AN OPTIMIZED FAMILY OF EXPONENTIAL RATIO ESTIMATOR FOR FINITE POPULATION MEAN IN SIMPLE RANDOM SAMPLING

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ABSTRACT
This paper proposes a new approach for estimating the population mean using a generalized exponential ratio class of estimators under simple random sampling without replacement, incorporating auxiliary information. The mean square error of the proposed estimators is derived under the retaining terms up to the first order, and the determining constants are chosen to minimize the resulting expression. The efficiency of the proposed estimators is assessed through analytical comparisons with existing estimators. The performance of the proposed estimators is further evaluated using three real data sets, graphical illustrations, and Monte Carlo simulation experiments conducted on both symmetric and asymmetric populations. The empirical and simulation results demonstrate that the proposed estimators outperform the usual unbiased estimator, ratio estimator, and several existing estimators in terms of efficiency.

KEYWORDS: Study variable; Auxiliary variable; Mean square error(MSE); Percentage relative efficiency(PRE); Monte Carlo simulation

MSC: 62D05.

RESUMEN
En este trabajo se propone un nuevo enfoque para estimar la media poblacional usnado estimadores de tipo razoón exponential generalizado sin remplazo e incorporando información auxiliar. El error cuadrático medio del estimador propuesto se deriva de los términos retenidos del primer orden y sus constants se escogen de forma que se minimice la expresión resultante. Mediante comparaciones analíticas con estimadores existentes, se obtiene la eficiencia de los estimadores propuestos y se evalua su desempeño en tres conjuntos de datos reales mediante ilustraciones graficas y simulacin̆ de Monte Carlo en poblaciones simétricas y asimetricas. Los resultados em[íricos y simulados muestran un mejor desempeño del estimador propuesto con respecto al estimador insesgado usual, el estimador de razón y otros estiadores existentas en cuanto a su eficiencia .

PALABRAS CLAVE: Variable de estudio; Variable auxiliar; error cuadrático medio (MSE); Porcentaje de eficiencia relativa (PRE); Simulación de Monte Carlo

1. INTRODUCTION

Simple random sampling without replacement (SRSWOR) entails selecting a specified number of units from a data set without replacement; each unit in the data set has an equal possibility of selection, and once selected, a unit cannot be chosen again. In recent years, classic methods such as ratios, regressions and exponentials

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have been commonly used to estimate population means due to their simplicity and ease of computation. Researchers have improved standard ratio and exponential estimators by incorporating auxiliary variables such as coefficients of variation, skewness, and kurtosis as well as the population mean, to estimate the unknown population mean of study variables. Ratio, regression and exponential estimation approaches require prior knowledge of the auxiliary variable's population characteristics. Watson [28] introduced the conventional regression estimator of population mean as the best linear unbiased (BLU) estimator for study variables, taking into account auxiliary information. Cochran [7] proposed a conventional ratio estimate for population mean under SRS, Prasad [19] gives some enhanced ratio-type population mean and ratio estimators for surveys with finite population samples, Bahl and Tuteja [5] proposed an exponential ratio and product estimate for population mean using SRS, Upadhyaya and Singh [27] provides an estimator of the finite population mean based on a transformed auxiliary parameter, Singh and Tailor [24] offered a mean estimate for a finite population with an auxiliary character's known coefficient of variation, Kadilar and Cingi [12] offered a estimator using correlation coefficient for estimating population mean, Koyuncu and Kadilar [14] gives an efficient estimators for estimating the population mean, Yan and Tian [31] provides a ratio estimator incorporating the skewness coefficient of an auxiliary variable, Subramani and Kumarapandian [25] introduce a decile-based estimator for an auxiliary variable, Swain [26] provides a better ratio-type estimate for sample survey for the population estimation, Kadilar [13] introduce a power, ratio and exponential type estimator, Lawson [17] offered a new ratio estimators using a coefficient of variation, Raja et al. [22] enhancing the mean ratio estimator for estimating population mean using conventional parameter, Gupta and Yadav [9] offered an enhanced population mean calculation based on sample size information, Ijaz and Ali [10] offered some improved ratio estimators, Baghel and Yadav [4] gives restructered group of estimators for the average of the population, Dansawad [8] offered a estimator of exponential, ratio-cum-product type for the population mean in SRS using the information of an auxiliary variable, Zakari et al. [32] offered an estimator that linearly integrates the product, exponential and ratio type estimators, Zaman [33] provided a broad exponential estimator for the finite population mean. Prasad [20] proposed some linear regression type ratio exponential estimators based on quartile deviation and deciles. Abiodun et al. [1] offered an efficient exponential type estimator, Adichwal et al. [3] provides an approximation of overall parameters leveraging auxiliary data, Bhusan et al. [6] gives predictive estimators of logarithmic type, Adejumobi et al. [2] put forward a group of exponential ratio type estimators utilizing deciles of an auxiliary parameter, Iqbal et al. [11] examines the use of generalized mixture estimators in the financial and health sectors using basic random sampling, Kumar and Siddiqui [15] offered an enhanced estimation of population mean, Kumar et al. [16] provided mean estimate based on basic random sampling with an effective class of estimators, Yadav et al. [29] offered an effective family of estimators of ratio type and Yadav and Prasad [30] who used the auxiliary data of deciles and quartile deviance to propose some new effective linear regression ratio type estimators.
In this paper, we propose an improved class of estimators for estimating the population mean of a study variable using auxiliary information under simple random sampling. The objectives of this paper are:

1. To propose an efficient class of estimators for population mean estimation that competes with existing estimators, particularly the recent estimator of Adejumobi et al. [2].

2. To compare the theoretical efficiency of proposed estimators with the existing ones,

3. To demonstrate the performance of the proposed estimators using real-world data sets and to provide graphical representations in support of the theoretical results.

4. To conduct a simulation study using artificially generated symmetric and asymmetric populations.

This paper's main goal is to efficiently include auxiliary information to offer an improved class of estimators for the finite population mean under simple random sampling without replacement (SRSWOR). Despite being frequently employed, current ratio and exponential-type estimators have limited flexibility and lower efficiency. These restrictions serve as motivation for the development of a generalized exponential ratio estimator with ideally selected constants. The mean square error (MSE) of the suggested estimator is derived under a linear (first-order) approximation. The suggested estimator outperforms a number of current estimators, including more modern exponential-type estimators, according to theoretical efficiency constraints backed by numerical comparisons, graphical representations, real data applications, and simulation tests.
The following parts provide further design for the paper. Section 2 provides an explanation of the notations and methodologies used. Section 3 discusses important estimators, including their MSE/minimum MSE expressions. In Section 4, we propose a generalized exponential ratio class of estimators for estimating the population mean using SRS icluding MSE and minimum MSE expressions. Section 5 details the efficiency conditions. Section 6 provides numerical study of the effectiveness of suggested estimators based on three real data sets. Section 7 discusses the results of the simulation study with symmetric and asymmetric populations, specifically the Normal and Weibull distributions. Section 8 summarizes the complete results of the study. The paper is concluded in Section 9.

