$, for a natural number m.
For a natural number n, the rising (ascending) factorial of t is defined by

$$t^{\overline{n}} = \prod_{j=0}^{n-1}(t+j) = \frac{\Gamma(t+n)}{\Gamma(t)}, \qquad t^{\overline{0}} = 1. \tag{4}$$

For any real number the α rising function of t is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \qquad t \in \mathbb{R} - \{\dots,-2,-1,0\}, \quad 0^{\overline{\alpha}} = 0. \tag{5}$$

For our purposes we list down the following properties,

$$\nabla_t(\rho(s)-t)^{\underline{\alpha-1}} = -(\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}}, \tag{6}$$

$$\Delta_t(\sigma(s)-t)^{\overline{\alpha-1}} = -(\alpha-1)(\sigma(s)-t)^{\overline{\alpha-2}}, \tag{7}$$

where $\sigma(s) = s+1$ and $\rho(s) = s-1$ are the forward and backward jumping operators, respectively. Also, if
$g: {}_{b-\alpha}\mathbb{N} \times {}_b\mathbb{N} \to \mathbb{R}$, then for $t \in {}_{b-\alpha}\mathbb{N}$ we have

$$\nabla\left(\sum_{s=t+\alpha}^{b} g(t,s)\right) = \sum_{s=t+\alpha}^{b} \nabla_t g(t,s) - g(t-1, t+\alpha-1), \tag{8}$$

and if $g: {}_b\mathbb{N} \times {}_b\mathbb{N} \to \mathbb{R}$, then for $t \in {}_b\mathbb{N}$ we have

$$\Delta\left(\sum_{s=t}^{b} g(t,s)\right) = \sum_{s=t}^{b} \Delta_t g(t,s) - g(t+1,t), \tag{9}$$

where $\mathbb{N}_a = \{a, a+1, \dots\}$ and ${}_b\mathbb{N} = \{\dots, b-1, b\}$, for real numbers a and b.

2. ESSENTIAL DEFINITION

DEFINITION 2.1. Let b be a real number and $f: {}_b\mathbb{N} \to \mathbb{R}$,
The (delta) Riemann left fractional sum of order $\alpha > 0$ is defined by Abdeljawad [1] as

$$\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s)-t)^{\underline{\alpha-1}} f(s), \qquad t \in {}_{b-\alpha}\mathbb{N}. \tag{10}$$

The (nabla) Riemann left fractional sum of order $\alpha > 0$ is defined by

$$\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b} (\sigma(s)-t)^{\overline{\alpha-1}} f(s), \qquad t \in {}_b\mathbb{N}. \tag{11}$$

The (delta) Riemann left fractional difference of order $\alpha > 0$ is defined by Abdeljawad [1] as

$$\Delta^{\alpha} f(t) = (-1)^n \nabla^n \Delta^{-(n-\alpha)} f(t) = \frac{(-1)^n \nabla^n}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} (\rho(s)-t)^{\underline{n-\alpha-1}} f(s), \tag{12}$$

for $t \in {}_{b-(n-\alpha)}\mathbb{N}$ and $n = [\alpha]+1$where $[\alpha]$ is the greatest integer less than α.
The (nabla) Riemann left fractional difference of order $\alpha > 0$ is defined by

$$\nabla^{\alpha} f(t) = (-1)^n \Delta^n \nabla^{-(n-\alpha)} f(t) = \frac{(-1)^n \Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b} (\sigma(s)-t)^{\overline{n-\alpha-1}} f(s), \qquad t \in {}_{b-n}\mathbb{N}. \tag{13}$$

3. THE GENERALIZED DISCRETE RANDOM VARIABLE TYPE I

Suppose X be a discrete random variable and α be the parameter of distribution. The generalized discrete random variable type I $(\mathrm{GDRV}^{\mathrm{I}})$ is represented as the function $\Phi_{\underline{\alpha}}(x)$, and defined by $\frac{x^{\overline{\alpha-1}}}{\Gamma(\alpha)}$. The $\mathrm{GDRV}^{\mathrm{I}}$ appears in most of distributions and we can rewrite these distributions in terms of it. For example, the Binomial distribution can be rewritten as following:

$$Bin(x,p) = \Phi_{\underline{x+1}}(n)\, \Phi_{x+1}(p)\, \Phi_{n-x+1}(q) B(x+1, n-x+1)\Gamma(n+2),$$

where $x = 0, 1, 2, \dots,\ 0 < p < 1, 1 - p = q$ and $B(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha)}$ is the beta function. The generalized discrete random variable type I, has the properties as:

