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INDEPENDENT EVIDENCE COORDINATES: REPRESENTATION, NORMALIZATION, AND INFERENCE IN PARACONSISTENT EVIDENCE SEMANTICS

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ABSTRACT
Evidence representations routinely append a coordinate for contradiction, ignorance or indeterminacy to the polar pair of supporting and opposing degrees that paraconsistent formalisms provide. Its presence in the state tuple does not by itself show that it changes logical inference. We study the question structurally on the non-normalized triple $\langle T, I, F\rangle \in [0,1]^3$ supplied by neutrosophic logic, comparing it with the paraconsistent annotated logic PAL2v, with Belnap–Dunn evidence pairs, with the interlaced bilattice $[0,1]\odot[0,1]$ and with the normalized simplex. Three structural results organize the analysis. A representation–inference factorization: when the polar components of every connective depend only on polar components and designation reads only (T,F), the projection onto the polar square is a strict surjective matrix homomorphism, so the consequence relation of the cube is that of PAL2v for every rule of I-propagation. An activation trilemma: once designation reads I, no pair of propagation rules satisfies De Morgan duality, disjunction introduction and extensivity of conjunctive indeterminacy at once, though each pair is achievable. A normalization fibre theorem: unit-sum normalization is exactly the quotient by positive rays and deletes exactly the magnitude $M = T + I + F$, which the image and M together reconstruct. Subsidiary results show that a derived coordinate adds no degree of freedom, that I admits no uniform truth direction compatible with negation, and that admissible contradiction measures lie between $max(0, T + F - 1)$ and $min(T, F)$. Two corollaries carry this to deterministic decision rules: one that reads only the polar pair cannot exploit an independent coordinate, and one that reads only normalized triples is constant along fibres.
KEYWORDS: paraconsistent logic; evidence semantics; annotated logic; Belnap–Dunn logic; bilattices; indeterminacy; normalization; logical consequence; logical matrices; decision rules.
MSC: 03B53; 03B50; 03B52; 68T27

RESUMEN
Las representaciones multidimensionales de la evidencia añaden con frecuencia una coordenada de contradicción, ignorancia, indeterminación o fiabilidad al par de grados a favor y en contra que los formalismos paraconsistentes ya proporcionan. Que la coordenada figure en la tupla de estado no muestra por sí mismo que altere la inferencia lógica. El problema se estudia de forma estructural sobre la terna no normalizada $\langle T, I, F\rangle \in [0,1]^3$ de la lógica neutrosófica, comparándola con la lógica anotada paraconsistente PAL2v, con los pares de evidencia de Belnap–Dunn, con el bilátice entrelazado $[0,1]\odot[0,1]$ y con el símplex normalizado. Tres resultados estructurales organizan el análisis. Una factorización representación–inferencia: si las componentes polares de todos los conectivos dependen solo de componentes polares y la designación lee solo (T,F), la proyección sobre el cuadrado polar es un homomorfismo de matrices estricto y sobreyectivo, de modo que la consecuencia del cubo es la de PAL2v para toda regla de propagación de I. Un trilema de activación: cuando la designación lee I, ningún par de reglas de propagación satisface a la vez dualidad de De Morgan, introducción de la disyunción y extensividad de la indeterminación conjuntiva, aunque cada par de las tres sí es realizable. Un teorema de fibras para la normalización: la normalización a suma unidad es exactamente el cociente por rayos positivos y elimina exactamente la magnitud $M = T + I + F$, que junto con la imagen normalizada reconstruye el estado. Dos corolarios trasladan el análisis a las reglas de decisión deterministas.
PALABRAS CLAVE: lógica paraconsistente; semántica de la evidencia; lógica anotada; lógica de Belnap–Dunn; bilátices; indeterminación; normalización; consecuencia lógica; matrices lógicas; reglas de decisión.

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1. INTRODUCTION

Paraconsistent formalisms represent simultaneous support for and opposition to a proposition without collapse: Belnap and Dunn's [6, 23, 24] four values, the annotated logics of da Costa [17, 18], Subrahmanian and Vago and their two-valued variant PAL2v [1, 2, 9, 19, 20], Ginsberg's and Fitting's [25, 27] bilattices, and the logics of formal inconsistency all do this, in crisp or graded form. A recurrent move in the literature on uncertainty is to append a further coordinate to the pair of support and opposition: an indeterminacy degree in neutrosophic logic, a hesitancy in intuitionistic fuzzy sets [54], a neutrality in picture fuzzy sets, an uncertainty mass in subjective logic, a reliability or classicality coordinate in the logics of evidence and truth. The question this paper asks is not whether such a coordinate can be appended; it obviously can. The question is what it buys: "When does an additional evidence coordinate represent genuinely new information, and when does it change what follows from what?"

These are different questions, and the central claim of the paper is that they have different answers: representational independence does not imply inferential relevance. A coordinate may vary freely while the others are fixed, so that states differing only in it are indistinguishable to any lower-dimensional representation, and yet make no difference to logical consequence until operators or designation are made to read it. Adding a coordinate to the state space and adding a dimension to consequence are different operations. A third operation, normalization to unit sum, is different from both: it neither adds nor removes a coordinate but quotients the state space along rays and removes evidence magnitude.

We investigate these distinctions on one principal case. Let T,F $\in [0,1]$ be degrees of supporting and opposing evidence for a proposition (the μ,λ of PAL2v with new names), and let I $\in [0,1]$ be a degree of explicitly indeterminate evidence, with no constraint on $T + I + F$. The state $\langle T, I, F \rangle$ is the single-valued neutrosophic triple under an evidential reading; we take it because it is the free, non-normalized three-coordinate representation that the literature already supplies, and because its relation to PAL2v was made explicit in prior work [32]. The example that drives the paper is the pair $\langle 0.8,0.1,0.8 \rangle$ and $\langle 0.8,0.9,0.8 \rangle$: same support, same opposition, different recorded indeterminacy. No construction that factors through (T,F) separates them (Section 3); no consequence relation with polar designation separates them either (Section 4); and a designation that does separate them must give up one of three natural properties of the connectives (Section 5).

Smarandache's [41, 42, 43, 44, 45, 46, 47, 48, 49] principle of independently varying neutrosophic components provides the representational starting point of the present analysis. Our aim is to determine which consequences of that principle are representational, which depend on normalization, and which require the semantics of logical consequence to read the additional coordinate. The non-normalized neutrosophic triple supplies a particularly clean test case because its third coordinate is not constrained to be a function of support and opposition. The paper does not claim novelty for three-component representations, for the independence of the components, or for contradiction without explosion; Section 8.2 delimits what is claimed. The contributions are:

The contributions are of three kinds. On representation: a derived coordinate adds no degree of freedom (Theorem 3.8), a primitive one separates states that no function of (T,F) separates (Theorem 3.9), and it admits no uniform truth direction compatible with negation (Theorem 3.13). On consequence: under polar designation the cube has the consequence relation of PAL2v for every I-rule (Theorem 4.8, from Lemma 4.6), extending the language by an I-reading operator is conservative over it (Corollary 4.10), and once designation reads I a trilemma binds the propagation rules (Theorem 5.1). On normalization: ν is the quotient by positive rays and deletes exactly the magnitude (Theorems 6.6, 6.8). Contradiction measures are sandwiched between $max(0, T + F - 1)$ and $min(T, F)$ and depend only on the polar pair (Theorems 6.2, 6.5); two corollaries carry the first and third groups to decision rules (Section 7.1).

The contribution is structural: the results organize representation, normalization and consequence within a common formal framework (Figure 1), and identify the assumptions under which an additional coordinate becomes semantically observable. They give the principle of independent components a formal content that can be tested; they separate which of its consequences hold at the level of states, which at the level of normalization and which require additional semantic machinery; and they apply to any representation that appends a non-polar coordinate to a

paraconsistent evidence pair. The universal statements are proved analytically in the text; the numerical computations are an independent check, not the proof. The distinction matters beyond this formal setting: several contributions to this special issue decide under evidence that is missing, conflicting or only partially identified rather than merely noisy, and the representation-versus-inference separation argued here is the general form of the question each of them answers for its own domain.

2. EVIDENCE STRUCTURES AND PRIOR ART

Annotated logics and PAL2v. Annotated logics were introduced for paraconsistent logic programming by Blair and Subrahmanian [8], generalized to arbitrary complete lattices of annotations by Kifer and Subrahmanian [29], and axiomatized as the family Pτ by da Costa, Subrahmanian and Vago. In PAL2v (also written Eτ) the annotation is a pair $(\mu,\lambda) \in [0,1]^2$ of favourable and contrary evidence degrees, and the pair is free: $\mu + \lambda$ may exceed 1. Two derived quantities organize the applied literature: $G_c = \mu - \lambda \in [-1,1]$ (certainty), $G_{ct} = \mu + \lambda - 1 \in [-1,1]$ (contradiction), and the four vertices of the square are read as true (1,0), false (0,1), inconsistent (1,1) and paracomplete (0,0). A tri-annotated variant with two favourable indicators and one unfavourable indicator exists; its third annotation is polar, not an indeterminacy channel.

Belnap–Dunn logic, bilattices, trilattices. Dunn and Belnap read {t,f,b,n} as what a source may have told a computer, so that b and n are informational states; the resulting logic FDE is paraconsistent and paracomplete [34]. Ginsberg abstracted this to bilattices, sets with a truth order $\leq t$ and a knowledge order ≤k and a negation reversing the first and preserving the second; Fitting showed that interlaced bilattices arise as products L1⊙L2, of which $[0,1]\odot[0,1]$ is the continuous instance used below; Arieli and Avron [3, 4] developed the logics of logical bilattices. Shramko and Wansing [38, 39, 40] generalized to trilattices with three orders, SIXTEEN3 being the paradigm. Dubois [22] criticized reading b and n as truth values rather than information states, and Dunn replied. That distinction is the one we need: our coordinates are evidence degrees, and the results concern what evidence degrees do to consequence.