2. NOTATIONS AND METHODOLOGY

Think about employing the simple random sampling without replacement (SRSWOR) strategy to select a sample of size n from a population, known as $V = (V_1, V_2, ..., V_N)$, where N is the population size. x and y indicate the study and auxiliary variable, respectively. Let Y_i and X_i be the values of the study variable y and the auxiliary variable x, respectively, linked to the i-th (i = 1, 2, …, N) population unit V. The observed values for the chosen sample units are indicated by y_i and x_i, respectively. $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ are sample and the population means of the primary variable y. $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ represents the population and sample means of variable x. β_2 is the population kurtosis coefficient of variable x. β_1 is the skewness coefficient of x. ρ is the correlation coefficient between variables x and y. $s_y = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2}$ and $S_y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$ are the sample and population standard deviation of variable y. $s_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$ and $S_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$ are the sample and population standard deviation of variable x. S_y^2 and S_x^2 are the population variance of variables (y, x). The coefficient of variation of variables y and x are represented by $C_y = \frac{S_y}{\bar{Y}}$ and $C_x = \frac{S_x}{\bar{X}}$. The existing and suggested estimators' mean square error (MSE) can be calculated using the following notation: $1 + e_0 = \frac{\bar{y}}{\bar{Y}}$, $1 + e_1 = \frac{\bar{x}}{\bar{X}}$ such that the expected values for these errors are, $E(e_0) = E(e_1) = 0$ and $E(e_0^2) = f_1 C_y^2$, $E(e_1^2) = f_1 C_x^2$, $E(e_0 e_1) = f_1 \rho C_y C_x$, where $f = \frac{n}{N}$ is the sampling fraction, $f_1 = \left(\frac{1}{n} - \frac{1}{N}\right)$ is the finite population correction factor. $\theta = \frac{a\bar{X}}{a\bar{X}+b}$ where a and b are real constants, and $R = \frac{\bar{X}}{\bar{Y}}$, where $\bar{X}$, $\bar{Y}$ are the population means of the auxiliary and study variables, respectively.

3. REVIEW OF SOME CONVENTIONAL ESTIMATORS

The Usual unbiased estimator of the population mean is represented as

$$\bar{Y}_0 = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1}$$

The variance of $\bar{Y}_0$ is formulated as

$$\mathrm{V}(\bar{Y}_0) = f_1 \bar{Y}^2 C_y^2 \tag{2}$$

Cochran's [7] offered a usual ratio estimator for population mean $\bar{Y}$, and is defined as:

$$\bar{Y}_R = \bar{y}\frac{\bar{X}}{\bar{x}} \tag{3}$$

The mean square error (MSE) of $\bar{Y}_R$ is given as

$$\text{MSE}(\bar{Y}_R) = f_1\bar{Y}^2\left[C_y^2 + C_x^2 - 2\rho C_y C_x\right] \tag{4}$$

Bahl and Tuteja [5] introduce an exponential-type ratio and product estimation approach for estimating $\bar{Y}$, which are expressed as follows:

$$\bar{Y}_{BT1} = \bar{y}\exp\left\{\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right\} \tag{5}$$

$$\bar{Y}_{BT2} = \bar{y}\exp\left\{\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right\} \tag{6}$$

The MSE of $\bar{Y}_{BT1}$ and $\bar{Y}_{BT2}$ are represented as

$$\text{MSE}(\bar{Y}_{BT1}) = f_1\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4} - \rho C_y C_x\right] \tag{7}$$

$$\text{MSE}(\bar{Y}_{BT2}) = f_1\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4} + \rho C_y C_x\right] \tag{8}$$

Yan and Tian [31] proposed a ratio-type estimator for the population mean using the coefficient of skewness of the auxiliary variable x as

$$\bar{Y}_{YT} = \frac{[\bar{y}+b(\bar{X}-\bar{x})]}{(\bar{x}\beta_1+\beta_2)}(\bar{X}\beta_1+\beta_2) \tag{9}$$

The mean square error (MSE) of $\bar{Y}_{YT}$ is provided as follows:

$$\text{MSE}(\bar{Y}_{YT}) = f_1[R_1^2 S_y^2 + S_y^2(1-\rho^2)] \tag{10}$$

where, $R_1 = \frac{\bar{Y}\beta_1}{\bar{X}\beta_1+\beta_2}$.

Subramani and Kumarapandiyan [25] proposed an estimation approach for estimating $\bar{Y}$ utilizing deciles of an auxiliary variable as follows:

$$\bar{Y}_{SKi} = \bar{y}\left[\frac{\bar{X}+D_i}{\bar{x}+D_i}\right], i = 1, 2, ..., 10 \tag{11}$$

The mean square error (MSE) of $\bar{Y}_{SKi}$ is provided as follows:

$$\text{MSE}(\bar{Y}_{SKi}) = f_1\bar{Y}^2[C_y^2 + \phi_i^2 C_x^2 - 2\phi_i\rho C_y C_x] \tag{12}$$

where, $\phi_i = \frac{\bar{X}}{\bar{X}+D_i}$, i=1,2,...,10.

Kadilar [13] provides a novel exponential estimation approach for estimating $\bar{Y}$ of the population, which is as follows:

$$\bar{Y}_K = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \tag{13}$$

The mean square error (MSE) of $\bar{Y}_K$ is given as follows:

$$MSE(\bar{Y}_K) = f_1\bar{Y}^2 C_y^2[1-\rho^2] \tag{14}$$

Zakari et al. [32] proposed an alternative estimator based on a linear combination of ratio, product, and exponential-type estimators, given as follows:

$$\bar{Y}_{GR} = \bar{y}\left(k\frac{\bar{X}}{\bar{x}} + (1-k)\frac{\bar{x}}{\bar{X}}\right)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{15}$$

Where k is a suitably chosen constant. The mean square error (MSE) of $\bar{Y}_{GR}$ is given by

$$\text{MSE}((\bar{Y}_{GR}) = f_1\bar{Y}^2C_y^2[1-\rho^2] \tag{16}$$

Dansawad [8] suggested an exponential-type ratio-cum-product estimator for estimating population means under SRSWOR, based on the following auxiliary information:

$$\bar{Y}_{ND} = \bar{y}\left[\alpha\exp\left\{\frac{2(\bar{X}-\bar{x})}{\bar{X}+\bar{x}}\right\} + (1-\alpha)\exp\left\{\frac{2(\bar{x}-\bar{X})}{\bar{x}+\bar{X}}\right\}\right] \tag{17}$$

The MSE of $\bar{Y}_{ND}$ is provided as

$$\text{MSE}((\bar{Y}_{ND}) = f_1\bar{Y}^2C_y^2[1-\rho^2] \tag{18}$$

Abiodun et al. [1] proposed an efficient exponential-type estimator for estimating the finite population mean, given by

$$\bar{Y}_{MA} = \frac{\bar{y}}{2}\left[\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha} + \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right] \tag{19}$$

The MSE of $\bar{Y}_{MA}$ is provided as

$$MSE(\bar{Y}_{MA}) = f_1\bar{Y}^2C_y^2[1-\rho^2] \tag{20}$$

Adejumobi et al. [2] proposed an exponential-based ratio estimation approach that employs auxiliary-variable deciles for estimating the $\bar{Y}$, given as follows:

$$\bar{Y}_{AM1} = \frac{[\bar{y}_h + u_1(\bar{X}-\bar{x}) + v_1\bar{y}]}{(\bar{x}+D_1)}(\bar{X}+D_1)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{21}$$

$$\bar{Y}_{AM2} = \frac{[\bar{y}_h + u_2(\bar{X}-\bar{x}) + v_2\bar{y}]}{(\bar{x}+D_2)}(\bar{X}+D_2)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{22}$$

$$\bar{Y}_{AM3} = \frac{[\bar{y}_h + u_3(\bar{X}-\bar{x}) + v_3\bar{y}]}{(\bar{x}+D_3)}(\bar{X}+D_3)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{23}$$

$$\bar{Y}_{AM4} = \frac{[\bar{y}_h + u_4(\bar{X}-\bar{x}) + v_4\bar{y}]}{(\bar{x}+D_4)}(\bar{X}+D_4)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{24}$$

$$\bar{Y}_{AM5} = \frac{[\bar{y}_h + u_5(\bar{X}-\bar{x}) + v_5\bar{y}]}{(\bar{x}+D_5)}(\bar{X}+D_5)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{25}$$