In the special case, when $\alpha = 2$ *;* it becomes ordinary discrete random variable X. We state the later property of the $\mathrm{GDRV}^{\mathrm{I}}$ as a theorem, i.e.

THEOREM 3.1. The expectation of the $\mathrm{GDRV}^{\mathrm{I}}$, $\Phi_{\underline{\alpha}}(x)$, coincides with (delta) Riemann left fractional sum of the probability function at -1 for $\alpha > 0$ and (delta) Riemann left fractional difference of the probability function at -1 for $\alpha < 0$, $\alpha \notin \{\dots, -2, -1\}$, i.e.

$$E\left[\Phi_{\underline{\alpha}}(x)\right] = \begin{cases} \Delta^{-\alpha} f(-1), & \alpha > 0 \\ \Delta^{\alpha} f(-1), & \alpha < 0, \alpha \notin \{\dots, -2, -1\} \end{cases} \tag{14}$$

where

$$\Delta^{-\alpha} f(t) = \sum_{x=\alpha-1}^{b-1-t} \frac{x^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(x+t-1)$$

is (delta) Riemann left fractional sum of order α, such that

$$\Delta^{\alpha} f(t) = \sum_{x=-\alpha-1}^{b-t-1} \frac{x^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} f(x+t-1),$$

is (delta) Riemann left fractional difference of order α.

In order to prove the theorem 3.1, we state and prove the following lemma.

LEMMA 3.1. Let $f: {}_{b}\mathbb{N} \to \mathbb{R}$ and $\alpha > 0$ be given, with $n - 1 < \alpha \le n$. The following two definitions for the fractional difference $\nabla_b^{\alpha} f: {}_{b-n+\alpha}\mathbb{N} \to \mathbb{R}$ are equivalent:

$$\Delta^{\alpha} f(t) = (-1)^n \nabla^n \Delta^{-(n-\alpha)} f(t), \tag{15}$$

and

$$\Delta^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \sum_{s=t-\alpha}^{b} (\rho(s)-t)^{\overline{-\alpha-1}} f(s), & n-1 < \alpha < n, \\ (-1)^n \nabla^n f(t), & \alpha = n. \end{cases} \tag{16}$$

PROOF: Let f and α be given as in the statement of the lemma 3.1 and we are showing that (16) is equivalent to (15) on the ${}_{b-n+\alpha}\mathbb{N}$.

If $\alpha = n$, then (15) and (16) are equivalent, since

$$\Delta^{\alpha} f(t) = (-1)^n \nabla^n \Delta^{-(n-\alpha)} f(t) = (-1)^n \nabla^n \Delta^{-0} f(t) = (-1)^n \nabla^n f(t).$$

If $\mathrm{n} - 1 < \alpha < n$, then a direct application of (15) implies that

$$\Delta^{\alpha} f(t) = (-1)^n \nabla^n \Delta^{-(n-\alpha)} f(t) = (-1)^n \nabla^n [\frac{1}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} (\rho(s)-t)^{\overline{n-\alpha-1}} f(s)$$

$$= \frac{(-1)^n \nabla^{n-1}}{\Gamma(n-\alpha)} \nabla [\sum_{s=t+(n-\alpha)}^{b} (\rho(s)-t)^{\overline{n-\alpha-1}} f(s)],$$

by using (6) and (8) we get it as

$$= \frac{(-1)^n \nabla^{n-1}}{\Gamma(n-\alpha)} [\sum_{s=t+(n-\alpha)}^{b} -(n-\alpha-1)(\rho(s)-t)^{\underline{n-\alpha-2}} f(s)$$

