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A NEW GENERALIZED BETA-X FAMILY OF DISTRIBUTIONS WITH APPLICATIONS

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ABSTRACT

In this paper, we introduce a new generalized Beta–X family of distributions, motivated by the need for greater flexibility in modeling complex and diverse real-world data. The proposed family of distributions is developed by compounding the Beta distribution with a baseline distribution through a transformation mechanism, enhancing both the shape and tail properties of the original model. We explore several mathematical and statistical properties of the new family, including its probability density function, distribution function, quantile function, moments and entropy. The parameters of the proposed densities are estimated using the method of maximum likelihood. The practical applicability and flexibility of the improved distributions as well as the effectiveness of the produced estimators are illustrated through two real-life datasets. The results highlight the advantages of the new family over existing distributions in terms of goodness-of-fit and interpretability.

KEYWORDS: Beta distribution; Entropy; Shannon entropy; Transformation function.

MSC: 62E15; 62F10.

RESUMEN

En este trabajo introducimos una nueva familia generalizada Beta–X de distribuciones, motivada por la necesidad de mayor flexibilidad para modelar datos reales complejos y diversos. La familia propuesta se construye al componer la distribución Beta con una distribución base mediante un mecanismo de transformación, lo que mejora tanto la forma como las propiedades de cola del modelo original. Estudiamos varias propiedades matemáticas y estadísticas de la nueva familia, incluyendo la función de densidad, la función de distribución, la función cuantil, los momentos y la entropía. Los parámetros se estiman mediante el método de máxima verosimilitud. La aplicabilidad y flexibilidad de las distribuciones propuestas, así como la eficacia de los estimadores, se ilustran mediante dos conjuntos de datos reales. Los resultados muestran ventajas frente a distribuciones existentes en términos de bondad de ajuste e interpretabilidad.

PALABRAS CLAVE: distribución Beta; entropía; entropía de Shannon; función de transformación.

1. INTRODUCTION

In many real-world applications, classical probability distributions often fail to provide adequate flexibility for modeling complex data. To address this limitation, numerous generalizations and extensions of

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existing distributions have been proposed in the statistical literature. One common strategy is to introduce additional parameters or transformation schemes that enhance the modeling flexibility of baseline distributions. This approach has led to the development of several families of generalized distributions. A comprehensive review of such families can be found in the monograph by [17].

These recent works further illustrate the continued development of flexible “generated” distribution families for improved data modeling. [4] proposes a logarithmic beta-generated family and derives key distributional properties along with likelihood-based inference and applications, emphasizing the practical advantages of beta-generation for capturing diverse shapes. [22] develop a novel T–X construction (built around a modified type-II half-logistic generator), establish algebraic properties, and apply maximum likelihood estimation and real-data illustrations in lifetime and reliability contexts. [23] introduce the Generalized Alpha–Beta–Power family via an exponentiated extension of an existing transformation, derive properties such as moments and entropy, and demonstrate flexibility through applications to reliability in engineering and medical datasets. Together, these papers motivate and support further investigation of generalized beta transformation-based X-families—like the generalized Beta–X framework proposed in our study—both theoretically and in applied modeling.

The Beta distribution is a particularly flexible model defined on the interval $[0, 1]$, making it ideal for representing proportions, rates, and probabilities. Parameterized by two positive shape parameters, α and β, it can exhibit various shapes including uniform, U-shaped, and bimodal forms. This flexibility allows the Beta distribution to be useful in diverse fields such as finance, engineering, and social sciences, particularly where outcomes involve bounded uncertainty [16, 9].

The practical utility of the Beta distribution has been well recognized in contexts like quality control and project management. For example, it is commonly used in risk assessment and decision analysis to model task durations, costs, and success probabilities, offering a probabilistic framework for managing uncertainties [1].

The probability density function (pdf) of the Beta distribution is given by

$$g(x; \alpha, \beta) = \frac{1}{\mathrm{B}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1,\ \alpha, \beta > 0, \tag{1.1}$$

where $\mathrm{B}(\alpha, \beta)$ is the Beta function, defined as $\mathrm{B}(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$. The corresponding cumulative distribution function (CDF) is expressed using the regularized incomplete Beta function as

$$G(x; \alpha, \beta) = I_x(\alpha, \beta) = \int_0^x \frac{1}{\mathrm{B}(\alpha, \beta)} w^{\alpha-1}(1-w)^{\beta-1}\, dw. \tag{1.2}$$

Various Beta-based distribution families have been introduced through composition with different baseline distributions. Notable examples include the Beta-Gumbel [18], Beta-Weibull [8], Beta-Pareto [3], and Beta Marshall-Olkin [5] distributions. Eugene et al. (2002) introduced the general framework of Beta-generated distributions, where the CDF of a Beta–G generated variable X is given by

$$G(x; \alpha, \beta) = \frac{1}{\mathrm{B}(\alpha, \beta)} \int_0^{F(x)} w^{\alpha-1}(1-w)^{\beta-1}\, dw, \tag{1.3}$$

with $F(x)$ representing the CDF of a baseline distribution.

When the identity transformation $T(F(x)) = F(x)$ is used, this framework yields the well-known density function of the ordered statistics. Specifically, if $F(x)$ corresponds to a uniform distribution, the resulting $g(x)$ is the standard Beta density. In the exponential case, it leads to interesting forms involving order statistics. For instance, [21] showed that if $F(x)$ is exponential with rate θ, the corresponding Beta-generated variable has a representation involving a linear combination of ordered from standard exponential variables.