Quantitative evidence semantics. Carnielli and Rodrigues' [12, 13, 14, 36] logics of evidence and truth, probabilities over Belnap–Dunn logic [10, 11, 30], belief functions over it [7], the six-valued logics LET^{N+} [15] and their probabilities, and modal treatments of ignorance [31] make the evidential reading quantitative. Two features matter for us. In these systems the polar pair is free (a pair of probabilities $(p(\varphi),p(\neg\varphi))$ with no sum constraint, i.e. the (μ,λ) of PAL2v in probabilistic dress), and where a third coordinate appears it is reliability or classicality, expressed by an operator ∘ in the language and, in twist-structure semantics, bounded by the polar coordinates (z3 ≤ z1⊔z2). Logics of formal inconsistency internalize consistency as an operator and admit paraconsistent probabilities.

Neutrosophic logic. Neutrosophic logic assigns a triple (T,I,F) of independent components with $0 \leq T + I + F \leq 3$; single-valued neutrosophic sets are the $[0,1]$-valued case [52]. The literature reads $T + F > 1$ as a paraconsistent regime, refines components into sub-components [51], measures dependence between components, and allows components outside $[0,1]$. Picture fuzzy sets are the sub-case [16] $T + I + F \leq 1$ and intuitionistic fuzzy sets the sub-case $I = 1 - T - F$. Rivieccio [35] examined neutrosophic connectives from the standpoint of Belnap's logic: negation should fix I; min and max are the natural I-rules for conjunction; ignoring I returns Belnap's connectives; a truth ordering incorporating I is problematic; and consequence rests on designated subsets. Several results below give these observations a precise form. The embedding of PAL2v with I = 0 and a bounded paraconsistency result are in prior work. That normalization to unit sum is not innocuous has also been observed: ranking of triples by score, accuracy and certainty functions can change after normalization.

Normalized representations. Subjective logic normalizes its opinion [28] (b,d,u) to sum 1 and keeps evidence magnitude in a separate parameter through the bijection with Beta evidence counts; Dempster–Shafer [37] theory normalizes mass and discards conflict in Dempster's rule. Normalization is the appropriate representation when the coordinates are shares of one total; Section 6.2 identifies what it removes when the components are independently accumulated evidence degrees. The distinction goes beyond normalization behavior. Dempster–Shafer's [37] unassigned mass is a property of the whole frame of discernment and is driven toward zero as more specific evidence

is assigned to proper subsets; the coordinate I studied here is, when primitive rather than derived (Theorem 3.9), independent of T and F by construction and need not shrink as supporting or opposing evidence accumulates. An evidence body can therefore be indeterminate without being incomplete, which is not a state Dempster–Shafer's mass function can represent on its own.

Logical matrices. The representation–inference separation result of Section 4 is an instance of a standard fact about logical matrices [26, 53]: a surjective homomorphism between matrices that is strict (the designated set of the source is the preimage of that of the target) preserves the consequence relation. We have not found this fact applied to the question of what a coordinate adds to an evidence representation, and we include a proof to make the instance explicit.

3. INDEPENDENT EVIDENCE COORDINATES

3.1 The evidence state space

Convention 3.1 (Evidential reading). Throughout, T,I,F ∈ [0,1] are degrees of supporting, indeterminate and opposing evidence for a proposition, each measured against the capacity of its own channel. Nothing is assumed about their sum.

Definition 3.2 (Evidence cube, polar projection). A = $[0,1]^3$ with elements a = $\langle T_a, I_a, F_a \rangle$. The polar projection is $\pi(a)$ = (T_a, F_a) ∈ Q = $[0,1]^2$; Q is the PAL2v annotation square. We write $M(a) = T_{a+} I_{a+} F_a$ ∈ [0,3] (magnitude), $E(a) = T_{a+} F_a$ (polarized magnitude), $D(a) = T_{a-} F_a$ (net evidence), and, for a ≠ 0, $\nu(a)$ = a/$M(a)$ (normalization), with values in the simplex Δ = {s ∈ A : $s_{T+} s_{I+} s_F$ = 1}. We call T,F the polar coordinates and I the non-polar coordinate.

$$\pi(a) = (T_a, F_a) \in Q = [0,1]^2 \tag{1}$$

$$M(a) = T_a + I_a + F_a, \;\; E(a) = T_a + F_a, \;\; D(a) = T_a - F_a \tag{2}$$

$$\nu(a) = \frac{a}{M(a)}, \;\; \Delta = \{s \in A : \; s_T + s_I + s_F = 1\} \tag{3}$$

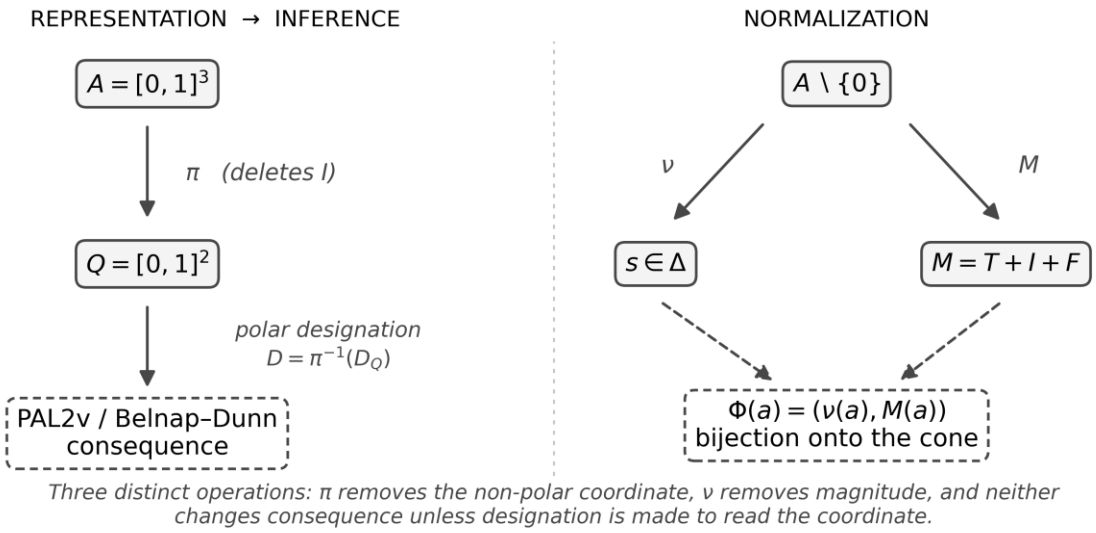


Figure 1. The three operations kept apart in this paper. The projection π deletes the non-polar coordinate and, under polar designation, leaves consequence unchanged (Theorem 4.8); the normalization ν deletes the magnitude M, which together with the normalized image reconstructs the state (Theorem 6.6). Representation, inference and normalization are three distinct operations.

Proposition 3.3 (Distributions versus evidence degrees). Let a finite body of evidence E carry element weights $w(e) \geq 0$, additive over subsets, and partitioned into supporting, indeterminate and opposing parts E+, E0, E−. Let K+, K0, K− > 0 be channel capacities and set T = $w(E+)$/K+, I = $w(E0)$/K0, F = $w(E-)$/K− (each clipped to [0,1]). (1) If K+ = K0 = K− = $w(E) > 0$, a common denominator, then $T + I + F = 1$ and the triple is a probability distribution over three exclusive and exhaustive states. (2) Fix a body E whose three parts all carry positive weight. As the capacities range over the positive reals, the magnitude $T + I + F$ takes every value of the half-open interval (0,3], the value 3 being attained at K+ = $w(E+)$, K0 = $w(E0)$, K− = $w(E-)$. The endpoint 0 is not attained for such a body: it is the magnitude of the empty body, and for a non-empty one it is approached only in the limit as the capacities grow without bound. (3) If some part of E is empty, its channel contributes 0 for every capacity; the attainable range is then (0,r], where r is the number of parts of positive weight.

$$T = \frac{w(E^+)}{K^+}, \quad I = \frac{w(E^0)}{K^0}, \quad F = \frac{w(E^-)}{K^-} \tag{4}$$

Proof. (1) is the additivity of w over a partition. (2) The bounds follow from T,I,F ∈ [0,1]. Taking each capacity equal to the weight of its own part gives T = I = F = 1 and hence magnitude 3. Scaling all three capacities by a common factor c ≥ 1 makes the magnitude a continuous and strictly decreasing function of c, equal to 3 at c = 1 and tending to 0 as c → ∞; by the intermediate value theorem every value of (0,3] is attained. No finite choice gives 0, because each part has positive weight and each capacity is finite, so every component is strictly positive. (3) A part of weight 0 gives a component identically 0, which removes one unit from the supremum. □

Normalization is thus a choice of denominator. Under a common denominator the components are competing shares of one total and the simplex is the appropriate state space; under channel-specific capacities they are independently accumulated degrees and the cube is. The conclusion that matters below is the second clause: independent channel capacities produce the non-normalized cube rather than a unit-sum simplex. Everything that follows concerns that case.

3.2 Redundancy, independence, inertness and activation

Definition 3.4 (Representational redundancy and independence). Let S be a set of states with coordinate maps $x_1,...,x_n : S \to X_i$, and let x_j be one of them. The coordinate xj is representationally redundant (relative to the others) if there is a function g with $x_j = g(x_1,...,\hat{x}_j,...,x_n)$ on S; it is representationally independent if there exist s,s′ ∈ S agreeing on all other coordinates and differing on x_j.