$$-(\rho(t+n-\alpha-1)-t+1)^{\underline{n-\alpha-1}} f(t+n-\alpha-1)]$$

$$= (-1)^n \nabla^{n-1}[-\sum_{s=t+(n-\alpha)}^{b} (\frac{(\rho(s)-t)^{\underline{n-\alpha-2}}}{\Gamma(n-\alpha-1)} f(s) + f(t+n-\alpha-1))]$$

$$= (-1)^n \nabla^{n-1}[\sum_{s=t+(n-\alpha-1)}^{b} \frac{-(\rho(s)-t)^{\underline{n-\alpha-2}}}{\Gamma(n-\alpha-1)} f(s)]$$

and by repeating this action $n-2$ times, which yields to

$$\Delta^{\alpha} f(t) = (-1)^n \nabla^{n-1}[\sum_{s=t+(n-\alpha-1)}^{b} \frac{-(\rho(s)-t)^{\underline{n-\alpha-2}}}{\Gamma(n-\alpha-1)} f(s)]$$

$$= (-1)^n \nabla^{n-2}[\sum_{s=t+(n-\alpha-2)}^{b} \frac{(\rho(s)-t)^{\underline{n-\alpha-3}}}{\Gamma(n-\alpha-2)} f(s)] =$$

$$= \nabla^{n-n}[\sum_{s=t+(n-\alpha-n)}^{b} \frac{(\rho(s)-t)^{\underline{n-\alpha-(n+1)}}}{\Gamma(n-\alpha-n)} f(s)] = \frac{1}{\Gamma(-\alpha)} \sum_{s=t-\alpha}^{b} (\rho(s)-t)^{\underline{-\alpha-1}} f(s).$$

■

Now, by using the lemma 3.1 we prove the theorem 3.1.

PROOF: For $\alpha > 0$, substitute $x = \rho(s) - t$ in the expression (10) and also for $\alpha < 0$, $\alpha \notin \{\dots, -2, -1\}$ in the expression (16).■

The Laplace transform of the GDRV$^{\mathrm{I}}$ is $\mathbb{L}_{\alpha-1}\left\{ \Phi_{\underline{\alpha}}(x), s\right\} = (s)^{-\alpha}$, and the discrete Laplace transform, defined by Atici and Eloe [2] as

$$\mathbb{L}_a\{f(t), s\} = \sum_{t=a}^{\infty} \left(\frac{1}{1+s}\right)^{t+1} f(t).$$

Let $\alpha > 0$, $\beta > -1$ and $x > 0$ be given. The relationship between the GDRV$^{\mathrm{I}}$ and GCRV is given by

$$(\beta+1)B(\alpha+1, \beta-\alpha+1)\ \Phi_{\underline{\alpha+1}}(\beta) = \frac{D^{\alpha}\ \Phi_{\beta+1}(x)}{\Phi_{\beta-\alpha+1}(x)},$$

Where D^{α} is the right Riemann- Liouville fractional derivative (see [9]), that by consideration $D^{\alpha} x^{\beta} = \beta^{\underline{\alpha}} x^{\beta-\alpha}$ and multiplying it by $\frac{1}{\Gamma(1+\beta)\Gamma(\alpha+1)\Gamma(\beta-\alpha+1)}$ we get

$$\Phi_{\underline{\alpha+1}}(\beta)\ \Phi_{\beta-\alpha+1}(x) \frac{1}{\Gamma(\beta+1)} = \frac{D^{\alpha}\ \Phi_{\beta+1}(x)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha+1)},$$

now, by a suitable simplification, we obtain our result. The similar result can be obtained by the right Riemann- Liouville fractional integral.

In property (b) of the GCRV, in [8], it can be seen that the bounds of integration equal to the support of ordinary random variable. By considering this point, in our present work, the bounds of summation in second property allow us to introduce the generalized type of the probability distributions. The example presented in this work is fractional discrete uniform distribution, such that its special case is the ordinary discrete uniform distribution. In the subsection, we discuss this new distribution and its statistical properties.