$$\bar{Y}_{AM6} = \frac{[\bar{y}_h + u_6(\bar{X}-\bar{x}) + v_6\bar{y}]}{(\bar{x}+D_6)}(\bar{X}+D_6)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{26}$$

$$\bar{Y}_{AM7} = \frac{[\bar{y}_h + u_7(\bar{X}-\bar{x}) + v_7\bar{y}]}{(\bar{x}+D_7)}(\bar{X}+D_7)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{27}$$

$$\bar{Y}_{AM8} = \frac{[\bar{y}_h + u_8(\bar{X}-\bar{x}) + v_8\bar{y}]}{(\bar{x}+D_8)}(\bar{X}+D_8)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{28}$$

$$\bar{Y}_{AM9} = \frac{[\bar{y}_h + u_9(\bar{X}-\bar{x}) + v_9\bar{y}]}{(\bar{x}+D_9)}(\bar{X}+D_9)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{29}$$

$$\bar{Y}_{AM10} = \frac{[\bar{y}_h + u_{10}(\bar{X}-\bar{x}) + v_{10}\bar{y}]}{(\bar{x}+D_{10})}(\bar{X}+D_{10})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{30}$$

where,

$$\bar{y}_h = \frac{\bar{y}}{2}\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right\} \tag{31}$$

More generally, the estimators discussed above are of the form:

$$\bar{Y}_{AMi} = \frac{[\bar{y}_h + u_i(\bar{X}-\bar{x}) + v_i\bar{y}]}{(\bar{x}+D_i)}(\bar{X}+D_i)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), i = 1, 2, ..., 10 \tag{32}$$

where, u_i and v_i are the real scalars. The MSE of $\bar{Y}_{AMi}$ is provided as

$$\text{MSE}(\bar{Y}_{AMi}) = \bar{Y}^2 \left[A_i + \frac{(C_i D_i^2 + B_i E_i^2 - 2D_i E_i F_i)}{F_i^2 - B_i C_i} \right] \tag{33}$$

where,
$A_i = f_1[C_y^2 + (\Delta_i + \frac{1}{2})^2 C_x^2 - 2(\Delta_i + \frac{1}{2})\rho C_y C_x]$,
$B_i = f_1(\frac{\bar{X}}{\bar{Y}})^2 C_x^2$,
$C_i = 1 + f_1[C_y^2 + (3\Delta_i^2 + 2\Delta_i + 1)C_x^2 - 4(\Delta_i + \frac{1}{2})\rho C_y C_x]$,
$D_i = f_1(\frac{\bar{X}}{\bar{Y}})[\rho C_y C_x - (\Delta_i + \frac{1}{2})C_x^2]$,
$E_i = f_1[C_y^2 - 3(\Delta_i + \frac{1}{2})\rho C_y C_x + (2\Delta_i^2 + \frac{3}{2}\Delta_i + \frac{3}{4})C_x^2]$,
$F_i = f_1(\frac{\bar{X}}{\bar{Y}})[\rho C_y C_x - 2(\Delta_i + \frac{1}{2})C_x^2]$,
$\Delta_i = \frac{\bar{X}}{\bar{X} + D_i}$.
Based on the review of existing estimators, this study aims to improve estimation efficiency and therefore proposes a generalized exponential ratio class of estimation approach for population mean estimation.

4. SUGGESTED ESTIMATORS

Motivated by existing estimators, this study develops an efficient estimators for estimating the population mean of the study variable by incorporating auxiliary information. Accordingly, an expanded class of efficient exponential ratio estimators is therefore proposed for finite population mean estimation, as defined below.

$$\bar{Y}_{DS} = (\bar{y}_h + \alpha(\bar{X} - \bar{x}) + \beta\bar{y}) \left(\frac{a\bar{X} + b}{a\bar{x} + b} \right)^{\gamma} \exp\left(\frac{a(\bar{X} - \bar{x})}{a(\bar{X} + \bar{x}) + 2b} \right) \tag{34}$$

where, $\bar{y}_h = \frac{\bar{y}}{2}\left[\exp\left(\frac{a(\bar{X}-\bar{x})}{a(\bar{X}+\bar{x})+2b}\right) + \exp\left(\frac{a(\bar{x}-\bar{X})}{a(\bar{X}+\bar{x})+2b}\right)\right]$
The scalar γ is assigned fixed numerical values to study its effect on the efficiency of the estimator. The scalars α and β are obtained through minimization of the MSE of $\bar{Y}_{DS}$, while the pair (a, b) represents known auxiliary parameters whose values define different estimators within the proposed class.
Different instances of the proposed estimator related to various choices of the parameters a and b are presented in Table 6..
Expanding the equation (34) using the notations given in Section 2, and considering the error terms under the first-order approximation, we derive,

$$\begin{aligned} \bar{Y}_{DS} = &\left[\frac{(1+e_0)\bar{Y}}{2} \left\{ \exp\left(\frac{a(\bar{X} - (1+e_1)\bar{X})}{a(\bar{X} + (1+e_1)\bar{X}) + 2b} \right) + \exp\left(\frac{a((1+e_1)\bar{X} - \bar{X})}{a(\bar{X} + (1+e_1)\bar{X}) + 2b} \right) \right\} \right. \\ &\left. + \alpha(\bar{X} - (1+e_1)\bar{X}) + \beta(1+e_0)\bar{Y} \right] \left(\frac{a\bar{X} + b}{a(1+e_1)\bar{X} + b} \right)^{\gamma} \exp\left(\frac{a(\bar{X} - (1+e_1)\bar{X})}{a(\bar{X} + (1+e_1)\bar{X}) + 2b} \right) \end{aligned} \tag{35}$$

On subtracting $\bar{Y}$ from both sides of the equation (35), we obtain,

$$\begin{aligned} \bar{Y}_{DS} - \bar{Y} = \bar{Y}\Big[&- \theta e_1 + \frac{3}{8}\theta^2 e_1^2 - \gamma\theta e_1 + \gamma\theta^2 e_1^2 + \gamma\frac{(\gamma+1)}{2}\theta^2 e_1^2 + e_0 - \theta e_0 e_1 - \gamma\theta e_0 e_1 + \frac{\theta^2}{8} e_1^2 \\ &- \alpha R e_1 - \alpha\frac{R}{2}\theta e_1^2 + \alpha\gamma R\theta e_1^2 + \beta + \beta e_0 - \beta\frac{\theta}{2} e_1 + \frac{3}{8}\theta^2 e_1^2 \beta - \beta\gamma\theta e_1 + \beta\frac{\gamma}{2}\theta^2 e_1^2 \\ &+ \beta\gamma\frac{(\gamma+1)}{2}\theta^2 e_1^2 - \beta\frac{\theta}{2} e_0 e_1 - \beta\gamma\theta e_0 e_1 \Big] \end{aligned} \tag{36}$$

where, $\theta = \frac{a\bar{X}}{a\bar{X}+b}$ and $R = \frac{\bar{X}}{\bar{Y}}$.
On squaring both sides of equation (36), and neglecting terms of order higher than two in e, we obtain

$$
\begin{aligned}
(\bar{Y}_{DS}-\bar{Y})^2 =& \bar{Y}^2\Big[\beta^2+e_1^2\Big(\theta^2+\gamma^2\theta^2+\alpha^2R^2+\frac{\beta^2\theta^2}{4}+\beta^2\gamma^2\theta^2+2\theta^2\gamma+2\theta R\alpha+\theta^2\beta \\
&+2\theta^2\beta\gamma+2\gamma\theta R\alpha+\gamma\beta\theta^2+2\gamma^2\beta\theta^2+\alpha R\beta\theta+2\alpha R\gamma\beta\theta+\beta^2\theta^2\gamma\Big) \\
&+e_0^2\left(1+\beta^2+2\beta\right)+2\beta e_0e_1\left(-\theta-\gamma\theta-\beta\theta-\beta\gamma\theta\right)+2\beta e_1^2\Big(\frac{3}{8}\theta^2+\gamma\theta^2 \\
&+\frac{\gamma(\gamma+1)}{2}\theta^2+\frac{\theta^2}{8}-\frac{\alpha}{2}\theta R+\alpha\gamma\theta R+\frac{3}{8}\theta^2\beta+\frac{\beta}{2}\gamma\theta^2+\beta\frac{\gamma(\gamma+1)}{2}\theta^2\Big)+ \\
&2e_0e_1\Big(-\theta-\theta\beta-\gamma\theta-\gamma\theta\beta-R\alpha-R\alpha\beta-\frac{\theta}{2}\beta^2-\frac{\theta}{2}\beta-\gamma\theta\beta-\gamma\theta\beta^2\Big)\Big] \qquad (37)
\end{aligned}
$$