Definition 3.5 (Inferential inertness and activation). Fix a propositional language $\mathscr{L}$, an algebra structure on S interpreting its connectives, and a designated set D ⊆ S; let ⊨ be the induced consequence relation. The coordinate xj is inferentially inert for ($\mathscr{L}$,S,D) if any two valuations that agree on all coordinates other than xj (on atoms) designate the same formulas, so that ⊨ is unchanged by variation in x_j alone; it is inferentially active otherwise. When the coordinates other than xj form a reduct S′ with designated set D′ and the projection π : S → S′ is a surjective homomorphism with $D = \pi^{-1}(D')$, semantic factorization through π implies inertness, which in turn implies the coincidence of the consequence relations of $\langle S, D\rangle$ and $\langle S', D'\rangle$ (Lemma 4.6); the converse implications are not claimed.

In the cube, I is representationally independent (Theorem 3.9), and whether it is inert depends on the operators and on D (Theorem 4.8, Section 5). The two properties are logically unrelated: Definition 3.4 mentions no language, and Definition 3.5 is trivially satisfied by a redundant coordinate but may also be satisfied by an independent one.

3.3 Derived coordinates: functional redundancy

Definition 3.6. For g : Q → [0,1] define ι_g : Q → A by $\iota_g(\mu,\lambda) = \langle \mu, g(\mu,\lambda), \lambda\rangle$, and let $\Gamma_g = \iota_g(Q)$ be its image, the graph of g. The embedding of prior work is ι_0. The orders used below are: the evidence order $a \leq_e b$ iff $T_a \leq T_b$, $I_a \leq I_b$, $F_a \leq F_b$; the truth preorder $a \leq_t b$ iff $T_a \leq T_b$ and $F_a \geq F_b$; on Q, the product order and the truth order of [0,1]⊙[0,1].

$$\iota_g(\mu,\lambda) = \langle \mu, g(\mu,\lambda), \lambda \rangle, \quad \Gamma_g = \iota_g(Q) \tag{5}$$

Theorem 3.7 (Embedding). For every g: (i) ι_g is injective and $\pi \circ \iota_g$ = idQ; (ii) $G_c(\mu,\lambda)$ = D($\iota_g(\mu,\lambda)$) and $G_{ct}(\mu,\lambda)$ = E($\iota_g(\mu,\lambda)$)−1; (iii) ι_g is an order embedding of the PAL2v truth order into (A,$\leq t$); (iv) ι_g is an order embedding of the product order of Q into (A,$\leq e$) iff g is coordinatewise nondecreasing; (v) ι_g commutes with negation ((μ,λ)↦(λ,μ) versus $\langle T, I, F \rangle \mapsto \langle F, I, T \rangle$) iff g is symmetric.

Proof. (i) $\pi \circ \iota_g(\mu,\lambda) = (\mu,\lambda)$. (ii) D and E read only T and F. (iii) $\leq t$ reads only (T,F). (iv) $\iota_g(\mu,\lambda) \leq e\ \iota_g(\mu',\lambda')$ requires $g(\mu,\lambda) \leq g(\mu',\lambda')$ whenever μ≤μ′, λ≤λ′, which is monotonicity; conversely monotonicity gives the implication. (v) $\iota_g(\lambda,\mu) = \langle \lambda, g(\lambda,\mu), \mu \rangle$ and $\neg\iota_g(\mu,\lambda) = \langle \lambda, g(\mu,\lambda), \mu \rangle$ agree iff $g(\lambda,\mu) = g(\mu,\lambda)$. □

Theorem 3.8 (Functional redundancy of a derived coordinate). Suppose I is defined as I = $g(T,F)$, so that the state set is Γ_g. Then for every g, π|Γg is a bijection onto Q with inverse ι_g, and it is an isomorphism of the truth preorder restricted to Γ_g onto the PAL2v truth order; if g is coordinatewise nondecreasing it is also an isomorphism of $\leq e$|Γg onto the product order of Q. In every case Γ_g has no degree of freedom beyond (T,F), every operator on Γ_g is the ι_g−conjugate of an operator on Q, and the derived coordinate is representationally redundant. For non-monotone g the evidence order is not transported: with g = $1 - T$, (0,0) ≤ (1,0) in Q but ⟨0,1,0⟩ ⋦e ⟨1,0,0⟩.

Proof. π|Γg is a bijection with inverse ι_g by Theorem 3.7(i); the truth preorder is transported by (iii) for every g, and the evidence order by (iv) exactly when g is monotone; the displayed pair witnesses failure otherwise. Redundancy is the definition of Γ_g. □

The candidates $1 - \max(T,F)$ (Atanassov's [5] hesitancy when $T + F \leq 1$), $1-|T-F|$ and $\min(1-T, 1-F)$ (which equals the first) are all PAL2v quantities; a representation that defines I by any of them coincides with PAL2v up to renaming. Representational content beyond PAL2v requires a primitive I, which is the neutrosophic option.

Theorem 3.9 (Representational independence of a primitive coordinate). (i) Γ_g has empty interior in A for every g, measurable or not. (ii) If a ≠ b and $\pi(a) = \pi(b)$, no function h : A → X that factors through π separates a from b. (iii) In particular ⟨0.8,0.1,0.8⟩ and ⟨0.8,0.9,0.8⟩ are identified by every PAL2v quantity (G_c, G_{ct}, the para-analyser state) and by every candidate derived indeterminacy: $1 - \max(T,F) = \min(1-T, 1-F)$ = 0.2, $1-|T-F|$ = 1. Hence I is representationally independent.

Proof. (i) Every open ball around (x,y,$g(x,y)$) contains points (x,y,z) with z ≠ $g(x,y)$. (ii) h = h′∘π and $\pi(a) = \pi(b)$ give $h(a) = h(b)$. (iii) is an instance of (ii); the identity $1 - \max(T,F) = \min(1-T, 1-F)$ is elementary and has also been checked numerically. □

Corollary 3.10 (Contradiction is preserved by every embedding). For every g, $\min(\mu,\lambda) = \min(T,F) \circ \iota_g(\mu,\lambda)$ and $\max(0, G_{ct}) = \max(0, E-1) \circ \iota_g$; the paraconsistent region $\mu + \lambda > 1$ of PAL2v is mapped onto $T + F > 1$ in A, whatever I is.

3.4 States: silence versus abstention

The vertices of the cube read as follows; the last two have no PAL2v counterpart. ⟨1,0,0⟩ = PAL2v (1,0) true: maximal support, no opposition, nothing undecided. ⟨0,0,1⟩ = (0,1) false: maximal opposition only. ⟨1,0,1⟩ = (1,1) inconsistent: maximal contradiction, nothing undecided. ⟨0,0,0⟩ = (0,0) paracomplete: absence of evidence of every kind — silence. ⟨0,1,0⟩: no PAL2v counterpart — maximal indeterminate evidence, no polar evidence: a recorded abstention. ⟨1,1,1⟩: no PAL2v counterpart — maximal support, opposition and indeterminacy at once.

The difference between ⟨0,0,0⟩ and ⟨0,1,0⟩ is the difference between silence and an explicit report of undecidedness; neither Belnap's n nor PAL2v's (0,0) nor the simplex (where ⟨1,1,1⟩ does not exist) expresses it. This is what the independent coordinate represents. Whether it infers anything is the subject of Section 4.

3.5 Ordered structure and negation

Definition 3.11 (Negation and three relations). $\neg\langle T,I,F\rangle = \langle F,I,T\rangle$ (the I-fixing negation of prior work). For a,b ∈ A: a $\leq_e$ b iff $T_a \leq T_b$, $I_a \leq I_b$, $F_a \leq F_b$ (evidence order); a $\leq_t$ b iff $T_a \leq T_b$ and $F_a \geq F_b$ (truth preorder); a $\leq_d$ b iff $I_a \geq I_b$ (determinacy preorder).

$$\neg\langle T,I,F\rangle = \langle F,I,T\rangle \tag{6}$$

Proposition 3.12. (A,$\leq_e$) is a complete distributive lattice with componentwise meets and joins, bottom ⟨0,0,0⟩ and top ⟨1,1,1⟩. $\leq_t$ and $\leq_d$ are preorders; the quotient of (A,$\leq_t$) by $\pi(a) = \pi(b)$ is Q with the truth order of $[0,1]\odot[0,1]$, and the quotient of (A,$\leq_d$) by $I_a = I_b$ is the chain ([0,1],≥). ¬ is an involution fixing I, with fixed points {T=F}; it is an automorphism of (A,$\leq_e$) and an anti-automorphism of (A,$\leq_t$); it preserves $\min(T,F)$, E and M and negates D.

Proof. $\leq_e$ is a product of complete chains. $\leq_t$ is reflexive and transitive but not antisymmetric (⟨0.5,0.1,0.5⟩ $\leq_t$ ⟨0.5,0.9,0.5⟩ and conversely); its quotient is the truth order of the square. For the anti-automorphism: a $\leq_t$ b iff $T_a \leq T_b$ and $F_a \geq F_b$; since $T_{\neg b} = F_b \leq F_a = T_{\neg a}$ and $F_{\neg b} = T_b \geq T_a = F_{\neg a}$, this is ¬b $\leq_t$ ¬a. The rest is direct. □

Theorem 3.13 (Uniform truth-direction impossibility). Let $\gtrsim$ be a partial order on A such that (U) there is a fixed ε ∈ {+,−} such that for all T,F and all i < i′: $\langle T,i,F\rangle \gtrsim \langle T,i',F\rangle$ if ε=+, and $\langle T,i',F\rangle \gtrsim \langle T,i,F\rangle$ if ε=−; and (R) a $\gtrsim$ b implies ¬b $\gtrsim$ ¬a. Then no such $\gtrsim$ exists. The same conclusion holds if (U) is required only on the diagonal T=F. Consequently, for every partial order on A that is reversed by ¬, distinct diagonal states that differ only in I are incomparable (an antichain under variation of I), and every partial order that is I-monotone on the diagonal is not reversed by ¬.