3.1 The fractional discrete uniform distribution type I

DEFINITION 3.1.1. The random variable X is said to have a fractional discrete uniform distribution type I with parameters (α, β), $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{N}_{-1}$ if its probability function is given by

$$f_X(x) = \frac{1}{\beta+2}, \quad x = \alpha-1, \alpha, \dots, \alpha+\beta, \ \alpha \in \mathbb{R}, \ \beta \in \mathbb{N}_{-1}, \qquad (17)$$

and denoted as $X \sim FDU^I\{\alpha-1, \alpha, \dots, \alpha+\beta\}$.

We classified this family of distributions FDU^I into $sub-FDU$ $(\beta \in \mathbb{N})$ and $super-FDU$ $(\beta \notin \mathbb{N})$ distributions. When $X \sim sup-FDU\ \{\alpha-1, \alpha, \dots, \alpha+\beta\}$, then the expectation, variance and moment generating function of this distribution are given by

$$E[X] = \alpha + \frac{\beta(\beta+1)-2}{2(\beta+2)}, \tag{18}$$
$$Var(X) = \frac{\beta^4+8\beta^3+23\beta^2+28\beta+12}{12(\beta+2)^2}, \tag{19}$$
$$M_X(t) = \frac{e^{t\alpha}}{\beta+2}\left\{\frac{e^{-t}-e^{t(\beta+1)}}{1-e^t}\right\}, \tag{20}$$
respectively.
The FDU^I for $\alpha = 2,\ \beta = N-2, (N \in \mathbb{N})$ yields the $DU\{1,2,\dots,N\}$ distribution and for $\alpha = 1,\ \beta = N-1, (N \in \mathbb{N}_0)$, the $DU\{0,1,\dots,N\}$ distribution. This result can be obtained separately for (18), (19) and (20).

3.1.1 MLE for parameters of the FDU^I distribution

The Likelihood function for the FDU^I distribution equals to
$$L(\alpha,\beta) = (\beta+2)^{-n}\prod_{i=1}^{n} I_{\{\alpha-1,\alpha,\dots,\alpha+\beta\}}(x_i),$$
when α is constant, it equals to
$$L(\beta) = (\beta+2)^{-n} I_{\{x_{(n)}-\alpha,\ x_{(n)}-\alpha+1,\dots\}}(\beta),$$
this is a decreasing function of β and it will be maximized if $\hat{\beta} = x_{(n)} - \alpha$. On the other hand, the Likelihood function
$$L(\alpha,\hat{\beta}) = \left(x_{(n)}-\alpha+2\right)^{-n} I_{[x_{(n)}-\hat{\beta},\ x_{(1)}+1]}(\alpha),$$
will be maximized if $\hat{\alpha} = x_{(1)} + 1$.
Therefore, generally the MLE estimators for α and β of FDU^I distribution equals to
$$(\hat{\alpha},\hat{\beta}) = \left(x_{(1)}+1, x_{(n)}-x_{(1)}-1\right). \tag{21}$$

3.1.2 The relationship with other distributions

Suppose that $X \sim U(-2,\beta)$, such that $\beta \in \mathbb{N}_{-1}$. Then, it can be easily proved that $Y = [X]$ have the FDU^I distribution with parameters β and $\alpha = -1$, where $[X]$ is the greatest integer less than or equal to X.