A further generalization was proposed by [6] and [17] through the so-called W–X family of distributions, defined by

$$G(x) = \int_{A}^{T(F(x))} r(w)\, dw, \tag{1.4}$$

where $r(w)$ is the generator density supported on $[A, B]$, and $T(F(x))$ is a transformer function applied to the baseline CDF $F(x)$. This framework allows great flexibility in defining new families of distributions. The corresponding pdf of (1.4) can be expressed as:

$$g(x) = \frac{d}{dx}\left[T(F(x))\right] \cdot r\left(T(F(x))\right). \tag{1.5}$$

The transformer function $T(F(x))$ must satisfy the following conditions:

(i) $T(F(x)) \in [A, B]$

(ii) $T(F(x))$ is differentiable and non-decreasing

(iii) $T(F(x)) \to A$ as $x \to -\infty$ and $T(F(x)) \to B$ as $x \to \infty$

While several transformer functions have been studied, including identity and power forms, these were typically applied with generators other than the Beta density. Table 1 lists a few examples of such transformations, assuming a Beta generator.

Transformer	Corresponding pdf $g(x; \alpha, \beta)$
$T(F(x)) = F(x)$	$\frac{1}{\mathrm{B}(\alpha,\beta)} f(x) F(x)^{\alpha-1}(1 - F(x))^{\beta-1}$
$T(F(x)) = F^c(x)$	$\frac{c}{\mathrm{B}(\alpha,\beta)} f(x) F(x)^{c(\alpha-1)}(1 - F^c(x))^{\beta-1}$

Table 1: Examples of transformers for Beta-generated distributions

In this paper, we develop a more general Beta–X family of distributions using a broad class of transformer functions, including both existing and novel transformations. Notably, we introduce two new transformer forms: the Power-Adjusted Function (PAF) and the Squared-Complement Logarithmic (SCL) transformer. This extended framework leads to a rich class of distributions with desirable theoretical and practical properties.

The remainder of the paper is organized as follows. Section 2. presents the definition of the new generalized Beta–X family. Section 3. provides illustrative examples including Beta-Uniform, Beta-Exponential, Beta-Beta, and the new Beta-Kumaraswamy distributions derived via the PAF and SCL transformers. Section 4. explores the basic mathematical properties, such as the expanded forms of the pdf and cdf, moments, and the moment-generating function. Section 5. addresses the behavior of ordered statistics while Section 6. defines the entropy concept for the new family of distributions. In Section 7., we derive the maximum likelihood estimators (MLEs) for the proposed models. Section 8. presents two practical applications to demonstrate the usefulness of the new family. The last section, Section 9., summarizes main conclusions.

2. NEW GENERALIZED BETA–X FAMILY OF DISTRIBUTIONS

Consider the Beta distribution with pdf given by

$$r(w) = \frac{1}{\mathrm{B}(\alpha,\beta)} w^{\alpha-1}(1-w)^{\beta-1}, \quad 0 < w < 1,\ \alpha, \beta > 0,$$

used as a generator distribution for constructing the distribution introduced in (1.4). The distributions produced are referred to in this paper as the Beta–X family of distributions and can be generally expressed as:

$$G(x;\alpha,\beta) = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^{T(F(x))} w^{\alpha-1}(1-w)^{\beta-1}\, dw = I_{T(F(x))}(\alpha,\beta), \tag{2.1}$$

where $I_{T(F(x))}(\alpha,\beta)$ is the regularized incomplete Beta function evaluated at the transformer $T(F(x))$ for a given baseline CDF $F(x)$. Additional parameters can be introduced via $F(x)$ to provide more flexibility to the resulting distributions.

Consequently, the corresponding family of pdfs is given by:

$$g(x;\alpha,\beta) = \frac{1}{\mathrm{B}(\alpha,\beta)} \left[T(F(x))\right]^{\alpha-1} \left[1 - T(F(x))\right]^{\beta-1} J(x) f(x), \tag{2.2}$$

where $J(x)f(x) = \frac{d}{dx}\left[T(F(x))\right]$.

In this research, two new transformers have been developed: the Power–Adjusted Fractional (PAF) transformer and the Scaled–Logistic (SCL) transformer.

To introduce enhanced flexibility and control over the distribution's shape, we define the PAF transformer as:

$$T(F(x)) = \frac{2F^n(x)}{1+F(x)}, \quad n \in \mathbb{N}, \tag{2.3}$$

where n serves as a tuning parameter. This transformer generalizes existing forms in the literature (e.g., for $n = 1$), and allows increasing stability as $n \to \infty$. The produced family of distributions maintains analytical tractability and provides a fruitful framework for modeling diverse data behaviors.

At the same time, the SCL transformer is defined as:

$$T(F(x)) = \frac{e^{\gamma F(x)} - 1}{e^{\gamma} - 1}, \quad \gamma > 0. \tag{2.4}$$

This transformation preserves theoretical soundness, maintains compatibility with Beta generators, and enhances the capacity to model complex real-world phenomena. The parameter γ controls skewness and tail thickness. For small γ, $T(F(x)) \approx F(x)$, recovering the identity transformation. As $\gamma \to \infty$, the transformation adds left skew and heavy tails; as $\gamma \to 0^+$, it behaves linearly.

Merging the new transformations in (2.3) and (2.4) with the general density in (2.2) yields the following new Beta–X distributions:

The Beta–X density with the PAF transformer is:

$$g(x;\alpha,\beta,n)=\frac{2^{\alpha}}{\mathrm{B}(\alpha,\beta)}\cdot\frac{nF^{n-1}(x)+(n-1)F^{n}(x)}{(1+F(x))^{\alpha+\beta}}\cdot F^{n(\alpha-1)}(x)\cdot(1+F(x)-2F^{n}(x))^{\beta-1}f(x). \tag{2.5}$$

And the Beta–X density with the SCL transformer is:

$$g(x;\alpha,\beta,\gamma)=\frac{1}{\mathrm{B}(\alpha,\beta)}\left(\frac{e^{\gamma F(x)}-1}{e^{\gamma}-1}\right)^{\alpha-1}\left(1-\frac{e^{\gamma F(x)}-1}{e^{\gamma}-1}\right)^{\beta-1}\cdot\frac{\gamma e^{\gamma F(x)}}{e^{\gamma}-1}f(x). \tag{2.6}$$

Table 2 introduces the two transformations for some specific baseline distributions.