Proof. Take a = $\langle t,i,t\rangle$, b = $\langle t,i',t\rangle$ with i < i′ (for instance t=0.5, i=0.2, i′=0.7). Then ¬a = a and ¬b = b. By (U), a $\gtrsim$ b (case ε=+; the other is symmetric). By (R), ¬b $\gtrsim$ ¬a, i.e. b $\gtrsim$ a. Antisymmetry gives a = b, contradicting i ≠ i′. The two consequences restate this: on the diagonal ¬ fixes every element, so (R) forces such states to be incomparable, which (U) forbids. □

Corollary 3.14 (Ordered structure of the cube). The three relations treat the non-polar coordinate differently, and the statement has to record the difference. (i) The evidence order $\leq_e$ is the product of the knowledge order of $[0,1]\odot[0,1]$ with the usual chain order on the I-factor, and it is preserved by ¬. (ii) The truth preorder $\leq_t$ is the pullback along π of the truth order of $[0,1]\odot[0,1]$; equivalently it is the product of that truth order with the indiscrete preorder on the I-factor, in which any two values of I are mutually related. It is reversed by ¬, and its quotient by $\pi(a) = \pi(b)$ is the truth order of the bilattice. (iii) ¬ acts as the polar negation on the Q-factor and as the identity on the I-factor. Consequently A is not the product of Fitting's interlaced bilattice with one chain participating in both bilattice orders: the I-factor carries the chain order in the knowledge direction and the indiscrete preorder in the truth direction. In particular the structure is not a trilattice with I as a third alethic order, because by Theorem 3.13 no partial order on A is at once uniformly monotone in I and reversed by negation.

Remark 3.15 (Sign-dependent attenuation is compatible). Theorem 3.13 forbids a uniform direction for I; it does not forbid I from attenuating polar evidence. The total preorder induced by $\sigma(a) = (T_a - F_a)(1 - I_a)$ is reversed by ¬ (since σ(¬a) = $-\sigma(a)$) and is monotone in I toward 0 on each side of the diagonal, in opposite directions on the two sides. So I may contract toward indifference, and thereby influence truth-sensitive rankings, without supplying a uniform direction of truth. Score functions adding $1 - I$ as a truth-like term give total orders useful for ranking but not reversed by ¬. The theorem explains the difficulty Rivieccio reported in defining a truth ordering that incorporates I: it constrains any non-polar coordinate paired with a polarity-swapping negation, and is no peculiarity of the neutrosophic triple.

$$\sigma(a) = (T_a - F_a)(1 - I_a) \tag{7}$$

4. REPRESENTATION AND LOGICAL CONSEQUENCE

4.1 Connectives with an I-rule

Definition 4.1 (Connectives). Let $\oplus, \oplus' : [0,1]^2 \to [0,1]$ be commutative and associative (I-rules). Define a∧b = $\langle \min(T_a, T_b), I_a \oplus I_b, \max(F_a, F_b) \rangle$, a∨b = $\langle \max(T_a, T_b), I_a \oplus' I_b, \min(F_a, F_b) \rangle$, a→b = ¬a∨b. The polar parts are the PAL2v (Belnap–Dunn, [0,1]⊙[0,1] truth-lattice) meet and join; the only design freedom is in I. The neutrosophic tradition takes $\oplus = \oplus'$ = max or a t-norm/t-conorm pair.

$$a \wedge b = \langle \min(T_a, T_b), I_a \oplus I_b, \max(F_a, F_b) \rangle \tag{8}$$

$$a \vee b = \langle \max(T_a, T_b), I_a \oplus' I_b, \min(F_a, F_b) \rangle, \quad a \to b = \neg a \vee b \tag{9}$$

Proposition 4.2. (i) ∧,∨ are monotone for $\leq e$ iff $\oplus, \oplus'$ are monotone, and always monotone for $\leq t$. (ii) ∧ is the $\leq t$-meet and ∨ the $\leq t$-join on the quotient Q. (iii) a∧a = a iff $\oplus$ is idempotent (e.g. max, min); it fails for the probabilistic sum $x + y - xy$. The arithmetic mean is idempotent but not associative and is not an admissible I-rule.

Proof. Componentwise checks; the polar components are lattice operations of Q. □

Theorem 4.3 (De Morgan). ¬(a∧b) = ¬a∨¬b for all a,b iff $\oplus = \oplus'$. Double negation ¬¬a = a holds unconditionally.

Proof. ¬(a∧b) = $\langle \max(F_a, F_b), I_a \oplus I_b, \min(T_a, T_b) \rangle$ and ¬a∨¬b = $\langle \max(F_a, F_b), I_a \oplus' I_b, \min(T_a, T_b) \rangle$; the polar parts agree, so equality for all a,b is equality of $\oplus$ and $\oplus'$ as functions. A numerical check covers the sixteen combinations of {max,min,prob-sum,mean} on a 48×48 grid (the mean is included only as a binary identity test): exactly the four diagonal combinations satisfy De Morgan. □

Counterexample 4.4 (Mixed I-rules). With $\oplus$ = max for ∧ and $\oplus'$ = min for ∨, a = ⟨0.3,0.4,0.5⟩, b = ⟨0.7,0.9,0⟩: ¬(a∧b) has I = 0.9 while ¬a∨¬b has I = 0.4. (Verified by direct computation: $a \wedge b = \langle 0.3, \max(0.4,0.9), 0.5 \rangle = \langle 0.3,0.9,0.5 \rangle$, so ¬(a∧b) = ⟨0.5,0.9,0.3⟩; and ¬a = ⟨0.5,0.4,0.3⟩, ¬b = ⟨0,0.9,0.7⟩, so ¬a∨¬b = $\langle \max(0.5,0), \min(0.4,0.9), \min(0.3,0.7) \rangle = \langle 0.5,0.4,0.3 \rangle$.)

4.2 A factorization lemma for logical matrices

Definition 4.5 (Matrices, valuations, consequence). Let $\mathscr{L}$ be a propositional language over a countable set of atoms with connectives ¬,∧,∨ (and defined →). A matrix for $\mathscr{L}$ is a pair M = $\langle S, D \rangle$ where S is an algebra of the same type as $\mathscr{L}$ and D ⊆ S. A valuation is a homomorphism v : $\mathscr{L}$ → S. Γ $\models_M$ φ iff every valuation with v[Γ] ⊆ D has $v(\varphi) \in D$.

Lemma 4.6 (Strict surjective homomorphisms preserve consequence). Let $M_A = \langle A, D_A \rangle$ and $M_Q = \langle Q, D_Q \rangle$ be matrices for $\mathscr{L}$ and let π : A → Q be a surjective homomorphism of algebras with $D_A = \pi^{-1}(D_Q)$. Then $\models_\{M_A\}$ and $\models_\{M_Q\}$ coincide.

Proof. ($\models_\{M_Q\} \subseteq \models_\{M_A\}$.) Let v be an A-valuation with v[Γ] ⊆ D_A and $v(\varphi) \notin D_A$. Then π∘v is a Q-valuation (composition of homomorphisms) with (π∘v)[Γ] ⊆ D_Q and (π∘v)(φ) ∉ D_Q, by $D_A = \pi^{-1}(D_Q)$. So an A-countermodel yields a Q-countermodel. ($\models_\{M_A\} \subseteq \models_\{M_Q\}$.) Let u be a Q-valuation with u[Γ] ⊆ D_Q and $u(\varphi) \notin D_Q$. Choose a section σ : Q → A with π∘σ = id_Q (surjectivity), and let v be the A-valuation with $v(p) = \sigma(u(p))$ on atoms. Since π is a homomorphism and π∘v = u on atoms, π∘v = u on all formulas. Hence v[Γ] ⊆ $\pi^{-1}(D_Q) = D_A$ and $v(\varphi) \notin D_A$: a Q-countermodel yields an A-countermodel. □

This is the standard fact that a strict surjective matrix homomorphism preserves the logic of the matrix; it is not claimed as new, and the proof is included only to make the instance explicit. Its role here is diagnostic: it separates enrichment of the semantic state space from enrichment of the induced consequence relation. First, the hypothesis "π is a homomorphism" says that the polar components of every connective depend only on polar components; the hypothesis "$D_A = \pi^{-1}(D_Q)$" says that designation reads only the polar components. Second, the lemma makes no assumption at all about how the non-polar coordinate is propagated: any I-rules whatever are allowed.

4.3 The separation theorem

Definition 4.7 (Polar designation). For θ ∈ (0,1] let D_θ = {a ∈ A : $T_a \geq \theta$}, and write $\models_\theta$ for the consequence of $\langle A, D_\theta \rangle$ with the connectives of Definition 4.1. More generally a designated set is polar if D = $\pi^{-1}(D_Q)$ for some D_Q ⊆ Q. With θ = 1 and crisp polar values, D_1 = {T=1} = {t,b} and the (T,F)-reduct of $\models_1$ is Belnap–Dunn consequence; for θ < 1 one obtains the graded analogue of PAL2v.

Theorem 4.8 (Representation–inference separation). For every choice of I-rules ⊕,⊕′ and every polar designated set D, the consequence relation of $\langle A, D \rangle$ coincides with that of $\langle Q, D_Q \rangle$, the (T,F)-reduct (continuous PAL2v / the truth lattice of [0,1]⊙[0,1] with the same designation). In particular $\models_\theta$ is the PAL2v consequence with designation T ≥ θ, and I is inferentially inert for ($\mathscr{L}$,A,D).