3.2 The generalized fractional discrete uniform distribution

The random variable FDU^I, that we introduced it in previous subsection, variants at both sides by different parameters, in such a way that one takes real values and the other one takes a subset of correct numbers. Moreover, the measure of variation of X is one. In this subsection, we introduce other fractional uniform discrete distribution, such that like before case, its random variable variants at both sides by different parameters. But in this case, the measure of variation of X is variable. We call it the generalized fractional discrete uniform distribution $(GFDU)$ and will show that, this distribution contain all of discrete uniform distributions that are introduced up now.
DEFINITION 3.2.1. The random variable X is said to have a fractional discrete uniform distribution type I with parameters (α,β,γ), $\alpha \in \mathbb{R}$, $\beta \in \mathbb{N}_{-1}$ and $\gamma > 0$ when its probability function is given by
$$f_X(x) = \frac{1}{\gamma[(\beta+1)\gamma^{-1}+1]},\quad x = \alpha - \frac{1}{\gamma},\ \alpha,\ \alpha+\frac{1}{\gamma}\dots,\ \alpha+\frac{\beta}{\gamma}, \tag{22}$$
and denoted as $X \sim GFDU\ \{\alpha - \frac{1}{\gamma}, \alpha, \dots, \alpha+\frac{\beta}{\gamma}\}$, where $\beta \in \mathbb{N}_{-1}$, $\gamma > 0$ and $\alpha \in \mathbb{R}$, also $[.]$ is the integer part.
We classified this family of distributions $GFDU^I$ into $\text{sub}-GFDU\ (\beta \in \mathbb{N})$ and $\text{super}-GFDU\ (\beta \notin \mathbb{N})$ distributions. When $X \sim sub-GFDU\ \left\{\alpha-\frac{1}{\gamma}, \alpha, \dots, \alpha+\frac{\beta}{\gamma}\right\}$, the expectation, variance and moment generating function of this distribution are given by
$$\mathrm{E(X)} = \frac{\beta^2+(2\alpha\gamma+1)\beta+2(2\alpha\gamma-1)}{2\gamma^2[(\beta+1)\gamma^{-1}+1]}, \tag{23}$$
$$\mathrm{Var(X)} = \frac{A\beta^4+B\beta^3+C\beta^2+D\beta+E}{12\gamma^3([(\beta+1)\gamma^{-1}+1])^2}, \tag{24}$$
$$M_X(t) = \frac{e^{t\alpha}}{\gamma[(\beta+1)\gamma^{-1}+1]}\left\{\frac{e^{-t\gamma^{-1}}-e^{t(\beta+1)\gamma^{-1}}}{1-e^{t\gamma^{-1}}}\right\}, \tag{25}$$

respectively, where

$$A=4-3\gamma,$$
$$B = 12\alpha\gamma(1-\gamma) - 2\gamma + 10,$$
$$C = 12\alpha(1-\gamma)(\alpha\gamma - 1) + 15\gamma + 8,$$
$$D = 12\alpha\gamma(1-\gamma)(3\alpha\gamma - 1) + 14\gamma + 14$$
$$E = 24\alpha\gamma(\alpha\gamma(\gamma-1) - 1) + 12.$$

The $GFDU$ distribution for $\gamma = 1$ yields the FDU^I distribution and for $\gamma = 1$ and $\alpha = 2, \beta = N-2, (N \in \mathbb{N})$ yields the $DU\{1,2,\dots,N\}$ distribution and for $\gamma = 1, \alpha = 1$ and $\beta = N-1, (N \in \mathbb{N}_0)$, the $DU\{0,1,\dots,N\}$ distribution. This result can be obtained separately for (23), (24) and (25).

3.2.1 MLE for parameters of the $GFDU$ distribution

Suppose that γ is known. The Likelihood function for the $GFDU$ distribution is

$$L(\alpha,\beta) = \left(\frac{1}{\gamma[(\beta+1)\gamma^{-1}+1]}\right)^n \prod_{i=1}^n I_{\left\{\alpha-\frac{1}{\gamma},\alpha,\dots,\alpha+\frac{\beta}{\gamma}\right\}}(x_i).$$

If α be constant, it is equal to

$$\left(\frac{1}{\gamma[(\beta+1)\gamma^{-1}+1]}\right)^n I_{\{\gamma(x_{(n)}-\alpha),\ \gamma(x_{(n)}-\alpha)+1,\dots\}}(\beta),$$

this function is a power of the integer part of a function of β. For considering its monotony, we know that the function inside the integer part is differentiable if it is continuous in its local minimum point. By considering domain of β, we see that $\gamma\left(x_{(n)}-\alpha\right)$ is only continuous point for this function. On the other hand, the derivative of function in this point is zero, namely the Likelihood function is constant function of β. Therefore, the estimator of parameter β is $\hat{\beta} = \gamma\left(x_{(n)}-\alpha\right)$. On the other hand, the Likelihood function

$$L\left(\alpha,\hat{\beta}\right) = \left(\frac{1}{\gamma[x_{(n)}-\alpha+\gamma^{-1}+1]}\right)^n I_{[\gamma x_{(n)}-\hat{\beta},\ x_{(1)}+\gamma^{-1}]}(\alpha),$$