Transformer Type	**Distribution**	**Transformer $T(F(x))$**
PAF	Uniform $(0,1)$	$\frac{2x^n}{1+x}$
	Exponential (θ)	$\frac{2(1-e^{-\theta x})^n}{1+(1-e^{-\theta x})}$
	Beta (a,b)	$\frac{2[I_{T(F(x))}(a,b)]^n}{1+I_{T(F(x))}(a,b)}$
	Kumaraswamy (a,b)	$\frac{2(1-(1-x^a)^b)^n}{2-(1-x^a)^b}$
SCL	Uniform $(0,1)$	$\frac{e^{\gamma x}-1}{e^{\gamma}-1}$
	Exponential (θ)	$\frac{e^{\gamma(1-e^{-\theta x})}-1}{e^{\gamma}-1}$
	Beta (a,b)	$\frac{e^{\gamma I_x(a,b)}-1}{e^{\gamma}-1}$
	Kumaraswamy (a,b)	$\frac{e^{\gamma(1-(1-x^a)^b)}-1}{e^{\gamma}-1}$

Table 2: A set of newly proposed transformation functions along with some distributions.

The hazard rate function, or failure rate, is a key concept in survival and reliability analysis. For the general Beta–X family of distributions defined in (2.2), the hazard function is given by (e.g. [13])

$$h(x;\alpha,\beta)=\frac{g(x;\alpha,\beta)}{1-G(x)},$$

where $G(x)$ is the cumulative distribution function associated with $g(x;\alpha,\beta)$.

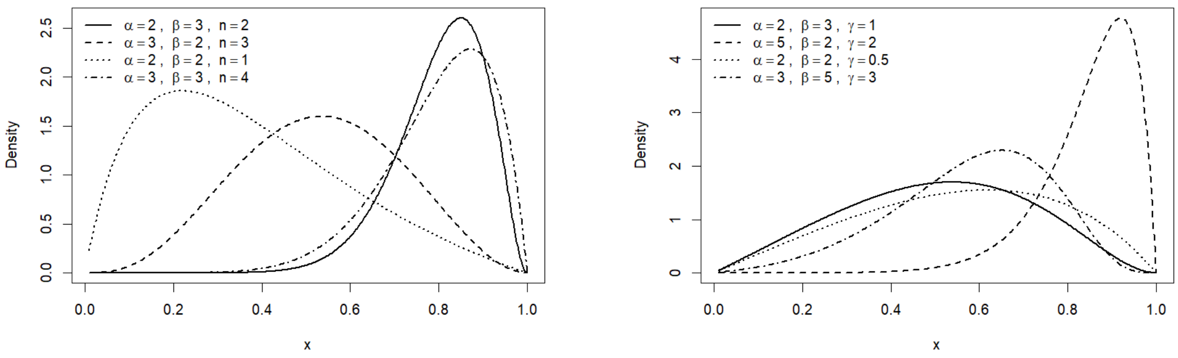
For the specific choices of the transformer function $T(\cdot)$ defined in (2.3) and (2.4), the corresponding hazard function respictively becomes

$$\begin{aligned}h_{\mathrm{PAF}}(x)=&\frac{1}{B(\alpha,\beta)}\cdot\frac{1}{1-G(x)}\left(\frac{2F^{n}(x)}{1+F(x)}\right)^{\alpha-1}\left(1-\frac{2F^{n}(x)}{1+F(x)}\right)^{\beta-1}\\&\times\frac{2f(x)\left[nF^{n-1}(x)(1+F(x))-F^{n}(x)\right]}{(1+F(x))^{2}}.\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}h_{\mathrm{SCL}}(x)=&\frac{1}{B(\alpha,\beta)}\cdot\frac{1}{1-G(x)}\left(\frac{e^{\gamma F(x)}-1}{e^{\gamma}-1}\right)^{\alpha-1}\left(1-\frac{e^{\gamma F(x)}-1}{e^{\gamma}-1}\right)^{\beta-1}\\&\times\frac{\gamma e^{\gamma F(x)}f(x)}{e^{\gamma}-1}.\end{aligned} \tag{2.8}$$

These expressions enable further analysis of the reliability behavior and tail properties of the proposed distributions.



(a)

(b)

Figure 1: Beta–Uniform density functions using (a) the PAF and (b) the SCL transformer for various combinations of parameters.

3. EXAMPLES

In this section, the two new Beta–X families of distributions, defined in (11) and (12), are introduced by implementing novel transformation functions: the PAF and the SCL transformers. These transformations are applied to specific baseline distributions as outlined in Table 2.

The resulting pdfs are most analytically tractable when they admit simple closed-form expressions. This is instantly achieved when the CDF $F(x)$ and the corresponding pdf $f(\cdot)$ are appropriately incorporated into the transformation functions.

In the sub-sections that follow, four specific examples of the distribution $F(x)$ are considered to illustrate the match transformers. These include the uniform, exponential, beta, and Kumaraswamy distributions.