The mean square error (MSE) of $\bar{Y}_{DS}$ is obtained by taking the expectation of equation (37), and retaining terms up to the first-order approximation, yielding

$$
\text{MSE}\left(\bar{Y}_{DS}\right)=E\left[\bar{Y}_{DS}-\bar{Y}\right]^2=f_1\bar{Y}^2\left[\alpha A_1+\beta A_2+\alpha^2A_3+\beta^2A_4+\alpha\beta A_5+A_6\right]. \qquad (38)
$$

where,
$A_1=2\theta RC_x^2+2\gamma\theta RC_x^2-2RC_xC_y\rho$,
$A_2=\theta^2C_x^2+6\gamma\theta^2C_x^2+3\gamma^2\theta^2C_x^2+2C_y^2-5C_xC_y\rho\theta-6C_xC_y\rho\theta\gamma+C_x^2\theta^2$,
$A_3=R^2C_x^2$,
$A_4=1+C_x^2\frac{\theta^2}{4}+2C_x^2\gamma^2\theta^2+3C_x^2\gamma\theta^2+C_y^2-3C_xC_y\rho\theta-4C_xC_y\rho\gamma\theta+\frac{3}{4}\theta^2C_x^2$,
$A_5=4C_x^2R\gamma\theta-2RC_xC_y\rho$,
$A_6=\theta^2C_x^2+\gamma^2\theta^2C_x^2+2\theta^2C_x^2\gamma+C_y^2-2C_xC_y\rho\theta-2C_xC_y\rho\gamma\theta$.

The optimum values of α and β are obtained by taking partial derivatives of equation (38) with respect to α and β and setting them equal to zero, yielding:
$\frac{\partial MSE(\bar{Y}_{DS})}{\partial\alpha}=0$ and $\frac{\partial MSE(\bar{Y}_{DS})}{\partial\beta}=0$

$$
\alpha_{opt}=\frac{2A_1A_4-A_2A_5}{A_5^2-4A_3A_4},\beta_{opt}=\frac{2A_2A_3-A_1A_5}{A_5^2-4A_3A_4}. \qquad (39)
$$

The minimum MSE of the proposed estimator $\bar{Y}_{DS}$ is obtained by substituting the optimal values of α and β from equation (39) into equation (38), and is given as follows:

$$
MSE_{min}(\bar{Y}_{DS})=\bar{Y}^2f_1\left[A_6+\frac{A_1^2A_4+A_2^2A_3-A_1A_2A_5}{A_5^2-4A_3A_4}\right]. \qquad (40)
$$

5. EFFICIENCY COMPARISON

The efficiency requirements listed below were defined by comparing the MSE_{min} of the suggested estimator $\bar{Y}_{DS}$ is with the variance/MSE of the current estimators.
The proposed estimator $\bar{Y}_{DS}$ exhibits a lower MSE compared to the estimators given in Section 3.

(i) The Usual unbiased mean estimator $\bar{Y}_0$ if $MSE_{min}(\bar{Y}_{DS})<Var(\bar{Y}_0)$,
i.e. if $\left[A_6+\frac{A_1^2A_4+A_2^2A_3-A_1A_2A_5}{A_5^2-4A_3A_4}\right]<C_y^2$.

(ii) Cochran [7] estimator $\bar{Y}_R$ if $MSE_{min}(\bar{Y}_{DS})<$ MSE($\bar{Y}_R$),
i.e. if $\left[A_6+\frac{A_1^2A_4+A_2^2A_3-A_1A_2A_5}{A_5^2-4A_3A_4}\right]<\left[C_y^2+C_x^2-2\rho C_yC_x\right]$

(iii) Bahl and Tuteja [5] estimators $\bar{Y}_{BT1}$ and $\bar{Y}_{BT2}$ if $MSE_{min}(\bar{Y}_{DS})<$ MSE($\bar{Y}_{BT1}$) and $MSE_{min}(\bar{Y}_{DS})<$ MSE($\bar{Y}_{BT2}$)
i.e. if

(a) $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < \left[C_y^2 + \frac{C_x^2}{4} - \rho C_x C_y\right]$

(b) $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < \left[C_y^2 + \frac{C_x^2}{4} + \rho C_x C_y\right]$

(iv) Yan and Tian [31] estimator if $MSE_{min}(\bar{Y}_{DS})$ < MSE($\bar{Y}_{YT}$),
i.e. if $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < [R_1 C_y^2 + C_y^2(1-\rho^2)]$

(v) Subramani and Kumarapandiyan [25] estimator if $MSE_{min}(\bar{Y}_{DS})$ < MSE($\bar{Y}_{SKi}$),
i.e. if $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < [C_y^2 + \phi_i^2 C_x^2 - 2\phi_i \rho C_x C_y]$

(vi) Kadilar [13] estimator Y_K if $MSE_{min}(\bar{Y}_{DS})$ < MSE($\bar{Y}_K$),
i.e. if $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < C_y^2[1-\rho^2]$

(vii) Zakari et al. [32] estimator $\bar{Y}_{GR}$ if $MSE_{min}(\bar{Y}_{DS})$ < MSE($\bar{Y}_{GR}$),
i.e. if $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < C_y^2[1-\rho^2]$

(viii) Dansawad [8] Estimator $\bar{Y}_{ND}$ if $MSE_{min}(\bar{Y}_{DS})$ < MSE($\bar{Y}_{ND}$),
i.e. if $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < C_y^2[1-\rho^2]$

(ix) Abiodun et al. [1] Estimator $\bar{Y}_{MA}$ if $MSE_{min}(\bar{Y}_{DS})$ < MSE($\bar{Y}_{MA}$),
i.e. if $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < C_y^2[1-\rho^2]$

(x) Adejumobi et al. [2] estimator $\bar{Y}_{AMi}$ if $MSE_{min}(\bar{Y}_{DS}) < MSE(\bar{Y}_{AMi})$,
i.e. if $\left[A_6 + \frac{A_1^2 A_4 + A_2^2 A_3 - A_1 A_2 A_5}{A_5^2 - 4A_3 A_4}\right] < \left[A_i + \frac{(C_i D_i^2 + B_i E_i^2 - 2D_i E_i F_i)}{F_i^2 - B_i C_i}\right]$

The efficiency conditions derived above indicate that the proposed estimator outperforms the existing estimators. Numerical illustrations, simulation studies, and graphical representations further demonstrate that the inclusion of adjustable constants and parameters enhances flexibility and reduces the mean square error across a wide range of population settings.