is a power of the integer part of a function of α and point $x_{(1)}+\gamma^{-1}$ is only continuous point of this function and derivative of function in this point is zero. That is, the Likelihood function is constant function of α. Then the MLE for α parameter is $x_{(1)}+\gamma^{-1}$ and in general we have

$$\left(\hat{\alpha},\hat{\beta}\right) = \left(x_{(1)}+\gamma^{-1}, \gamma\left(x_{(n)}-x_{(1)}\right)-1\right). \tag{26}$$

4. THE GENERALIZED DISCRETE RANDOM VARIABLE TYPE II

Suppose that X is a discrete random variable and α is the parameter of distribution. The generalized discrete random variable type II ($GDRV^{II}$) is represented as the function $\Phi_{\overline{\alpha}}(x)$, and defined by $\frac{x^{\overline{\alpha-1}}}{\Gamma(\alpha)}$. The $GDRV^{II}$ Appears in some of distributions and we can rewrite these distributions in terms of it. For example, the Negative Binomial distribution can be written as following:

$$NB(x,p) = \Phi_{\overline{x+1}}(n)\,\Phi_{x+1}(q)\,\Phi_{n+1}(p)B(n+1,x+1)\Gamma(x+n+2),$$

Where $x = 0,1,\dots$, $0 < P < 1$ and $1-p = q$. The generalized discrete random variable type II, has the properties as:
In the special case for $\alpha = 2$, it becomes ordinary discrete random variable X. We state the later property of the $GDRV^{II}$ as a theorem, i.e.

THEOREM 4.1. The expectation of the $GDRV^{II}$, $\Phi_{\overline{\alpha}}(x)$, coincides with (nabla) Riemann left fractional sum of the probability function at 1 for $\alpha > 0$ and (nabla) Riemann left fractional difference of the probability function at and we have at 1 for $\alpha < 0$, $\alpha \notin \{\dots,-2,-1\}$, i.e.

$$E\left[\Phi_{\overline{\alpha}}(x)\right] = \begin{cases} \nabla^{-\alpha}f(1), & \alpha > 0 \\ \nabla^{\alpha}f(1), & \alpha < 0, \alpha \notin \{\dots,-2,-1\} \end{cases} \tag{27}$$

where

$$\nabla^{-\alpha}f(t) = \sum_{x=1}^{b+1-t} \frac{x^{\overline{\alpha-1}}}{\Gamma(\alpha)}.f(x+t-1),$$

is (nabla) Riemann left fractional sum of order α and

$$\nabla^{\alpha} f(t) = \sum_{x=1}^{b-t+1} \frac{x^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} . f(x+t-1),$$

is (nabla) Riemann left fractional difference of order α.
In order to prove the theorem 4.1, we state and prove the following lemma.

LEMMA 4.1. Let $f: {}_{b}\mathbb{N} \to \mathbb{R}$ and $\alpha > 0$ be given, with $n-1 < \alpha \leq n$. The following two definitions for the fractional difference $\nabla_b^{\alpha} f: {}_{b-n}\mathbb{N} \to \mathbb{R}$ are equivalent:

$$\nabla^{\alpha} f(t) = (-1)^n \Delta^n \nabla^{-(n-\alpha)} f(t), \tag{28}$$

and

$$\nabla^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \sum_{s=t}^{b} (\sigma(s)-t)^{\overline{-\alpha-1}} f(s), & n-1 < \alpha < n, \\ (-1)^n \Delta^n f(t), & \alpha = n. \end{cases} \tag{29}$$

PROOF: Let f and α be given as in the statement of the lemma 4.1 and we are showing that (29) is equivalent to (28) on the ${}_{b-n}\mathbb{N}$.
If $\alpha = n$, then (28) and (29) are obviously equivalent, since in this case

$$\nabla^{\alpha} f(t) = (-1)^n \Delta^n \nabla^{-(n-\alpha)} f(t) = (-1)^n \Delta^n \nabla^{-0} f(t) = (-1)^n \Delta^n f(t)$$