3.1. Beta–Uniform Distribution via PAF and SCL Transformers

As a first example, consider the baseline function to be the standard uniform distribution (i.e., $f(x) = 1$, $0 \le x \le 1$). Substituting it into the density in (11), where the PAF transformer is used, gives:

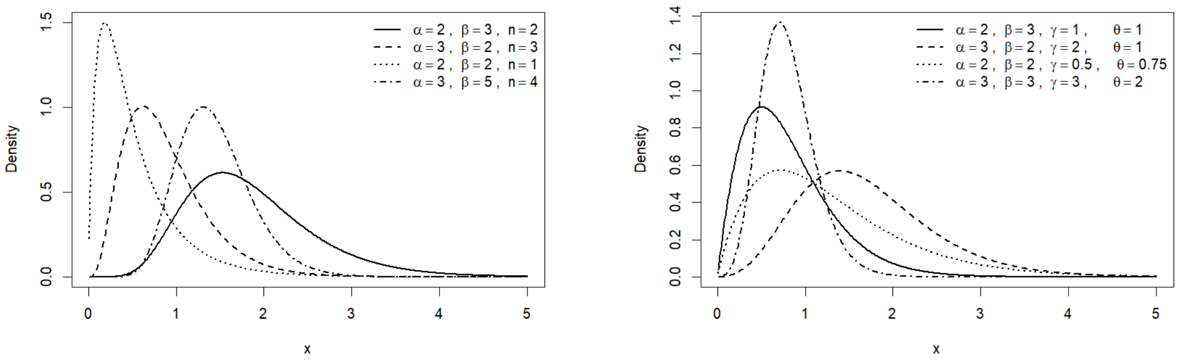
$$g(x;\alpha,\beta,n) = \frac{1}{B(\alpha,\beta)}\left(\frac{2x^n}{1+x}\right)^{\alpha-1}\left(1-\frac{2x^n}{1+x}\right)^{\beta-1}\cdot\frac{2[nx^{n-1}(1+x)-x^n]}{(1+x)^2}, \tag{3.1}$$

where $x \in [0,1]$, $\alpha, \beta > 0$, and $n \in \mathbb{N}$.

Applying the SCL transformer as developed in (12) and using the standard uniform distribution as baseline density gives:

$$g(x;\alpha,\beta,\gamma) = \frac{1}{B(\alpha,\beta)}\left(\frac{e^{\gamma x}-1}{e^{\gamma}-1}\right)^{\alpha-1}\left[1-\left(\frac{e^{\gamma x}-1}{e^{\gamma}-1}\right)\right]^{\beta-1}\cdot\frac{\gamma e^{\gamma x}}{e^{\gamma}-1}, \tag{3.2}$$

such that $x \in [0,1]$, $\alpha, \beta > 0$, and $\gamma > 0$.



(a) (b)

Figure 2: Beta–Exponential density functions using (a) the PAF and (b) the SCL transformer for various combinations of parameters.

As observed in Figure 1, panel (a) transitions from right-skewed to left-skewed shapes, passing through symmetric forms. In contrast, panel (b) presents pronounced left skewness, which enhances the heavy tail and may contribute to more robust estimations in the presence of outliers.

3.2. Beta–Exponential Distribution via PAF and SCL Transformers

Using the exponential distribution with pdf $f(x) = \theta e^{-\theta x}$ and $F(x) = 1 - e^{-\theta x}$ in (11) gives the following density with the PAF transformer:

$$g(x;\alpha,\beta,n) = \frac{1}{B(\alpha,\beta)} \left(\frac{2(1-e^{-\theta x})^n}{1+(1-e^{-\theta x})}\right)^{\alpha-1} \left(1 - \frac{2(1-e^{-\theta x})^n}{1+(1-e^{-\theta x})}\right)^{\beta-1} \cdot \text{PAF-J}(x), \tag{3.3}$$

where PAF-J(x) denotes the derivative of the PAF transformer times $f(x)$.
Similarly, using the SCL transformer gives:

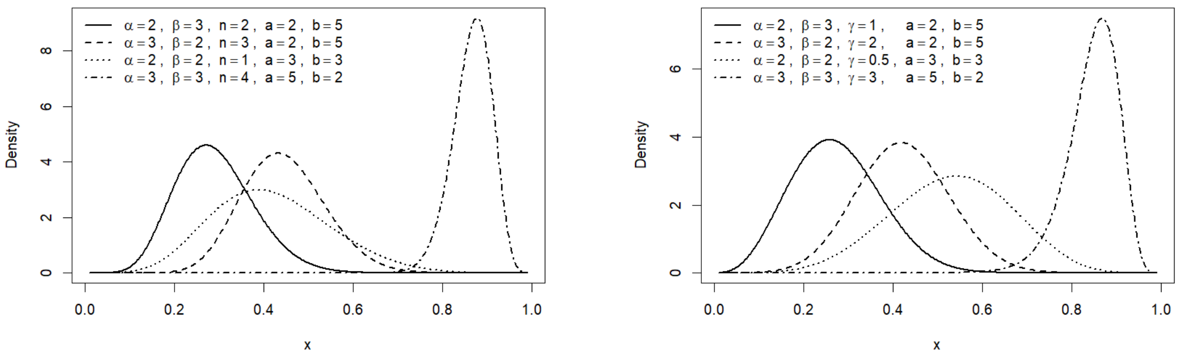
$$g(x;\alpha,\beta,\gamma) = \frac{1}{B(\alpha,\beta)} \left(\frac{e^{\gamma(1-e^{-\theta x})}-1}{e^{\gamma}-1}\right)^{\alpha-1} \left[1 - \left(\frac{e^{\gamma(1-e^{-\theta x})}-1}{e^{\gamma}-1}\right)\right]^{\beta-1} \cdot \frac{\gamma e^{\gamma(1-e^{-\theta x})}\theta e^{-\theta x}}{e^{\gamma}-1}. \tag{3.4}$$

3.3. Beta–Beta Distribution via PAF and SCL Transformers

Let the baseline be the Beta(a, b) distribution with $F(x) = I_x(a, b)$ and $f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$. Substituting into (11) gives:

$$g(x;\alpha,\beta,n) = \frac{1}{B(\alpha,\beta)} \left(\frac{2I_x^n(a,b)}{1+I_x(a,b)}\right)^{\alpha-1} \left(1 - \frac{2I_x^n(a,b)}{1+I_x(a,b)}\right)^{\beta-1} \cdot \text{PAF-J}(x), \tag{3.5}$$

where PAF-J(x) denotes the derivative of the PAF transformer times $f(x)$.