6. NUMERICAL ILLUSTRATION

In this section, we have conducted an empirical assessment on three actual data sets in order to strengthen the theoretical foundation. Table 2 displays the descriptive data for these populations. The PRE of several estimators of $\bar{Y}$ in comparison to the standard unbiased estimator $\bar{Y}_0$ is computed using the following expression:

$$PRE(Y^*, \bar{Y}_0) = \frac{Var(\bar{Y}_0)}{MSE(Y^*)} \times 100$$

where, $Y^* = \bar{Y}_0$, $\bar{Y}_R$, $\bar{Y}_{BT1}$, $\bar{Y}_{BT2}$, $\bar{Y}_{YT}$, $\bar{Y}_{SKi}$, $\bar{Y}_{GR}$, $\bar{Y}_{ND}$, $\bar{Y}_{MA}$, $\bar{Y}_{AMi}$ (i=1,2,...,10) and $\bar{Y}_{DSi}$ (i=1,2,...,10) while the PREs for existing and proposed estimators $\bar{Y}_{DS}$ for ten combination of (a,b) and for $\gamma = 0, 1$ are shown in Table 3. To show the efficiency of the proposed class of estimators, we computed the PREs of the proposed and considered estimators using the three natural data sets.
Two sets of figures are provided for better clarity. For the three real data sets, Figures 1, 2, and 3 offer a thorough comparison of all suggested and current estimators, while Figures 4, 5, and 6 show the suggested estimator with the highest PRE value when compared to current estimators. Confidence intervals are not provided because the analysis is based on fixed real data sets.

Estimator	a	b
$\bar{Y}_{DS1} = (\bar{y}_{h1} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\bar{X}}{\bar{x}}\right)^{\gamma}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$ where, $\bar{y}_{h1} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right)\right]$	1	0
$\bar{Y}_{DS2} = (\bar{y}_{h2} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\bar{X}+\beta_2(x)}{\bar{x}+\beta_2(x)}\right)^{\gamma}\exp\left(\frac{(\bar{X}-\bar{x})}{(\bar{x}+\bar{X})+2\beta_2(x)}\right)$ where, $\bar{y}_{h2} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\bar{X}-\bar{x}}{(\bar{X}+\bar{x})+2\beta_2(x)}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{(\bar{X}+\bar{x})+2\beta_2(x)}\right)\right]$	1	$\beta_2(x)$
$\bar{Y}_{DS3} = (\bar{y}_{h3} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)^{\gamma}\exp\left(\frac{(\bar{X}-\bar{x})}{(\bar{X}+\bar{x})+2C_x}\right)$ $\bar{y}_{h3} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\bar{X}-\bar{x}}{(\bar{X}+\bar{x})+2C_x}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{(\bar{X}+\bar{x})+2C_x}\right)\right]$	1	C_x
$\bar{Y}_{DS4} = (\bar{y}_{h4} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{C_x\bar{X}+\beta_2(x)}{C_x\bar{x}+\beta_2(x)}\right)^{\gamma}\exp\left(\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right)$ $\bar{y}_{h4} = \frac{\bar{y}}{2}\left[\exp\left(\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right)+\exp\left(\frac{C_x(\bar{x}-\bar{X})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right)\right]$	C_x	$\beta_2(x)$
$\bar{Y}_{DS5} = (\bar{y}_{h5} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\beta_2(x)\bar{X}+C_x}{\beta_2(x)\bar{x}+C_x}\right)^{\gamma}\exp\left(\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right)$ $\bar{y}_{h5} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right)+\exp\left(\frac{\beta_2(x)(\bar{x}-\bar{X})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right)\right]$	$\beta_2(x)$	C_x
$\bar{Y}_{DS6} = (\bar{y}_{h6} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\bar{X}\bar{X}+\rho}{\bar{X}\bar{x}+\rho}\right)^{\gamma}\exp\left(\frac{\bar{X}(\bar{X}-\bar{x})}{\bar{X}(\bar{X}+\bar{x})+2\rho}\right)$ $\bar{y}_{h6} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\bar{X}(\bar{X}-\bar{x})}{\bar{X}(\bar{X}+\bar{x})+2\rho}\right)+\exp\left(\frac{\bar{X}(\bar{x}-\bar{X})}{\bar{X}(\bar{X}+\bar{x})+2\rho}\right)\right]$	$\bar{X}$	ρ
$\bar{Y}_{DS7} = (\bar{y}_{h7} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\bar{X}\bar{X}+S_x^2}{\bar{X}\bar{x}+S_x^2}\right)^{\gamma}\exp\left(\frac{\bar{X}(\bar{X}-\bar{x})}{\bar{X}(\bar{X}+\bar{x})+2S_x^2}\right)$ $\bar{y}_{h7} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\bar{X}(\bar{X}-\bar{x})}{\bar{X}(\bar{X}+\bar{x})+2S_x^2}\right)+\exp\left(\frac{\bar{X}(\bar{x}-\bar{X})}{\bar{X}(\bar{X}+\bar{x})+2S_x^2}\right)\right]$	$\bar{X}$	S_x^2
$\bar{Y}_{DS8} = (\bar{y}_{h8} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{C_x^2\bar{X}+\rho}{C_x^2\bar{x}+\rho}\right)^{\gamma}\exp\left(\frac{C_x^2(\bar{X}-\bar{x})}{C_x^2(\bar{X}+\bar{x})+2\rho}\right)$ $\bar{y}_{h8} = \frac{\bar{y}}{2}\left[\exp\left(\frac{C_x^2(\bar{X}-\bar{x})}{C_x^2(\bar{X}+\bar{x})+2\rho}\right)+\exp\left(\frac{C_x^2(\bar{x}-\bar{X})}{C_x^2(\bar{X}+\bar{x})+2\rho}\right)\right]$	C_x^2	ρ
$\bar{Y}_{DS9} = (\bar{y}_{h9} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\bar{X}\bar{X}+\beta_2(x)}{\bar{X}\bar{x}+\beta_2(x)}\right)^{\gamma}\exp\left(\frac{\bar{X}(\bar{X}-\bar{x})}{\bar{X}(\bar{X}+\bar{x})+2\beta_2(x)}\right)$ $\bar{y}_{h9} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\bar{X}(\bar{X}-\bar{x})}{\bar{X}(\bar{X}+\bar{x})+2\beta_2(x)}\right)+\exp\left(\frac{\bar{X}(\bar{x}-\bar{X})}{\bar{X}(\bar{X}+\bar{x})+2\beta_2(x)}\right)\right]$	$\bar{X}$	$\beta_2(x)$
$\bar{Y}_{DS10} = (\bar{y}_{h10} + \alpha(\bar{X}-\bar{x}) + \beta\bar{y})\left(\frac{\rho\bar{X}+\beta_2(x)}{\rho\bar{x}+\beta_2(x)}\right)^{\gamma}\exp\left(\frac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right)$ $\bar{y}_{h10} = \frac{\bar{y}}{2}\left[\exp\left(\frac{\rho(\bar{X}-\bar{x})}{\rho((\bar{X}+\bar{x})+2\beta_2(x)}\right)+\exp\left(\frac{\rho(\bar{x}-\bar{X})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right)\right]$	ρ	$\beta_2(x)$

Table 1: Member of the suggested class of estimators $\bar{Y}_{DS}$ for various values of (a, b).

7. SIMULATION STUDY

We performed a simulation research based on hypothetically created symmetric and asymmetric populations in order to generalize the theoretical conclusions and to support the findings of the numerical analysis. Monte Carlo simulation is used to produce synthetic data by specifying parameter values from symmetric and asymmetric distributions. The PREs of proposed estimators and their existing counterparts are evaluated using generated (hypothetical) data. Here, we looked at the first model, which is Y = 0.5X + rpois(N, $\lambda = 10$), and assumed that $X \sim$ N(10, 2), which produced a normal population of N=1500 units. Next, we looked at the second model, which is Y = 0.5X + rpois(N, $\lambda = 10$), and assumed that $X \sim$ Weibull(N, 31, 9), which produced a Weibull population containing N=1500 units. From both populations, we use SRSWOR to choose n=30 units.