If $\mathrm{n}-1 < \alpha < n$, then a direct application of (28) implies that

$$\nabla^{\alpha} f(t) = (-1)^n \Delta^n \nabla^{-(n-\alpha)} f(t) = (-1)^n \Delta^n [\tfrac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b} (\sigma(s)-t)^{\overline{n-\alpha-1}} f(s)] = \tfrac{(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha)} \Delta [\sum_{s=t}^{b} (\sigma(s)-t)^{\overline{n-\alpha-1}} f(s)],$$

and by using of (7) and (9) we get it as

$$= \tfrac{(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha)} [\sum_{s=t}^{b} -(n-\alpha-1)(\sigma(s)-t)^{\overline{n-\alpha-2}} f(s) = (-1)^n \Delta^{n-1} [-\sum_{s=t}^{b} (\tfrac{(\sigma(s)-t)^{\overline{n-\alpha-2}}}{\Gamma(n-\alpha-1)} f(s)]$$

and by repeating this action $n-2$ times, which yields to

$$\nabla^{\alpha} f(t) = (-1)^n \Delta^{n-1} [\sum_{s=t}^{b} \tfrac{-(\sigma(s)-t)^{\overline{n-\alpha-2}}}{\Gamma(n-\alpha-1)} f(s)] = (-1)^n \Delta^{n-2} [\sum_{s=t}^{b} \tfrac{(\sigma(s)-t)^{\overline{n-\alpha-3}}}{\Gamma(n-\alpha-2)} f(s)] =$$

$$\Delta^{n-n} [\sum_{s=t}^{b} \tfrac{(\sigma(s)-t)^{\overline{n-\alpha-(n+1)}}}{\Gamma(n-\alpha-n)} f(s)] = \tfrac{1}{\Gamma(-\alpha)} \sum_{s=t}^{b} (\sigma(s)-t)^{\overline{-\alpha-1}} f(s). \blacksquare$$

Now by using the above statement, we can prove the theorem 4.1.
PROOF: For $\alpha > 0$, substitute $x = \sigma(s) - t$ in the expression (11) and also for $\alpha < 0$, $\alpha \notin \{\dots, -2, -1\}$ in the expression (29).■

The Laplace transform of the GDRV$^{\mathrm{II}}$ is given by $\mathbb{L}_1 \left\{ \Phi_{\overline{\alpha}}(x), s \right\} = (s)^{-\alpha}$, such that the discrete Laplace transform, defined by Atici and Eloe [3], as

$$\mathbb{L}_a\{f(t), s\} = \sum_{t=a}^{\infty} (1-s)^{t-1} f(t).$$

Let $\beta > 0$ and $x > 0$ be given. The relationship between the GDRV$^{\mathrm{II}}$ and GCRV is given by

$$(1-\beta) B(\alpha+1, 1-\beta-\alpha)(-1)^{\alpha} \, \Phi_{\overline{\alpha+1}}(\beta) = \frac{D^{\alpha} \, \Phi_{1-\beta}(x)}{\Phi_{1-\beta-\alpha}(x)},$$

and by consideration $D^{\alpha} x^{-\beta} = (-1)^{\alpha} \beta^{\overline{\alpha}} x^{-\beta-\alpha}$ and multiplying it by $\frac{1}{\Gamma(1-\beta)\Gamma(\alpha+1)\Gamma(1-\beta-\alpha)}$ we get

$$(-1)^{\alpha} \Phi_{\overline{\alpha+1}}(\beta) \, \Phi_{1-\beta-\alpha}(x) \tfrac{1}{\Gamma(1-\beta)} = \frac{D^{\alpha} \, \Phi_{1-\beta}(x)}{\Gamma(\alpha+1)\Gamma(1-\beta-\alpha)},$$

now, by a suitable simplification, we have our result. Also, the relationship between the GDRV$^{\mathrm{II}}$ and GDRV$^{\mathrm{I}}$ is given by

$$\Phi_{\overline{\alpha+1}}(x) = \Phi_{\underline{\alpha+1}}(x+\alpha-1).$$

Similar to previous cases, by using a property of the GDRV$^{\mathrm{II}}$, here the bounds of summation let us introduce the other type of discrete uniform distribution. In the following subsection, we discuss this new distribution and its statistical properties.