(a) (b)

Figure 3: Beta–Beta density functions using (a) the PAF and (b) the SCL transformer for various combinations of parameters.

The corresponding density using the SCL transformer becomes:

$$g(x;\alpha,\beta,\gamma) = \frac{1}{B(\alpha,\beta)}\left(\frac{e^{\gamma I_x(a,b)}-1}{e^{\gamma}-1}\right)^{\alpha-1}\left[1-\left(\frac{e^{\gamma I_x(a,b)}-1}{e^{\gamma}-1}\right)\right]^{\beta-1}\cdot\frac{\gamma e^{\gamma I_x(a,b)}}{e^{\gamma}-1}f(x). \tag{3.6}$$

3.4. Beta–Kumaraswamy Distribution via PAF and SCL Transformers

Using the Kumaraswamy distribution $F(x) = 1-(1-x^a)^b$ and $f(x) = abx^{a-1}(1-x^a)^{b-1}$ in (11), the PAF-based density becomes:

$$g(x;\alpha,\beta,n) = \frac{1}{B(\alpha,\beta)}\left(\frac{2F^n(x)}{1+F(x)}\right)^{\alpha-1}\left(1-\frac{2F^n(x)}{1+F(x)}\right)^{\beta-1}\cdot \text{PAF-J}(x), \tag{3.7}$$

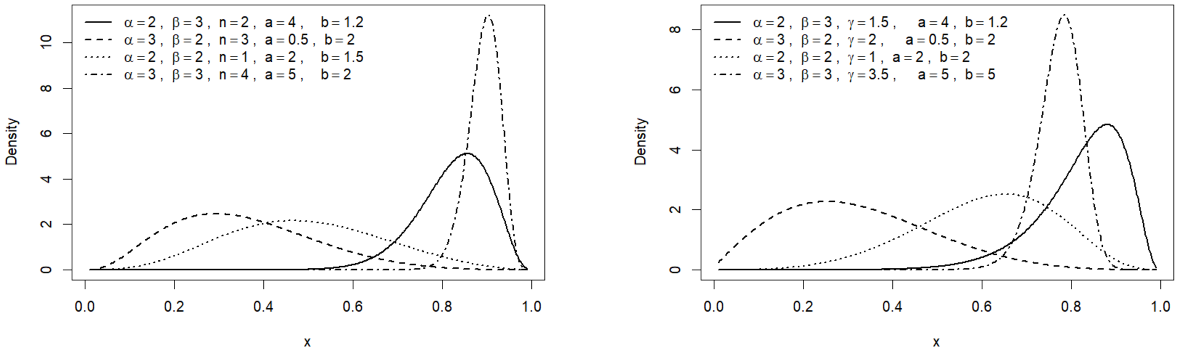
where $F(x) = 1-(1-x^a)^b$ and PAF-J(x) again denotes the appropriate derivative term.
Using the SCL transformer yields:

$$g(x;\alpha,\beta,\gamma) = \frac{1}{B(\alpha,\beta)}\left(\frac{e^{\gamma F(x)}-1}{e^{\gamma}-1}\right)^{\alpha-1}\left[1-\left(\frac{e^{\gamma F(x)}-1}{e^{\gamma}-1}\right)\right]^{\beta-1}\cdot\frac{\gamma e^{\gamma F(x)}}{e^{\gamma}-1}f(x), \tag{3.8}$$

where $F(x) = 1-(1-x^a)^b$ and $f(x) = abx^{a-1}(1-x^a)^{b-1}$.

4. BASIC MATHEMATICAL PROPERTIES

Even though the Beta-X density function depends on mathematical functions that are easily found in modern statistical software, numerical derivations often benefit from expressing the transformer $T(F(x))$ using well-known series expansions.



(a) (b)

Figure 4: Beta–Kumaraswamy density functions using (a) the PAF and (b) the SCL transformer for various combinations of parameters.

By applying the generalized binomial series expansion to the term $[1 - T(x)]^{\beta-1}$, we obtain

$$T(x)^{\alpha-1}[1 - T(x)]^{\beta-1} = \sum_{k=0}^{\infty} C_k T(x)^{\alpha+k-1}, \tag{4.1}$$

where $C_k = \binom{\beta-1}{k}(-1)^k$. This expansion satisfies the condition $|T(x)| < 1$, which holds in the current context since $T(x) \in [0, 1]$. It is worth noting that $T(x)$ is used here as a simplified form of the transformation $T(F(x))$, for ease of notation and interpretation.