Parameter	**Data Set-1 [18]**	**Data Set-2 [23]**	**Data Set-3 [23]**
N	40	34	34
n	20	20	20
$\bar{Y}$	5141.5363	856.4117	856.4117
$\bar{X}$	1221.6463	208.8823	199.4412
$\beta_1(x)$	0.3761	0.9782	1.1823
$\beta_2(x)$	-1.5154	0.0978	1.0445
C_y	0.0557	0.8561	0.8561
C_x	0.0839	0.7205	0.7531
ρ	0.9244	0.4491	0.4453
D_1	1111.8150	70.3000	60.6000
D_2	1119.4800	76.8000	83.0000
D_3	1139.2000	108.2000	102.7000
D_4	1159.8400	129.4000	111.2000
D_5	1184.2250	150.0000	142.5000
D_6	1252.5500	227.2000	210.2000
D_7	1307.1000	250.4000	264.5000
D_8	1345.7200	335.6000	304.4000
D_9	1366.7880	436.4000	373.2000
D_{10}	1389.3000	564.0000	634.0000

Table 2: Descriptive Metrics of Three Data Sets

Following 10,000 repetitions, the PREs were calculated using the following formula

$$PRE(Y^*, \bar{Y}_0) = \frac{Var(\bar{Y}_0)}{MSE(Y^*)} \times 100$$

where, Y^*=$\bar{Y}_0$, $\bar{Y}_R$, $\bar{Y}_{BT1}$, $\bar{Y}_{BT2}$, $\bar{Y}_{YT}$, $\bar{Y}_{SKi}$,$\bar{Y}_{GR}$, $\bar{Y}_{ND}$, $\bar{Y}_{MA}$, $\bar{Y}_{AMi}$ since(i=1,2,...,10) and $\bar{Y}_{DSi}$ (i=1,2,..,10) for $\gamma = 0, 1$ which demonstrates the existing and the suggested estimators. Table 4 displays the findings of PREs for both the simulated normal and Weibull populations. It demonstrates that the suggested estimators have the maximum PRE value in comparison to the current ones, This shows that the recommended estimators outperform the current ones in terms of PRE values under both the normal and Weibull distributions. To address variability across parameter settings, the simulation study uses both symmetric (normal) and asymmetric (Weibull) populations, which reflect the auxiliary variable's various distributional features. The efficiency of the estimators is investigated using PRE, which provide a full evaluation of estimation accuracy. The stability of the results across these several population scenarios suggests that the proposed estimators remain more efficient across different data-generation techniques.

Table 3: PREs of the proposed estimators $\bar{Y}_{DSi}$, (i=1,2,...,10) and existing estimators with respect to $\bar{Y}_0$ for real data sets at $\gamma = 0$, 1.

Estimators	$\gamma = 1$			$\gamma = 0$		
Data Sets→	Data Set - 1	Data Set - 2	Data Set - 3	Data Set - 1	Data Set - 2	Data Set - 3
$\mathbf{\bar{Y}_{DS1}}$	**720.047**	**878.816**	**1751.652**	**688.925**	**344.801**	**402.210**
$\mathbf{\bar{Y}_{DS2}}$	**720.322**	**874.599**	**1564.635**	**688.955**	**344.336**	**394.333**
$\mathbf{\bar{Y}_{DS3}}$	**720.032**	**848.821**	**1612.343**	**688.924**	**341.428**	**396.483**
$\mathbf{\bar{Y}_{DS4}}$	**723.494**	**872.976**	**1512.340**	**689.305**	**344.157**	**391.851**
$\mathbf{\bar{Y}_{DS5}}$	**720.057**	**654.889**	**1617.790**	**688.926**	**314.906**	**396.721**
$\mathbf{\bar{Y}_{DS6}}$	**720.047**	**878.723**	**1751.200**	**688.925**	**344.791**	**402.192**
$\mathbf{\bar{Y}_{DS7}}$	**718.532**	**242.666**	**246.048**	**688.766**	**205.192**	**206.534**
$\mathbf{\bar{Y}_{DS8}}$	**703.402**	**843.087**	**1606.943**	**687.534**	**340.764**	**396.245**
$\mathbf{\bar{Y}_{DS9}}$	**720.047**	**878.796**	**1750.593**	**688.926**	**344.799**	**402.169**
$\mathbf{\bar{Y}_{DS10}}$	**720.344**	**869.489**	**1384.274**	**688.957**	**343.769**	**385.151**
$\bar{Y}_0$	100.000	100.000	100.000	100.000	100.000	100.000
$\bar{Y}_R$	206.580	105.000	100.970	206.580	105.000	100.970
$\bar{Y}_{BT1}$	572.037	125.139	124.729	572.037	125.139	124.729
$\bar{Y}_{BT2}$	33.788	64.310	63.084	33.788	64.310	63.084
$\bar{Y}_{YT}$	5.563	5.684	5.241	5.563	5.684	5.241
$\bar{Y}_{SK1}$	610.021	119.102	117.052	610.021	119.102	117.051
$\bar{Y}_{SK2}$	607.398	121.688	120.099	607.398	121.688	120.099
$\bar{Y}_{SK3}$	600.611	123.219	121.946	600.611	123.219	121.946
$\bar{Y}_{SK4}$	593.478	123.771	122.551	593.468	123.771	122.551
$\bar{Y}_{SK5}$	585.005	124.855	124.023	585.005	124.855	124.023
$\bar{Y}_{SK6}$	561.405	125.127	124.689	561.405	125.127	124.689
$\bar{Y}_{SK7}$	542.949	124.324	124.036	542.949	124.324	124.036
$\bar{Y}_{SK8}$	530.208	122.444	123.284	530.207	122.444	123.284
$\bar{Y}_{SK9}$	523.389	121.959	121.802	523.389	121.959	121.802
$\bar{Y}_{SK10}$	516.217	116.799	116.709	516.216	116.799	116.708
$\bar{Y}_K$	687.358	125.265	124.733	687.357	125.264	124.733
$\bar{Y}_{GR}$	687.357	125.264	124.734	687.358	125.265	124.734
$\bar{Y}_{ND}$	687.358	125.265	124.734	687.358	125.265	124.734
$\bar{Y}_{MA}$	687.358	125.265	124.734	687.358	125.265	124.734
$\bar{Y}_{AM1}$	687.363	126.452	125.905	687.363	126.452	125.905
$\bar{Y}_{AM2}$	687.363	126.522	125.982	687.363	126.522	125.982
$\bar{Y}_{AM3}$	687.364	126.577	126.043	687.364	126.577	126.043
$\bar{Y}_{AM4}$	687.364	126.599	126.067	687.364	126.599	126.067
$\bar{Y}_{AM5}$	687.365	126.672	126.147	687.365	126.672	126.147
$\bar{Y}_{AM6}$	687.366	126.796	126.284	687.366	126.792	126.284
$\bar{Y}_{AM7}$	687.368	126.873	126.368	687.368	126.873	126.368
$\bar{Y}_{AM8}$	687.369	126.919	126.418	687.369	126.919	126.419
$\bar{Y}_{AM9}$	687.369	126.987	126.492	687.369	126.987	126.492
$\bar{Y}_{AM10}$	687.369	127.147	126.665	687.369	127.147	126.665

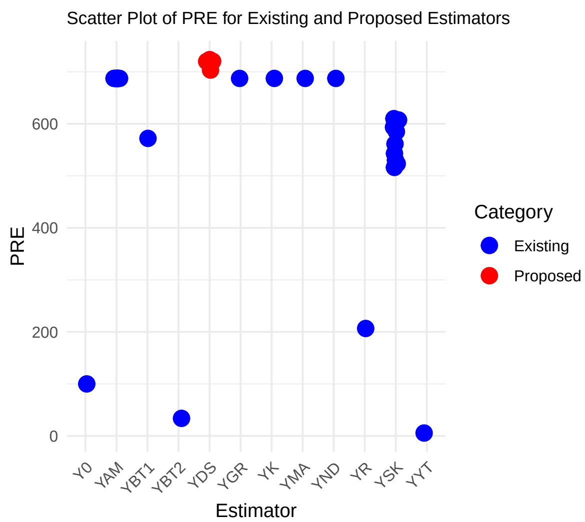

Figure 1: Comparison of PRE values between the existing estimators and the proposed estimators $\bar{Y}_{DSi}$, (i=1,2,...,10) for Data Set–1 under SRSWOR when $\gamma = 1$.