4.1 The fractional discrete uniform distribution type II

DEFINITION 4.1.1. The random variable X is said to have a fractional discrete uniform distribution type II with parameters (α, β), $\alpha \in \mathbb{R}$ and $\beta \in {}_{\alpha}\mathbb{N}$ when its probability function is given

$$f_X(x) = \frac{1}{\alpha-\beta+1}, \quad x = 1, \dots, \alpha-\beta+1, \ \alpha \in \mathbb{R}, \ \beta \in {}_{\alpha}\mathbb{N}, \tag{30}$$

which can be denoted by $X \sim FDU^{II}\{1, \dots, \alpha-\beta+1\}$.
Suppose that $X \sim FDU^{II}\{1, \dots, \alpha-\beta+1\}$, then the expectation, variance and moment generating function of this distribution are given by

$$E[X] = 1 + \frac{\alpha-\beta}{2}, \tag{31}$$

$$Var(X) = \frac{\beta^2+\alpha^2+2(\alpha+\beta)-14\alpha\beta}{12}, \tag{32}$$

$$M_X(t) = \frac{e^t}{\alpha-\beta+1}\left\{\frac{1-e^t}{1-e^{(\alpha-\beta+1)t}}\right\}, \tag{33}$$

respectively.

The FDU^{II} distribution for $\beta = \alpha - (N-1)$, gives the $DU\{1,2, \dots, N\}$ distribution. It can be yield this results separately for (31), (32) and (3).

5. RESULTS

In this work, by introduction of two type of discrete generalized random variables, we showed that the expected value of its first type equals to (delta) Riemann left fractional sum of the probability function at -1 for α > 0 and (delta) Riemann left fractional difference of the probability function at -1 for $\alpha < 0$, $\alpha \notin \{\dots, -2, -1\}$ and the expected value of its second type equals to (nabla) Riemann left fractional sum of the probability function at 1 f for α > 0 and (nabla) Riemann left fractional difference of the probability function at 1 for $\alpha < 0$, $\alpha \notin \{\dots, -2, -1\}$. Finally, we introduce the fractional versions of discrete uniform distribution and discuss its statistical properties.

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REFERENCES

[1] ABDELJAWAD, T. (2011): On Riemann and Caputo fractional differences, **Computers and Mathematics with Applications**, 62, 1602-1611.
[2] ATICI, F.M. and ELOE, P. W. (2007): A transform method in discrete fractional calculus, **International Journal of Difference Equations**, 2, 165-176.
[3] ATICI, F.M. and ELOE, P. W. (2009): Discrete Fractional Calculus with the Nabla Operator, 4, **EJQTDE** Spec. Ed. I, No. 3.
[4] ATICI, F.M. and ELOE, P. W. (2009): Initial value problems in discrete fractional calculus, **Proceedings of the American Mathematical Society**, 137, 981-989.
[5] ATICI, F.M. and SENGUL, S., (2010): Modeling with fractional difference equations, **Journal of Mathematical Analysis and Applications**, 369, 1-9.
[6] GANJI, M. and GHARARI, F. (2013): Some Fractional Special Functions and Fractional Moments, **General Mathematics Notes**, 18, 120-127.
[7] GANJI, M. and GHARARI, F., (2014): The Generalized Fourier Transform for the Generation of Complex Fractional Moments, **International Journal of Mathematical Archive** , 5, 293-299.
[8] GANJI, M. and GHARARI, F., (2015): The Generalized Random Variable Appears the Trace of Fractional Calculus in Statistics, **Applied Mathematics Information Sciences Letters**, 3, 61-67.
[9] KILBAS, A., SRIVASTAVA, M. H. and TRUJILLO, J. J. (2006): **Theory and Application of Fractional Differential Equations**, North Holland Mathematics Studies, Vol. 204.
[10] MILLER, K. S., and ROSS, B., (1989): Fractional difference calculus, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications, **Nihon University, Koriyama, Japan**,139-152.