Proof. π is surjective, and it is a homomorphism because the T and F components of ¬,∧,∨ in Definition 4.1 are functions of the T and F components of the arguments only; the I-rules do not enter. Polar designation is D = $\pi^{-1}(D_Q)$ by definition. Apply Lemma 4.6. Inertness: two valuations agreeing on (T,F) have the same image under π, hence satisfy the same sequents. □

This is the centre of the paper. Enlarging the state space and enlarging the consequence relation are different operations: the coordinate enlarges the first (Theorem 3.9) and leaves the second untouched for every I-rule, so long as designation stays polar. The second enlargement requires the additional semantic step of Section 5.

Corollary 4.9 (Non-explosion is inherited). For every θ ∈ (0,1], {p,¬p} ⊭_θ q.

Proof. $v(p) = \langle 1,0,1 \rangle$, $v(q) = \langle 0,0,0 \rangle$: v(¬p) = ⟨1,0,1⟩, so p,¬p ∈ D_θ while $T_{v(q)}$ = 0 < θ. Equivalently, by Theorem 4.8, this is the non-explosion of PAL2v and of Belnap–Dunn logic. □

Corollary 4.10 (Conservative extension by an I-reading connective). Extend the language with an operator ∂ given by $\partial\langle T, I, F \rangle = \langle I, 0, 1 - I \rangle$, which reads the non-polar coordinate into a polar position. Two claims must be kept apart. (i) Conservativity over the old signature: if designation remains polar, every sequent whose formulas are built from ¬,∧,∨ alone has the same status with and without ∂ in the language. This is Theorem 4.8 applied to the ∂-free reduct, which is where π is a homomorphism; the theorem is not applied to the expanded algebra, in which π∘ ∂ reads I and π is therefore no longer a homomorphism. (ii) Activation: it occurs only when formulas built with ∂ are themselves considered, or when designation is made I-sensitive. Take $v(p) = \langle 0,1,0 \rangle$ with polar designation and θ ∈ (0,1]. Then p is not designated, since its T-component is 0 < θ, whereas $\partial p = \langle 1,0,0 \rangle$ is designated; a designated set that reads ∂-formulas therefore designates p. The polar reduct cannot predict this, because ⟨0,1,0⟩ and ⟨0,0,0⟩ have the same polar image (0,0) while their ∂-images ⟨1,0,0⟩ and ⟨0,0,1⟩ differ in designation. Lemma 4.6 does not apply to the expanded matrix, and the separation of Theorem 4.8 is a statement about designation reading polar components only, not about which connectives the language contains.

5. ACTIVATING THE NON-POLAR COORDINATE

Section 4 showed that I is inert under polar designation, for every I-rule. Activation requires designation that reads it, or operators that read it. Corollary 4.10 is conservative precisely because its designated set stays polar; genuine activation needs a designated set such as $D_{\theta,\kappa}$ = {a : $T_a \geq \theta$, $I_a \leq \kappa$} that is not of the form $\pi^{-1}(D_Q)$ (Definition 4.7), so Lemma 4.6 does not apply, and the I-rules now matter to consequence.

Theorem 5.1 (Activation trilemma: contractivity versus extensivity). Let 0 < θ ≤ 1 and 0 ≤ κ < 1, and let Dθ,κ = {a ∈ A : Ta ≥ θ and Ia ≤ κ} be the designated set. Consider three properties of a pair of I-rules ⊕ for ∧ and ⊕′ for ∨. (DM) De Morgan duality, which under the I-fixing negation requires ⊕ = ⊕′ (Theorem 4.3). (DI) Disjunction introduction, a ⊨ a∨b for the induced consequence. (EXT) Extensivity of conjunctive indeterminacy, $\oplus(x,y) \geq \max(x,y)$: a conjunction never records less indeterminacy than either conjunct. No pair of I-rules satisfies DM, DI and EXT simultaneously, and each pair of the three is satisfiable.

Proof. Assume DM, so that ⊕ = ⊕′ = h. Since κ < 1, choose x ≤ κ < y ≤ 1 and states a, b with Ia = x, Ib = y and Ta ≥ θ, so that a is designated. DI requires a∨b to be designated, hence $Ia \vee b = h(x,y) \leq \kappa$. EXT requires

$h(x,y) \geq \max(x,y) = y > \kappa$. The two requirements are contradictory, which proves the incompatibility. For the pairwise satisfiability: $\oplus = \oplus' = \min$ satisfies DM and DI, since $\min(x,y) \leq x \leq \kappa$ whenever a is designated, and violates EXT; $\oplus = \oplus' = \max$ satisfies DM and EXT and violates DI, because $\max(x,y) = y > \kappa$; and $\oplus = \max$ with $\oplus' = \min$ satisfies DI and EXT and violates DM. ■

The trilemma isolates the price of making I operational. Contractive rules such as min keep designation stable under disjunction but let a conjunction of one determinate and one indeterminate premise be recorded as determinate; extensive rules such as max keep that record honest but break disjunction introduction, since a designated disjunct no longer guarantees a designated disjunction; and keeping both forces the two connectives apart and with them De Morgan duality. Which horn to give up is a modelling decision and not a mathematical one: applications that reason forward from designated premises will protect DI, whereas applications that audit how much indeterminacy a conclusion carries will protect EXT. The three instances named in the proof are the witnesses; the argument is analytic and depends on no grid search.

Remark 5.2 (Pair plus operator). Let $\mathscr{L}\partial$ be as in Corollary 4.10. A ∂-valuation is a pair (w,δ) in which w is a Q-valuation and δ assigns the read-off value of ∂ on atoms, extended componentwise by the connectives of Definition 4.1; the consequence relation designates a when $Ta \geq \theta$ and $\delta(a) \leq \kappa$. This adds no result. It is a renaming of the I-sensitive designation of Definition 4.1 and Corollary 4.10, and Theorem 5.1 applies to it verbatim. It is recorded because the pair-plus-operator presentation is the one used in the logics of evidence and truth, so the translation makes the scope of the trilemma explicit there.

6. CONTRADICTION, INDETERMINACY AND NORMALIZATION

6.1 Contradiction measures

Definition 6.1 (Admissible contradiction measure). $C : Q \to [0,1]$ is admissible if (C1) $C(T,0) = C(0,F) = 0$; (C2) $C(T,F) = C(F,T)$; (C3) C is nondecreasing in each argument; (C4) $C(1,1) = 1$; (C5) $C(T,F) \leq \min(T,F)$ (contradiction cannot exceed the weaker side); (C6) $C(T,F) \geq T + F - 1$ (forced overlap: if the two sides together exceed unit capacity, at least the excess is contradictory). Optionally (C7), strictness: $C(T,F) > 0$ whenever $T > 0$ and $F > 0$.

The axioms encode a polar-evidence conception of contradiction as co-presence of supporting and opposing evidence. They are not claimed to be the only reasonable ones; the theorem organizes the family they admit. Because the definition fixes the domain of C as the polar square Q, the statement that admissible measures do not depend on I is part of the conception adopted, not a consequence derived from it; and because (C5) and (C6) are among the axioms, the bounds of the theorem below follow directly from them.

Theorem 6.2 (Sandwich). Every admissible C satisfies $\max(0,T+F-1) \leq C(T,F) \leq \min(T,F)$ on Q. Both bounds are admissible; the lower bound $\max(0,G_{ct})$ (threshold, or forced-overlap, contradiction) is the smallest admissible measure and is not strict; the upper bound min (co-presence contradiction) is the largest and is strict.

$$\max(0,T+F-1) \leq C(T,F) \leq \min(T,F) \tag{10}$$

Proof. The inequalities are (C6) with $C \geq 0$, and (C5). Admissibility of the bounds is a direct check: (C1)–(C6) hold for both; (C7) holds for min and fails for $\max(0,T+F-1)$ at (0.4,0.4), where it gives 0 against 0.4. Minimality is (C6) with $C \geq 0$; maximality is (C5). □

Every entry of Table 1 is established analytically; the grid computation reported in Appendix B is an independent check and not the argument. For $min(T,F)$, axioms (C1)–(C4) are immediate, (C5) holds with equality, and (C6) holds because when $T + F > 1$ and, without loss of generality, $T \leq F$, one has $min(T,F) = T \geq T + F - 1$ precisely when $F \leq 1$; (C7) is clear. For the product T·F, (C5) follows from $F \leq 1$, which gives $T \cdot F \leq T$, together with $T \leq 1$, which gives $T \cdot F \leq F$; and (C6) follows from $(1-T)(1-F) \geq 0$, that is, from $T \cdot F \geq T + F - 1$. For $max(0,T+$

$F - 1$), axiom (C6) holds by definition and (C5) by the computation just given for min, while (C7) fails at (0.4,0.4), where the value is 0. Each violation is witnessed by a single point. The geometric mean $\sqrt{(TF)}$ violates (C5) at (1,0.25), where it takes the value 0.5 > 0.25. The conflict balance $2min(T,F)/(T+F)$ violates (C3), since it equals 1 at (0.1,0.1) and 2/11 ≈ 0.18 at (0.1,1), and violates (C5) at (0.1,0.1). The directional balance $|T-F|$ violates (C1) at every (T,0) with T > 0, (C4) at (1,1) where it is 0, (C5) at (1,0), (C6) at (0.9,0.9) where 0 < 0.8, and (C7) on the whole diagonal.

Candidate	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**
$\min(T,F)$	Y	Y	Y	Y	Y	Y	Y
T·F	Y	Y	Y	Y	Y	Y	Y
$\max(0,T+F-1)$ (=$G_{ct}{}^{+}$)	Y	Y	Y	Y	Y	Y	n
$\sqrt{(TF)}$	Y	Y	Y	Y	n	Y	Y
$2\min(T,F)/(T+F)$ (conflict balance)	Y	Y	n	Y	n	Y	Y
$\lvert T-F\rvert$ (directional balance)	n	Y	n	n	n	n	n

Table 1. Candidate contradiction measures against (C1)–(C7). Y = satisfied, n = violated. Analytically verified; counterexamples given for violations.