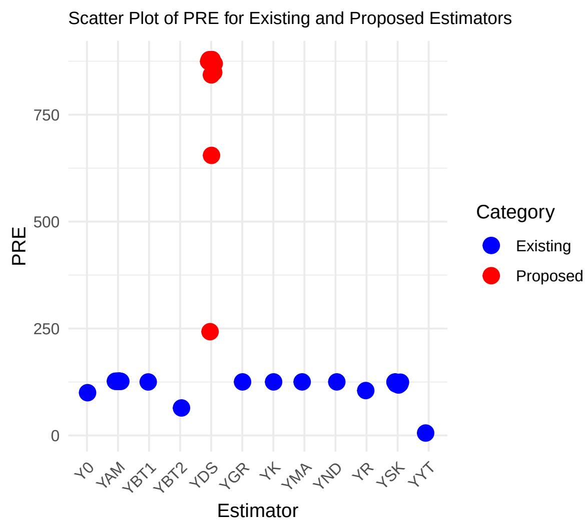

Figure 2: Comparison of PRE values between the existing estimators and the proposed estimators $\bar{Y}_{DSi}$, (i=1,2,...,10) for Data Set–2 under SRSWOR when $\gamma = 1$.

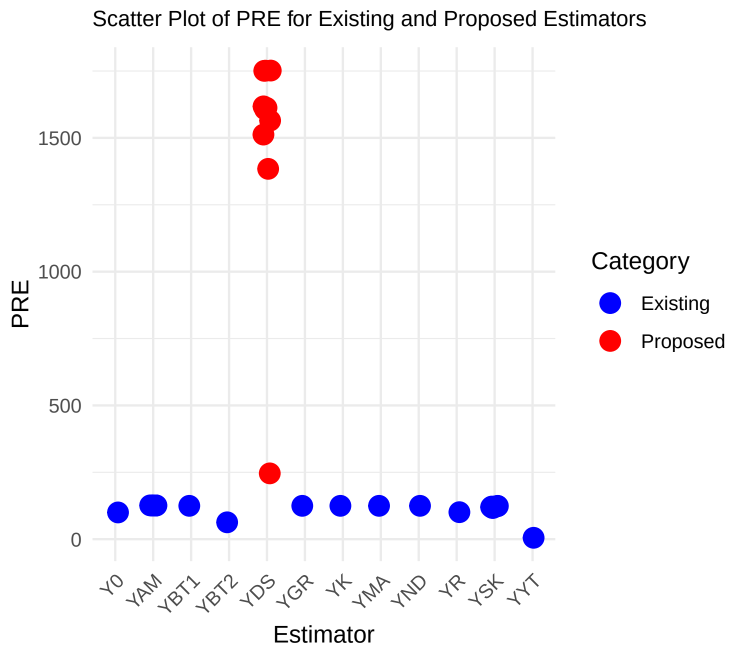

Figure 3: Comparison of PRE values between the existing estimators and the proposed estimators $\bar{Y}_{DSi}$, (i=1,2,...,10) for Data Set–3 under SRSWOR when $\gamma = 1$.

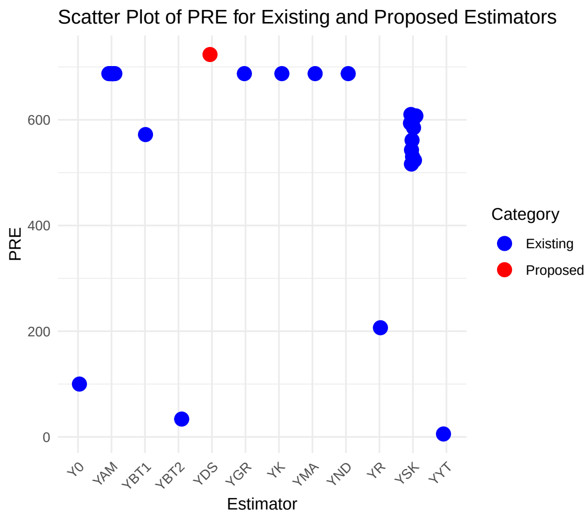

Figure 4: Comparison of PRE values between the existing estimators and the proposed estimator $\bar{Y}_{DS4}$ (highest PRE) for Data Set–1 under SRSWOR when $\gamma = 1$.

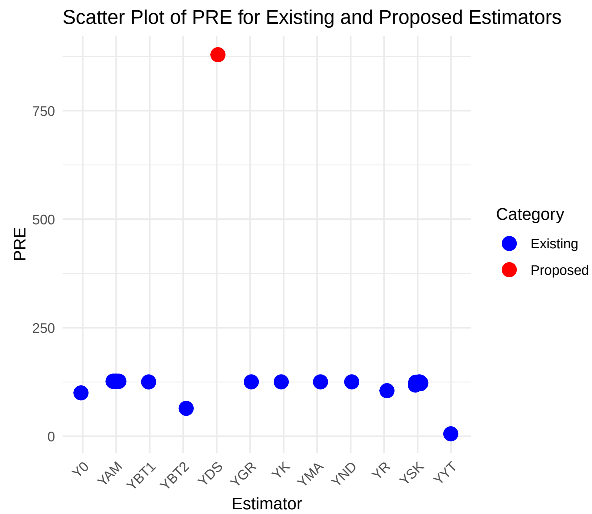

Figure 5: Comparison of PRE values between the existing estimators and the proposed estimator $\bar{Y}_{DS1}$ (highest PRE) for Data Set–2 under SRSWOR when $\gamma = 1$.

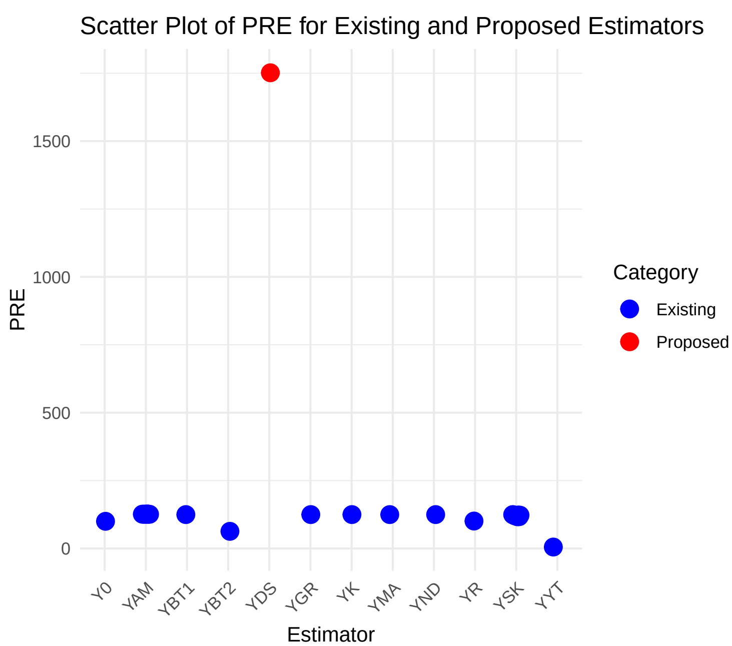

Figure 6: Comparison of PRE values between the existing estimators and the proposed estimator $\bar{Y}_{DS1}$ (highest PRE) for Data Set–3 under SRSWOR when $\gamma = 1$.