Thus, the pdf in (2.2) can be reshaped as:

$$g(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} f(x)J(x) \sum_{k=0}^{\infty} C_k T(x)^{\alpha+k-1}, \tag{4.2}$$

and the associated CDF can be written as:

$$G(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \sum_{k=0}^{\infty} C_k \frac{T(x)^{\alpha+k}}{\alpha + k}. \tag{4.3}$$

Assuming the PAF transformer is applied to $T(x)$, it simplifies as:

$$T(x) = \frac{2F^n(x)}{1 + F(x)} = \sum_{j=0}^{\infty} a_j F(x)^{n+j},$$

where $a_j = 2(-1)^j$. Raising this series to the power $\alpha + k - 1$ yields:

$$T(x)^{\alpha+k-1} = \left(\sum_{j=0}^{\infty} a_j F(x)^{n+j} \right)^{\alpha+k-1} = \sum_{m=0}^{\infty} A_{k,m} F(x)^m,$$

where $A_{k,m}$ denotes the mth coefficient in the expansion and is defined as:

$$A_{k,m} = \sum_{\substack{j_1+\cdots+j_{\alpha+k-1}=m \\ j_i \geq 0}} a_{j_1} a_{j_2} \cdots a_{j_{\alpha+k-1}}, \quad \text{with } j \geq n.$$

The term $J(x)$, defined as:

$$J(x) = \frac{2F(x)^{n-1}(n(1+F(x)) - F(x))}{(1+F(x))^2},$$

can be expanded by applying the generalized binomial expansion to both the numerator and denominator in terms of $F(x)$. The full expression for $J(x)$ can be presented as a power series:

$$J(x) = \sum_{k=0}^{\infty} B_k F(x)^{n+k-1}, \quad \text{where } B_k = 2(-1)^k(n+k).$$

Substituting these results into the density function $g(x)$ in (4.2) leads to:

$$g(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} f(x) \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} B_k A_{k,m} F(x)^{n+k-1+m}. \tag{4.4}$$

By introducing the index $r = n+k-1+m$, we can write this expression more compactly as:

$$g(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} f(x) \sum_{r=0}^{\infty} S_r F(x)^r, \tag{4.5}$$

where the coefficients S_r are defined by:

$$S_r = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} B_k A_{k,m}.\mathbf{1}_{\{r=n+k-1+m\}} = \sum_{k=0}^{r-n+1} B_k A_{k,r-n-k+1}.$$

Here the indicator function $\mathbf{1}_{\{r=n+k-1+m\}} = 1$ ensures the correct indexing of the coefficients.We denote this resulting pdf in (4.5) as $g_{PAF}(x;\alpha,\beta)$.
Analogously, for the SCL transformation:

$$T(x) = \sum_{k=0}^{\infty} a_k F(x)^k, \quad J(x) = \frac{\gamma}{e^{\gamma}-1}\sum_{k=0}^{\infty}\frac{(\gamma F(x))^k}{k!}, \quad a_k = \frac{\gamma^k}{k!(e^{\gamma}-1)}.$$

Then the full density in (2.2) becomes:

$$g(x;\alpha,\beta) = \frac{f(x)}{B(\alpha,\beta)} \cdot \frac{\gamma}{e^{\gamma}-1} \sum_{r=0}^{\infty} S_r^* F(x)^r, \tag{4.6}$$

where

$$S_r^* = \sum_{m=0}^{r}\sum_{n=0}^{r-m} A_m C_n D_{r-m-n},$$

such that

$$\begin{aligned} \left(\sum_{k=0}^{\infty} a_k F(x)^k\right)^{\alpha-1} &= \sum_{m=0}^{\infty} A_m F(x)^m, \\ \left(1 - \sum_{k=0}^{\infty} a_k F(x)^k\right)^{\beta-1} &= \sum_{n=0}^{\infty} C_n F(x)^n, \\ \sum_{j=0}^{\infty}\frac{(\gamma F(x))^j}{j!} &= \sum_{j=0}^{\infty} D_j F(x)^j. \end{aligned}$$

We denote the resulting pdf in (4.6) as $g_{SCL}(x;\alpha,\beta)$.
Finally, the following lemma settles both weighted densities in 4.5 and 4.6 in one standard form.

Lemma 4.1. (Integrated Weighted Density Representation for Beta - X distributions) *The density functions $g_{PAF}(x;\alpha,\beta)$ and $g_{SCL}(x;\alpha,\beta)$, corresponding to the Beta - X distributions with PAF and SCL transformers respectively, can be expressed in a unified weighted form as:*

$$g(x;\Theta) = \sum_{r=0}^{\infty} \omega_r(\Theta) h_r(x;\Theta), \tag{4.7}$$

where Θ denotes the parameter vector associated with the generator and baseline densities. The component functions are defined by $h_r(x;\Theta) = f(x)F(x)^r$, and the weights $\omega_r(\Theta)$ are given by:

$$\omega_r(\Theta) = \begin{cases} S_r/B(\alpha,\beta), & \textit{for } g_{PAF}, \\ S_r^*/B(\alpha,\beta)\cdot\gamma/(e^{\gamma}-1), & \textit{for } g_{SCL}. \end{cases}$$

This unified representation facilitates the dervation of various mathematical properties including moments and the moment generating function can be derived.

Thus, the rth moment of X is given by:

$$\mathbb{E}[X^r] = \sum_{i=0}^{\infty} \omega_i(\Theta) \int_0^1 x^r h_i(x;\Theta)\,dx.$$

Alternatively, the expectation $\mathbb{E}[X]$ can be computed directly via the transformer:

$$\mathbb{E}[X] = \frac{1}{B(\alpha,\beta)} \sum_{k=0}^{\infty} \frac{C_k}{\alpha+k} \int_0^1 T(x)^{\alpha+k} dx. \tag{4.8}$$

This follows from the identity $\mathbb{E}[X] = \int_0^1 xg(x)dx = 1 - \int_0^1 G(x)dx$.