Counterexample 6.3. $\sqrt{(TF)}$ violates (C5): at (1,0.25) it gives 0.5 > 0.25. The conflict balance $2\min/(T+F)$ violates (C3): it equals 1 at (0.1,0.1) and 0.18 at (0.1,1), so increasing opposition decreases it. These are balance measures, not contradiction measures.

Proposition 6.4 (Rank of the secondary measures). Support T, opposition F, any admissible C, polarized magnitude E, directional balance B = $|T-F|$, net evidence D and conflict balance are all functions of $\pi(a)$. (D,E) is a linear bijection of Q onto its image (T = $(E+D)/2$, F = $(E-D)/2$); (min,B) loses exactly the sign of D (both (0.9,0.6) and (0.6,0.9) give (0.6,0.3)). The family has rank two.

Proof. Each formula reads only T,F; the inversion and the collision are displayed. □

Theorem 6.5 (Contradiction–indeterminacy independence). For C = $\min(T,F)$ (or any admissible C with $C(c,c)$ = c), a ↦ $(C(a),I_a)$ is onto $[0,1]^2$; neither coordinate is a function of the other. In particular ⟨0.9,0.1,0.9⟩ and ⟨0.1,0.9,0.1⟩ exchange the values (C,I) = (0.9,0.1) and (0.1,0.9).

Proof. (c,i) is attained by $\langle c,i,c\rangle$; non-functionality follows from surjectivity onto a product. □

Under the polar-evidence conception of contradiction adopted here, admissible contradiction measures depend on the support–opposition plane, while I represents an orthogonal, explicitly indeterminate channel; the pair (C,I) spans a two-dimensional product. Recording conflict in I rather than in the polar plane would identify states that the two coordinates keep apart, in the same way that normalization identifies states along a ray (Section 6.2).

6.2 Normalization fibres and magnitude

Let A* = A\{0}. The multiplicative group (0,∞) acts partially on A* by c·a = ca (defined when ca ∈ A), and ν is constant on orbits. The next theorem says that ν is exactly the quotient by this action, and identifies the invariant that it deletes.

Theorem 6.6 (Normalization fibre theorem). (1) $\nu(a) = \nu(b)$ iff b = ca for some c > 0. The equivalence classes are the positive rays intersected with the cube; ν is not injective. (2) For s ∈ Δ, ν^{-1}(s) = {cs : 0 < c ≤ 1/max(s_T,s_I,s_F)}, a segment of Euclidean length ‖s‖₂/max(s), uncountable for every s. (3) $\Phi(a) = (\nu(a),M(a))$ is a bijection from A* onto the truncated cone {(s,m) : s ∈ Δ, 0 < m ≤ 1/max(s)}, with inverse (s,m) ↦ ms. (4) Hence normalization preserves composition (the point of Δ) and deletes exactly the magnitude M: states with the same normalized image differ only in M, and M with the image reconstructs the state.

$$\nu^{-1}(s) = \{cs : 0 < c \leq \tfrac{1}{\max(s)}\},\ \ \Phi(a) = (\nu(a),M(a)) \tag{11}$$

Proof. (1) If b = ca then $M(b)$ = cM(a) and $\nu(b) = \nu(a)$; conversely $\nu(a) = \nu(b)$ gives b = $(M(b)/M(a))$a. (2) cs ∈ A iff c·max$(s) \leq 1$. (3) Injective by (1) and $M(a)\nu(a)$ = a; surjective since for (s,m) in the cone a = ms ∈ A has $\nu(a)$ = s and $M(a)$ = m. (4) restates (3). A numerical check confirms a = Mν(a) and M ≤ 1/maxν(a) on 20,000 random points. □

Corollary 6.7 (Dimensional loss). A has dimension 3 and Δ dimension 2; ν loses one degree of freedom, which Theorem 6.6(3) identifies with M. Any other scalar coordinate that reconstructs the position along each fibre is an injective reparametrization of M on that fibre; if it is continuous, it is strictly monotone.

Theorem 6.8 (Conflation along a fibre). Let s ∈ Δ. (1) Along ν^{-1}(s), $\min(T,F)$ takes every value in (0, $\min(s_T, s_F)$/max(s)], and is identically 0 iff $s_T \cdot s_F = 0$. (2) The fibre contains both an overdetermined state ($T + F >$ 1) and an underdetermined state ($T + F <$ 1) iff $s_{T+}s_F > \max(s_T, s_I, s_F)$.

Proof. (1) $\min(T,F)$(cs) = c·$\min(s_T, s_F)$ for c ∈ (0,1/max(s)]. (2) $T + F$ along the fibre is $c(s_{T+s_F})$, below 1 for small c and maximal, equal to $(s_{T+}s_F)$/max(s), at the top; the maximum exceeds 1 iff the condition holds. The criterion has also been verified numerically on a simplex grid of step 1/60. □

Counterexample 6.9 (Three things normalization conflates). (a) ⟨0.1,0.1,0.1⟩ and ⟨0.9,0.9,0.9⟩ both normalize to (1/3,1/3,1/3); $\min(T,F)$ is 0.1 and 0.9. (b) ⟨0.1,0,0.1⟩ and ⟨0.9,0,0.9⟩ both normalize to (0.5,0,0.5); E is 0.2 and 1.8. (c) s = (0.45,0.10,0.45) satisfies $s_{T+}s_F$ = 0.9 > 0.45: ⟨0.45,0.10,0.45⟩ ($T + F$ = 0.9 < 1, G_{ct}+ = 0) and ⟨0.9,0.2,0.9⟩ ($T + F$ = 1.8, G_{ct}+ = 0.8) have the same image; the first is PAL2v-consistent and the second is not. For s = (0.2,0.6,0.2) the condition fails and the whole fibre is underdetermined. (d) ⟨0.9,0.7,0.1⟩ normalizes to (0.5294,0.4118,0.0588), as does ⟨0.45,0.35,0.05⟩ (exactly half of the first, hence on the same ray); the point on this ray that survives normalization unchanged is the simplex point itself, ⟨0.5294,0.4118,0.0588⟩, since it already has M = 1. More generally, any state with M = 1 is its own image under ν; an unrelated example of such a fixed point is ⟨0.05,0.9,0.05⟩ (which does not lie on the ⟨0.9,0.7,0.1⟩ ray, but independently satisfies 0.05 + 0.9 + 0.05 = 1).

Two distinct dimensional questions must be kept apart. What the cube adds to the simplex is M (this section). What I adds to the polar pair is a different question, answered in Sections 3 and 4. Neither addition changes consequence by itself: M does not enter designation or connectives at all, and I enters only when read. Two remarks temper the reading of M. It adds polar and non-polar evidence, so it is a volume rather than a strength: ⟨0.9,0,0.9⟩ and ⟨0.6,0.6,0.6⟩ share M = 1.8. The chart (T,I,F) ↦ (D,E,I) separates net evidence, polarized magnitude and indeterminate magnitude, and shows that the simplex constraint $E + I$ = 1 ties polarized to non-polarized magnitude. Nothing here depends on the bound 1 except fibre length: over- and under-sets change the lengths, not the ray structure.

Proposition 6.10 (The loci E = 1 and M = 1). (i) {E = 1} = $\{G_{ct} = 0\}$ is the image of fuzzy scalars t ↦ $\langle t, I, 1-t\rangle$ and is closed under ¬,∧,∨ for every I-rule. (ii) {M = 1} = Δ is the image of picture fuzzy sets with equality, i.e. of distributions over three exclusive states, and is not closed under ∧ or ∨ with $\oplus = \oplus' =$ max. (iii) Neither locus plays any role in $\models_\theta$ (Theorem 4.8).

Proof. (i) If $F_a = 1 - T_a$, $F_b = 1 - T_b$ then $\max(F_a, F_b) = 1 - \min(T_a, T_b)$ and $\min(F_a, F_b) = 1 - \max(T_a, T_b)$; ¬ swaps t and $1 - t$. (ii) ⟨0.5,0,0.5⟩∧⟨0,1,0⟩ = ⟨0,1,0.5⟩ has M = 1.5, and the disjunction ⟨0.5,1,0⟩ likewise. (iii) is Theorem 4.8. □

Threshold classifications involving $T + I + F$, such as the strong and weak paraconsistent regimes noted in the literature, encode additional information about the total evidential state; their systematic relation to the contradiction measures of Section 6.1 and to different notions of paraconsistency deserves a separate treatment (Section 8.3).

6.3 Interpreting the non-polar coordinate

Under Convention 3.1 the notions called indeterminacy sort as follows. Ignorance, meaning no evidence, lives in small M and not in I: ⟨0,0,0⟩, not ⟨0,1,0⟩. Recorded indeterminacy lives in I, which is the coordinate's proper content;

incompleteness in small E with I arbitrary; neutrality or ambiguity in high I, where a source says "neither"; and conflict in high $\min(T, F)$ — not I: the polar plane (Theorem 6.5).

The essential points are two. Silence and explicit abstention are different states ($\langle 0,0,0\rangle$ versus $\langle 0,1,0\rangle$), and I is the coordinate that records the second; and the coordinates keep distinct kinds of uncertainty apart: conflict is read in the polar plane, absence of evidence in low magnitude, and recorded indeterminacy in I. Whether the indeterminacy recorded in I would decrease under further evidence or is constitutive of the proposition is not distinguishable by a scalar I; the refined syntax of prior work can express such a split, and its logic is left to future work (Section 8.3).