Table 4: PREs of the proposed $\bar{Y}_{DSi}$, (i=1,2,...,10) and existing estimators for simulated data at $\gamma = 1$ and $\gamma = 0$.

Distributions →	Normal		Weibull	
Estimators	$\gamma = 1$	$\gamma = 0$	$\gamma = 1$	$\gamma = 0$
Proposed Estimators				
$\mathbf{\bar{Y}_{DS1}}$	**127.388**	**119.609**	**105.985**	**105.812**
$\mathbf{\bar{Y}_{DS2}}$	**119.194**	**115.902**	**105.783**	**105.706**
$\mathbf{\bar{Y}_{DS3}}$	**126.501**	**119.222**	**105.982**	**105.810**
$\mathbf{\bar{Y}_{DS4}}$	**114.151**	**113.539**	**105.706**	**105.705**
$\mathbf{\bar{Y}_{DS5}}$	**127.099**	**119.483**	**105.984**	**105.812**
$\mathbf{\bar{Y}_{DS6}}$	**127.253**	**119.551**	**105.984**	**105.812**
$\mathbf{\bar{Y}_{DS7}}$	**125.727**	**118.881**	**105.984**	**105.811**
$\mathbf{\bar{Y}_{DS8}}$	**115.878**	**114.328**	**105.697**	**105.694**
$\mathbf{\bar{Y}_{DS9}}$	**125.992**	**118.998**	**105.945**	**105.789**
$\mathbf{\bar{Y}_{DS10}}$	**119.207**	**115.908**	**105.854**	**105.742**
Existing Estimators				
$\bar{Y}_0$	100.000	100.000	100.000	100.000
$\bar{Y}_R$	78.076	78.076	100.056	100.056
$\bar{Y}_{BT1}$	106.112	106.112	100.910	100.910
$\bar{Y}_{BT2}$	68.796	68.796	97.417	97.417
$\bar{Y}_{YT}$	102.825	102.825	18.886	18.886
$\bar{Y}_{SK1}$	100.448	100.448	100.028	100.028
$\bar{Y}_{SK2}$	100.448	100.448	100.028	100.028
$\bar{Y}_{SK3}$	100.440	100.440	100.027	100.027
$\bar{Y}_{SK4}$	100.432	100.432	100.027	100.027
$\bar{Y}_{SK5}$	100.424	100.424	100.026	100.026
$\bar{Y}_{SK6}$	100.401	100.401	100.025	100.025
$\bar{Y}_{SK7}$	100.384	100.384	100.024	100.024
$\bar{Y}_{SK8}$	100.374	100.374	100.023	100.023
$\bar{Y}_{SK9}$	100.368	100.368	100.023	100.023
$\bar{Y}_{SK10}$	100.362	100.362	100.022	100.022
$\bar{Y}_K$	108.990	108.990	100.911	100.911
$\bar{Y}_{GR}$	108.990	108.990	100.911	100.911
$\bar{Y}_{ND}$	108.990	108.990	100.911	100.911
$\bar{Y}_{MA}$	108.990	108.990	100.911	100.911
$\bar{Y}_{AM1}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM2}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM3}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM4}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM5}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM6}$	109.223	109.223	101.071	101.071

Continued on next page

Table 4 (continued)

Distributions →	Normal		Weibull	
Estimators	$\gamma = 1$	$\gamma = 0$	$\gamma = 1$	$\gamma = 0$
$\bar{Y}_{AM7}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM8}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM9}$	109.223	109.223	101.071	101.071
$\bar{Y}_{AM10}$	109.223	109.223	101.071	101.071

8. RESULTS

Based on a thorough analysis of the numerical, simulation and graphical study results, we have made the following observations:

i) The results from Table 3 for each population indicate that the members $\bar{Y}_{DSi}$, i=1,2,...,10 of the suggested estimator $\bar{Y}_{DS}$, acquire the highest PRE in comparison to the current estimators, including the traditional mean estimator, Cochran [7] estimator, Bahl and Tuteja [5] estimators, Yan and Tian [31] estimator, Subramani and Kumarapandiyan [25] estimator, Kadilar [13] estimator, Zakari et al. [32] estimator, Dansawad [8] estimator, Abiodun et al. [1] estimator, Adejumobi et al. [2] estimator respectively.

ii) Table 6. presents some members of the proposed estimator corresponding to different combinations of the parameters a and b, while Table 3 shows that, among the suggested class of estimators, member $\bar{Y}_{DS4}$ for data set-1 and $\bar{Y}_{DS1}$ for data set - 2 and 3 are best for $\gamma = 0, 1$, Also the graphical representation of the heighest value of the suggested estimator can be shown in Figure 4, 5 and 6 for $\gamma = 1$.

iii) Figures 1, 2 and 3 shows the scatterplot of the PREs for both the existing and suggested estimators $\bar{Y}_{DSi}$ (i=1,2,...,10) for $\gamma = 1$ of data set - 1, 2 and 3. The x-axis depicts the estimators, and the y-axis displays the PRE values. The blue points correspond to the existing estimators, while the red points reflect the proposed estimators. The graphic clearly shows that the proposed estimators (represented by red points) routinely achieve greater PRE values than all other estimators.

iv) Table 4 shows the results for both the simulated normal and Weibull populations, which show that the suggested estimator exhibit superior PRE values in comparison to the existing estimators, indicating that it surpasses traditional methods in terms of efficiency. The table also shows that the proposed estimator $\bar{Y}_{DS1}$ performs best in normal and weibull populations for $\gamma = 0, 1$. All computations and simulations were performed using R statistical software, version 4.4.1 [21].

9. CONCLUSION

Using simple random sampling technique, this work suggests a generalized exponential ratio class of estimators under simple random sampling for estimating $\bar{Y}$. The minimum MSE for suggested estimators is mathematically approximated to the first order. Efficiency requirements are determined by comparing the proposed estimators' minimum MSE expression to the MSE/minimum MSE of current estimators. A numerical experiment with three actual datasets and a simulation experiment with artificially generated normal

and weibull populations support the efficiency conditions described in "Efficiency Conditions". Real world data shows that the proposed estimator suppresses traditional estimators in each population, which is further supported by graphical representations and the simulation results. It is suggested to use the indicated estimators for population mean estimation in practical situations.
Furthermore, the suggested estimator is ideal for real-world survey applications because it is simple to build and only requires knowledge of one auxiliary variable. Efficiency gains are made without expanding the sample size, which is especially advantageous for surveys with tight budgets. Consequently, under SRSWOR, the proposed generalized exponential ratio class of estimators provides a reliable and efficient alternative for population mean estimation. The flexibility of the proposed class allows the generation of numerous estimators through different combinations of the parameters a, b, and the tuning parameter γ. Although ten representative members corresponding to $\gamma = 0$ and $\gamma = 1$ are examined in this study, the framework admits many additional estimators. This methodology may be extended to various sampling designs in future research, or it may take into account the use of several auxiliary variables.

DATA AVAILABILITY

The manuscript provides all of the necessary data information.

CONFLICT OF INTEREST

No conflicts of interest, financial or otherwise, were identified by the authors in connection with this research.

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