The moment generating function when applying Lemma 4.1. can be written as:

$$M_X(t) = \sum_{m=0}^{\infty} \omega_m(\Theta)\mathbb{E}[e^{tX_m}].$$

5. ORDERED STATISTICS

Ordered statistics are central to statistical inference. Let $X_{(i)}$ be the ith order statistic from a sample of size n drawn from the Beta-X distribution. Its density is:

$$g_{(i)}(x) = \frac{n!}{(i-1)!(n-i)!} G(x)^{i-1}[1-G(x)]^{n-i} g(x). \tag{5.1}$$

Substituting $G(x)$ and $g(x)$ gives:

$$\begin{aligned} g_{(i)}(x) = {} & \frac{n!}{(i-1)!(n-i)!} \cdot \frac{1}{B(\alpha,\beta)} I_{T(F(x))}(\alpha,\beta)^{i-1} \\ & \cdot [1 - I_{T(F(x))}(\alpha,\beta)]^{n-i} T(x)^{\alpha-1}[1-T(x)]^{\beta-1} J(x) f(x). \end{aligned}$$

For the PAF transformation:

$$\begin{aligned} g_{(i)}(x) = {} & \frac{n!}{(i-1)!(n-i)!} \cdot \frac{f(x)}{B(\alpha,\beta)} I_{T(F(x))}(\alpha,\beta)^{i-1} \\ & \cdot [1 - I_{T(F(x))}(\alpha,\beta)]^{n-i} \sum_{m=n-1}^{\infty} \sum_{k=0}^{\infty} a_m C_k \cdot 2^{\alpha+k-1} \frac{F(x)^{m+\alpha+k-1}}{(1+F(x))^{\alpha+k-1}}. \end{aligned}$$

For the SCL transformation:

$$\begin{aligned} g_{(i)}(x) = \frac{n!}{(i-1)!(n-i)!} \cdot \frac{f(x)}{B(\alpha,\beta)} I_{T(F(x))}(\alpha,\beta)^{i-1} \\ \cdot [1 - I_{T(F(x))}(\alpha,\beta)]^{n-i} \left(\sum_{k=0}^{\infty} a_k F(x)^k\right)^{\alpha-1} \\ \cdot \left(1 - \sum_{k=0}^{\infty} a_k F(x)^k\right)^{\beta-1} \cdot \frac{\gamma}{e^{\gamma}-1} \sum_{k=0}^{\infty} \frac{(\gamma F(x))^k}{k!}. \end{aligned}$$

The results in the above sections, which involve series expansions, can be computed numerically using software such as `R` or `Mathematica`.This is achieved by substituting a sufficiently large positive integer in place of ∞ in the relative series expansions and employing numerical integration (e.g., Lentz's method) to evaluate the regularized incomplete beta function $I_{T(F(x))}(\alpha,\beta)$.

6. ENTROPY

Shannon Entropy, originally introduced by [24], is a fundamental measure in information theory used to quantify the uncertainty associated with a random variable. It has applications across a broad range of domains, including data compression, cryptography, machine learning, genetics, communications, and ecology (see, e.g., [10]).

The Shannon entropy can be defined for the Beta–X family of distributions with pdf $g(x)$, as:

$$H(X) = \mathbb{E}[-\ln g(X)] = -\int g(x) \ln g(x)\, dx. \tag{6.1}$$

Substituting $g(x)$, that has been defined in 2.2, into the expression, we obtain:

$$H(X) = \ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}[\ln T(X)] - (\beta-1)\mathbb{E}[\ln(1-T(X))] - \mathbb{E}[\ln T'(X)] + 2H_f(X), \tag{6.2}$$

where $T(X)$ is the transformer function, $T'(X)$ is its derivative, and $H_f(X)$ is the entropy of the baseline density $f(x)$.

The following two lemmas give entropy when applying the PAF and SCL transformations.

Lemma 6.1. (Entropy when using PAF transformation)
The Shannon entropy of the Beta–X distribution with PAF transformer, that has been defined in (2.5), is given by:

$$\begin{aligned} H_{PAF}(X) = \ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}\left[\ln\left(\frac{2F(X)^n}{1+F(X)}\right)\right] \\ - (\beta-1)\mathbb{E}\left[\ln\left(1 - \frac{2F(X)^n}{1+F(X)}\right)\right] - \mathbb{E}[\ln T'(X)] + 2H_f(X). \end{aligned} \tag{6.3}$$

Lemma 6.2. (Entropy when using SCL transformation)
The Shannon entropy of the Beta–X distribution with SCL transformer, that has been defined in (2.6), is