7. OPERATIONAL IMPLICATIONS AND ILLUSTRATION

7.1 Decision rules that read the coordinates

The separation theorem and the fibre theorem are statements about logical consequence, but each has an immediate reading for evidence-based decision support, where the object of interest is not a consequence relation but a policy. Let D be a finite set of actions and let R : A → D be a deterministic decision rule on the evidence cube, assigning to each evidential state the action to be taken in it. Two architectural choices are common in practice, and each of them is a factorization of R through one of the two maps studied above.

Corollary 7.1 (Polar decision factorization). Suppose R = $R_Q \circ \pi$ for some R_Q : Q → D, that is, the rule reads only the polar pair. Then $\pi(a) = \pi(b)$ implies $R(a) = R(b)$ for all a,b ∈ A.

Proof. $R(a) = R_Q(\pi(a)) = R_Q(\pi(b)) = R(b)$. □

Corollary 7.2 (Normalization invariance). Suppose R = $R_\Delta \circ \nu$ for some R_Δ : Δ → D, that is, the rule reads only the normalized triple. Then $\nu(a) = \nu(b)$ implies $R(a) = R(b)$; by Theorem 6.6(1) this happens exactly when b = ca for some c > 0, so R is constant along every normalization fibre and cannot react to the magnitude M.

Proof. As above, with Theorem 6.6(1) identifying the fibres with the positive rays. □

Both statements are one line long, and that is the point: they are the decision-theoretic shadow of Theorems 4.8 and 6.6, and they say something a designer can act on. The states $\langle 0.8,0.1,0.8\rangle$ and $\langle 0.8,0.9,0.8\rangle$ of Section 1 differ in recorded indeterminacy, and Corollary 7.1 says that no rule reading support and opposition alone can ever treat them differently, however elaborate its internal structure. Likewise $\langle 0.1,0,0.1\rangle$ and $\langle 0.9,0,0.9\rangle$ differ by a factor of nine in the amount of evidence behind the same balance, and Corollary 7.2 says that a rule operating on normalized triples must issue the same action in both: it distinguishes neither a thin record from a thick one nor, on a single ray, an underdetermined state from an overdetermined one (Theorem 6.8).

The operational conclusion is symmetrical with the logical one. An added coordinate, or a magnitude parameter, matters to a decision-support system only if the policy is designed to read it: the coordinate has to appear in the argument of the rule, exactly as it has to appear in the designated set for it to matter to consequence. This yields a criterion that can be applied at design time. If the distinctions the system must draw include «the sources disagree» versus «the sources are silent», then R cannot factor through π; if they include «one study» versus «twenty concordant studies», then R cannot factor through ν. Conversely, if a system is built on normalized triples and is later asked to react to the volume of evidence behind them, no adjustment of its decision thresholds will achieve it, because the information is absent from the argument of the rule rather than badly weighted within it.

No new data are needed for these observations and none are claimed: they are the two theorems restated in the vocabulary of decision rules. Their content is that the design question — which coordinates does the policy read? — has the same answer structure as the semantic one.

7.2 Source-structured evidence

When the evidence comes from several sources the annotation may be indexed by source, $\langle T1,\dots,Tp \mid I1,\dots,Ir \mid F1,\dots,Fs\rangle$, the refined neutrosophic syntax of the literature, with coarse projection $\langle \max_j Tj, \max_k Ik, \max_l Fl\rangle$. One distinction then becomes expressible that the coarse triple cannot state: within-source versus between-source contradiction. We record only this distinction here.

Definition 7.3. For p = s (sub-components indexed by the same sources), define the within-source and between-source co-presence measures $C_m = \max_i \min(Ti, Fi)$ and $C_x = \max_{i\neq j} \min(Ti, Fj)$.

$$C_m = \max_i \min(T_i, F_i),\ \ C_x = \max_{i\neq j} \min(T_i, F_j) \tag{12}$$

Proposition 7.4. C_m and C_x are independent, and the coarse projection identifies states they separate: with two sources, R1 (T1,T2) = (0.9,0.1), (F1,F2) = (0.1,0.9) has C_m = 0.1, C_x = 0.9; R2 (0.9,0.9), (0.9,0.9) has $C_m = C_x$ = 0.9; R3 (0.9,0.9), (0.1,0.1) has $C_m = C_x$ = 0.1; R1 and R2 both project to (0.9,0.9).

Proof. The values are computed numerically; independence follows from R1 versus R2 (same C_x, different C_m) and R1 versus R3 (same C_m, different C_x). □

The coarse projection removes the within/between distinction in the same way that normalization removes magnitude: in both cases an aggregation identifies states that carry different information. Threshold versions of these measures and their relation to refined paraconsistent regimes are left to a separate treatment.

7.3 Illustration: a body of studies

The illustration, summarised in Table 2, is schematic: the studies are labelled, not real, and no domain claim is made. Let a claim be assessed by studies of equal quality, each reporting three independent degrees (w+,w0,$w-$): strength of support found, extent of inconclusive results, strength of opposition found (a study may report both support and opposition strongly, so these are not shares). The synthesis annotation accumulates the reported degrees against channel capacities (Proposition 3.3(2)); in the table the capacity of each channel is taken equal to the number of studies, so the annotation is the channelwise mean and the magnitude M compares literatures of the same size. With a fixed capacity K, as in the companion study of Section 8.3, the magnitude also grows with the number of documents, which is the reading under which Theorem 6.6 separates a thin record from a duplicated one.

Literature	**Studies**	$\langle T, I, F\rangle$	**M**	**v**	C_m, C_x
L1 weak	3×(0.1,0.0,0.1)	⟨0.1,0,0.1⟩	0.2	(0.5,0,0.5)	—
L2 contradictory	3×(0.9,0.0,0.9)	⟨0.9,0,0.9⟩	1.8	(0.5,0,0.5)	—
L3 studies disagree	2×(0.9,0,0.1), 2×(0.1,0,0.9)	⟨0.5,0,0.5⟩	1.0	(0.5,0,0.5)	0.1, 0.9
L4 each study split	4×(0.5,0,0.5)	⟨0.5,0,0.5⟩	1.0	(0.5,0,0.5)	0.5, 0.5
L5 inconclusive	4×(0.2,0.8,0.1)	⟨0.2,0.8,0.1⟩	1.1	(0.18,0.73,0.09)	—

Table 2. A schematic body-of-studies illustration**.**

L1 and L2 have the same normalized image; M (0.2 versus 1.8) separates them (Theorem 6.6). L2 and L5 are both "unresolved"; the first has $\min(T, F)$ = 0.9, I = 0, the second 0.1 and 0.8 (Theorem 6.5). L3 and L4 have the same coarse triple but different (C_m,C_x) (Proposition 7.4). Adding two supporting studies (0.9,0,0.1) to L3 moves the polar pair to ⟨0.63,0,0.37⟩ and leaves I untouched; adding them to L5 lowers I from 0.8 to 0.53: indeterminacy that recedes as evidence accumulates. And whatever follows from the claim by the connectives with polar designation is the same for L5 as for ⟨0.2,0,0.1⟩ (Theorem 4.8); the recorded inconclusiveness affects conclusions only if designation reads I, and then Theorem 5.1 decides what is given up.

8. RELATION TO PRIOR WORK AND DISCUSSION

8.1 Comparison with related frameworks

Tables 3 and 4 are organized around the questions of this paper rather than around formalisms, and they separate two things that the literature often reports together. Table 3 records representational features: what the state tuple contains. Table 4 records inferential features: what the semantics does with it. The distinction matters because a set representation, a logic, a logical matrix and an evidence calculus are objects of different kinds, and only the last three determine a consequence relation. Fuzzy, intuitionistic fuzzy and picture fuzzy sets are representations: they fix a state space and its operations but do not by themselves fix a designated set, so no consequence relation — explosive or otherwise — is intrinsic to them, and the corresponding entries of Table 4 record that the answer depends on the logic with which the representation is paired. Entries marked with an asterisk are theory-dependent and are explained in the notes below: (a) FDE keeps only the crisp distinction between b and n. (b) Subjective logic recovers magnitude through the bijection with Beta evidence counts, but it is absent from the opinion itself. (c) In $LETK+$ the third coordinate is read by the operator $\circ$ and is inferentially active, and it is bounded by the polar coordinates. (d) Neutrosophic logic specifies operator tables and leaves the designated set open: with polar designation the answer is «inert» by Theorem 4.8, and with I-sensitive designation «active». (e) Refined components index sources, but the coarse triple does not.

Framework	Polar pair	Non-polar component	Normalization	Magnitude retained	Conflict vs. ignorance
Classical logic	—	—	—	—	not expressible
Fuzzy sets	no (F = 1−T)	—	required	no	not expressible
Intuitionistic fuzzy sets	bounded (T+F ≤ 1)	derived hesitancy	required	no	not expressible
Picture fuzzy sets	bounded	bounded by T+F	required	no	partly
Belnap–Dunn / FDE	crisp	none	none	no	crisp only *(a)
PAL2v	free	none	none	polar (E)	yes (Gct)
Interlaced bilattice [0,1]⊙[0,1]	free	none	none	polar	yes
Logics of formal inconsistency	crisp	consistency operator ∘	—	—	crisp
LETK+	free	reliability, bounded *(c)	none	—	yes
Subjective logic	bounded (b+d ≤ 1)	u, derived	required	via Beta counts *(b)	partly (u)
Neutrosophic, non-normalized	free	free	none	yes (M)	yes
Neutrosophic, normalized	bounded	bounded	required	no	no
This paper (evidence cube)	free	free	none	yes (M)	yes

Table 3. Representational features of the frameworks compared**.**

Framework	Non-polar component read by the semantics?	Consequence relation	Source or reliability structure	Comment
Classical logic	—	Explosive	no	consequence fixed by the logic itself
Fuzzy sets	—	not intrinsic to the representation	no	depends on the logic the representation is paired with
Intuitionistic fuzzy sets	—	not intrinsic to the representation	no	depends on the associated logic
Picture fuzzy sets	—	not intrinsic to the representation	no	depends on the associated logic
Belnap–Dunn / FDE	—	paraconsistent, paracomplete	no	matrix and designated set given
PAL2v	no	paraconsistent, paracomplete	no	annotation lattice with designation
Interlaced bilattice [0,1]⊙[0,1]	no	as FDE, graded	no	truth and knowledge orders
Logics of formal inconsistency	yes, through the operator	paraconsistent	no	consistency internalized in the language
LETK+	yes *(c)	paraconsistent, paracomplete	reliability	third coordinate bounded by the polar ones
Subjective logic	yes, in fusion	Probabilistic	base rate	an evidence calculus, not a matrix semantics
Neutrosophic, non-normalized	unspecified *(d)	operator tables; designation left open	refined components *(e)	the designated set is the open choice
Neutrosophic, normalized	no	as picture fuzzy	no	magnitude removed by construction
This paper (evidence cube)	inert under polar designation; active under $D\theta,\kappa$ or ∂	coincides with PAL2v (Theorem 4.8); trilemma (Theorem 5.1)	via Cm and Cx	the two regimes are separated by a theorem

Table 4. Inferential features of the frameworks compared.

8.2 Scope of the claims

To delimit what is claimed: novelty is claimed for none of the ingredients. Not for paraconsistency, the PAL2v degrees $\mu - \lambda$ and $\mu + \lambda - 1$, three-component neutrosophic representations and the independence of their components, the paraconsistent reading of $T + F > 1$, the embedding of PAL2v with I = 0, bilattice or trilattice orders, refined components, the picture and intuitionistic fuzzy cases, normalized representations that keep magnitude apart, the information-state/truth-value distinction, probabilistic measures over Belnap–Dunn logic, or the preservation of consequence by strict matrix homomorphisms. Nor is it claimed that I has a probabilistic foundation, that the cube outruns a trilattice other than in cardinality, or that states with independent I arise in any given application: the last is empirical.

8.3 Discussion

The four questions now have separate answers. What I adds is a representationally independent coordinate, separating states that no function of (T,F) separates (Theorem 3.9) — silence from explicit abstention (Section 3.4) — along a direction independent of contradiction (Theorem 6.5). What its inferential effect depends on is whether the semantics reads it: under polar designation consequence coincides with PAL2v for every I-rule (Theorem 4.8), and no uniform truth direction survives negation reversal (Theorem 3.13), though I may attenuate polar evidence in a sign-dependent way (Remark 3.15). What normalization removes is magnitude, and with it everything varying along a ray (Theorems 6.6, 6.8): it is the right representation when the coordinates are shares of one total, the cube when the channels accumulate independently. And representation becomes inference when designation or the language reads the coordinate, at the price fixed by the trilemma (Theorem 5.1); which horn to give up the mathematics does not decide.

Empirical companion. A companion study applies these distinctions [50] to conflict-aware retrieval-augmented generation, where each retrieved document receives a triple from separate judgements and the coordinates are accumulated against a fixed channel capacity, so that magnitude records the amount of evidence rather than its average strength (Proposition 3.3(2)). Its outcomes are not used as evidence for any theorem in the present article: every result above is proved here, from the definitions given here. What the companion illustrates is where the four evidential states that the present results keep apart — indeterminacy, insufficiency, contradiction and magnitude — become operational in a decision policy, which is the question Section 7 addresses in the abstract.

Relation to the principle of independent components. The present results do not challenge the principle of independent neutrosophic components; rather, they identify its precise mathematical consequences at the levels of representation (Section 3), normalization (Section 6.2) and inference (Sections 4 and 5).

Future work. Three lines are left outside the theorem chain: a proof theory for the language with and for the I-sensitive consequence, with the relation between a free and the bounded classicality coordinate of +; a typed split of I into a part that recedes under further evidence and a part that does not; and the empirical question of whether states with I independent of (T,F) arise in concrete evidential processes. The first of these lines has a practitioner-facing counterpart: [33] develops the typed split of an indeterminacy coordinate into components that further evidence of the same kind would remove and components that are constitutive of how the evidential question was posed, and carries that typing through a full decision pipeline rather than leaving it at the representational level addressed here.$\partial\partial LETK$ Likewise, the integration of independent evidence coordinates with plithogenic probability and multivalued hypothesis logic reflects the analytical structures explored by De la Cantera et al. [21].

9. CONCLUSIONS

Independent neutrosophic components provide a genuine enlargement of the evidential state space. A primitive indeterminacy coordinate distinguishes states that no function of support and opposition can distinguish (Theorem

3.9), while the non-normalized representation retains the evidence magnitude that is lost under unit-sum normalization (Theorem 6.6). These representational gains do not automatically alter logical consequence: with polar designation the consequence relation is that of the two-coordinate paraconsistent annotated logic for every way of propagating I (Theorem 4.8), and inferential relevance arises only when operators or designation explicitly read the additional coordinate, at which point a trade-off among natural properties of the connectives has to be made (Theorem 5.1). Contradiction, read in the polar plane, and explicitly recorded indeterminacy are distinct evidential dimensions (Theorem 6.5). The distinction between representation and inference therefore clarifies, rather than weakens, the mathematical significance of independent neutrosophic components: the free neutrosophic triple makes composition, magnitude, contradiction and indeterminacy simultaneously explicit within a single state representation, and the results above say at which level each of these distinctions acts.

The methodological principle is general. Before adding an evidence coordinate, one should ask separately whether it is representationally independent, transformation-preserved, semantically observable, and inferentially active. The four questions have separate answers, and the present case shows how each is settled: the first by the state space, the second by the normalization map, the third and fourth by the designation and the operators of the logic.

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APPENDIX A. NOTATION

Symbol	Meaning
A = $[0,1]^3$	evidence cube; a = $\langle T, I, F \rangle$
Q = $[0,1]^2$	polar square, the PAL2v annotation lattice
Δ	simplex $T + I + F = 1$
$\pi(a)$ = (T,F)	polar projection
$\nu(a)$ = a/$M(a)$	normalization, a ≠ 0
M = $T + I + F$, E = $T + F$, D = $T - F$	magnitude, polarized magnitude, net evidence
$G_c = \mu - \lambda, G_{ct} = \mu + \lambda - 1$	PAL2v degrees of certainty and contradiction
ι_g, Γ_g	embedding (μ,λ) ↦ $\langle \mu, g(\mu,\lambda), \lambda \rangle$ and its image
$\leq e, \leq t, \leq d$	evidence order, truth preorder, determinacy preorder
$\neg, \wedge, \vee, \rightarrow; \oplus, \oplus'$	connectives; I-rules
M = $\langle S, D \rangle$, $\models_M$	logical matrix and its consequence
$D_\theta, D_{\theta,\kappa}$	polar and I-sensitive designated sets
∂	I-reading connective, $\partial\langle T, I, F \rangle = \langle I, 0, 1 - I \rangle$
C, C_m, C_x	contradiction measures; within-source and between-source co-presence
$\Phi(a)$ = ($\nu(a)$,$M(a)$)	the cone bijection of Theorem 6.6

Table A.1. Notation.

APPENDIX B. STATUS OF THE RESULTS AND REPRODUCIBILITY

B.1 Status of the results

The table below records, for each result, whether an equivalent statement was identified in the sources reviewed. «Equivalent statement identified» means that the statement, or one equivalent to it, appears in the cited source. «Standard result specialized here» means that the mathematics is standard and only its placement in this argument is ours. «No equivalent statement identified in the sources reviewed» records the outcome of a bibliographic search (OpenAlex and Semantic Scholar together with the primary sources, search closed 2 September 2026); it is a statement about that search and not a claim of priority, and a bibliographic search cannot establish one.

Result	Status	Source / comment
Non-explosion (Cor. 4.9)	equivalent statement identified	inherited
Embedding ι0 (Thm 3.7)	equivalent statement identified	ιg is a reformulation
Functional redundancy (Thm 3.8)	no equivalent identified	elementary; closes the “derived I” alternative
Independence of primitive I (Thm 3.9)	standard result specialized here	the neutrosophic principle of independent components, given a formal statement
Uniform truth-direction impossibility (Thm 3.13)	no equivalent identified	difficulty reported in prior work; impossibility statement ours
De Morgan iff same I-rule (Thm 4.3)	standard result specialized here	prior work discusses min/max rules and De Morgan
Factorization lemma (Lem. 4.6)	equivalent statement identified	strict matrix homomorphisms
Separation theorem (Thm 4.8)	no equivalent identified	prior work notes that ignoring I returns Belnap's connectives; the consequence-level statement and its derivation from the lemma are ours
Conservative extension by ∂ (Cor. 4.10)	standard result specialized here	immediate from the separation theorem
Activation trilemma (Thm 5.1)	no equivalent identified	
Sandwich (Thm 6.2)	no equivalent identified	axioms new as a package; both bounds classical
(C,I) independence (Thm 6.5)	standard result specialized here	
Fibre theorem (Thm 6.6)	no equivalent identified	elementary; accounts for the rank changes after normalization observed in prior work; magnitude recovery in subjective logic is the analogue
Conflation criterion (Thm 6.8)	no equivalent identified	
Within/between-source contradiction (Prop. 7.4)	no equivalent identified	refined syntax is prior work

Table B.1. Status of the results against the sources reviewed.

B.2 Reproducibility

Every numerical statement — Counterexamples 4.4, 6.3 and 6.9, the three witnesses of Theorem 5.1, the entries of Table 1, the values in Propositions 6.4 and 7.4, the sampling checks for Theorems 4.3, 6.6 and 6.8, and the illustration of Section 7.3 — has been checked computationally. The computation is a secondary check: every universal statement is proved analytically in the text, and no universal claim rests on computation.