Baseline density	$\boldsymbol{H}_{\mathrm{PAF}}(\boldsymbol{X}) =$	$\boldsymbol{H}_{\mathrm{SCL}}(\boldsymbol{X}) =$
$U(0,1)$	$\ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}[\ln(2x^n - \ln(1+x))]$ $-(\beta-1)\mathbb{E}\Big[\ln\Big(1-\frac{2x^n}{1+x}\Big)\Big]$ $-\mathbb{E}\big[\ln\big(2+(n-1)\ln x\big]$ $+\mathbb{E}\big[\ln(n+(n-1)x) - 2\ln(1+x)\big)\big]$	$\ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}[\ln(e^{\gamma x}-1)]$ $-(\beta-1)\mathbb{E}[\ln(e^{\gamma}-e^{\gamma x})]$ $-\gamma\mathbb{E}[x] - \ln(\gamma)$ $+(\alpha+\beta-2)\ln(e^{\gamma}-1)$
Exponential(θ)	$\ln B(\alpha,\beta)$ $-(\alpha-1)\mathbb{E}[\ln(2(1-e^{-\theta x})^n - \ln(1+(1-e^{-\theta x})))]$ $-(\beta-1)\mathbb{E}\Big[\ln\Big(1-\frac{2(1-e^{-\theta x})^n}{2-e^{-\theta x}}\Big)\Big]$ $-\mathbb{E}\Big[\ln(2+(n-1)\ln(1-e^{-\theta x})$ $+\ln(n+(n-1)(1-e^{-\theta x})) - 2\ln(2-e^{-\theta x}))\Big]$ $-2\mathbb{E}[\ln(\theta e^{-\theta x})]$	$\ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}[\ln(e^{\gamma(1-e^{-\theta x})}-1)]$ $-(\beta-1)\mathbb{E}[\ln(e^{\gamma}-e^{\gamma(1-e^{-\theta x})})]$ $-\ln(\gamma) - \gamma + \mathbb{E}[e^{-\theta x}]$ $+(\alpha+\beta-2)\ln(e^{\gamma}-1) - 2\mathbb{E}[\ln(\theta e^{-\theta x})]$
Beta(a,b)	$\ln B(\alpha,\beta)$ $-(\alpha-1)\mathbb{E}[\ln(2I_x(a,b)^n - \ln(1+I_x(a,b)))]$ $-(\beta-1)\mathbb{E}\Big[\ln\Big(1-\frac{2I_x(a,b)^n}{1+I_x(a,b)}\Big)\Big]$ $-\mathbb{E}\,[\ln(2+(n-1)\ln I_x(a,b)$ $+\ln(n+(n-1)x) - 2\ln(1+I_x(a,b)))]$ $-2\mathbb{E}\Big[\ln\Big(\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\Big)\Big]$	$\ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}[\ln(e^{\gamma I_x(a,b)}-1)]$ $-(\beta-1)\mathbb{E}[\ln(e^{\gamma}-e^{\gamma I_x(a,b)})]$ $-\gamma\mathbb{E}[I_x(a,b)] - \ln(\gamma)$ $+(\alpha+\beta-2)\ln(e^{\gamma}-1)$ $-2\mathbb{E}\Big[\ln\Big(\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\Big)\Big]$
Kumaraswamy(a,b)	$\ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}[\ln(2[1-(1-x^a)^b]^n$ $-\ln(2-(1-x^a)^b))]$ $-(\beta-1)\mathbb{E}\Big[\ln\Big(1-\frac{2[1-(1-x^a)^b]^n}{2-(1-x^a)^b}\Big)\Big]$ $-\mathbb{E}[\ln(2+(n-1)\ln(1-(1-x^a)^b)$ $+\ln(n+(n-1)(1-(1-x^a)^b))$ $-2\ln(1+(1-(1-x^a)^b)))]$ $-2\mathbb{E}\Big[\ln\Big(\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\Big)\Big]$	$\ln B(\alpha,\beta) - (\alpha-1)\mathbb{E}[\ln(e^{\gamma(1-(1-x^a)^b)}-1)]$ $-(\beta-1)\mathbb{E}[\ln(e^{\gamma}-e^{\gamma(1-(1-x^a)^b)})]$ $-\gamma+\gamma\mathbb{E}[(1-x^a)^b] - \ln(\gamma)$ $+(\alpha+\beta-2)\ln(e^{\gamma}-1)$

Table 3: Shannon entropy for the Beta–X distributions with specific baseline density $f(x)$ examples.

given by:

$$\begin{aligned} H_{SCL}(X) &= \ln B(\alpha,\beta) - \ln\gamma + (\alpha+\beta-1)\ln(e^{\gamma}-1) \\ &\quad - (\alpha-1)\mathbb{E}[\ln(e^{\gamma F(X)}-1)] - (\beta-1)\mathbb{E}[\ln(e^{\gamma}-e^{\gamma F(X)})] + 2H_f(X). \end{aligned} \tag{6.4}$$

Table (3) summarizes Shannon entropy for the Beta – X distributions when applying the specific baseline densities that have been used in Section (3.), and employing the PAF and SCL transformations.

7. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we derive the maximum likelihood estimators (MLEs) for the parameters of the Beta–X family based on complete sample data. Let $x_1, x_2, \ldots, x_n$ be a random sample from a Beta–X distribution with parameters α, β and $\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ represents parameters of the baseline distribution $f(x)$. Define the full parameter vector as $\boldsymbol{\Theta} = (\alpha, \beta, \boldsymbol{\mu})^T$ of dimension $p \times 1$.
The log-likelihood function is given by:

$$
\begin{aligned}
\ell(\boldsymbol{\Theta}) &= \sum_{i=1}^{n} \log g(x_i; \alpha, \beta, \boldsymbol{\mu}) && (7.1)\\
&= -n \log B(\alpha, \beta) + (\alpha - 1) \sum_{i=1}^{n} \log T(x_i; \boldsymbol{\mu}) + (\beta - 1) \sum_{i=1}^{n} \log[1 - T(x_i; \boldsymbol{\mu})]\\
&\quad + \sum_{i=1}^{n} \log J(x_i; \boldsymbol{\mu}) + \sum_{i=1}^{n} \log f(x_i; \boldsymbol{\mu}), && (7.2)
\end{aligned}
$$

where $J(x_i; \boldsymbol{\mu}) = \frac{d}{dx} T(x_i; \boldsymbol{\mu})$ is the derivative of the transformation.
Let the score function be $\boldsymbol{\Omega}(\boldsymbol{\Theta}) = (\Omega_\alpha, \Omega_\beta, \Omega_{\boldsymbol{\mu}})^T$ where:

$$
\begin{aligned}
\Omega_\alpha &= \frac{\partial \ell}{\partial \alpha} = n[\psi(\alpha) - \psi(\alpha + \beta)] + \sum_{i=1}^{n} \log T(x_i; \boldsymbol{\mu}), && (7.3)\\
\Omega_\beta &= \frac{\partial \ell}{\partial \beta} = n[\psi(\beta) - \psi(\alpha + \beta)] + \sum_{i=1}^{n} \log[1 - T(x_i; \boldsymbol{\mu})], && (7.4)\\
\Omega_{\boldsymbol{\mu}} &= \frac{\partial \ell}{\partial \boldsymbol{\mu}} = \sum_{i=1}^{n} \left[\left(\frac{\alpha - 1}{T(x_i; \boldsymbol{\mu})} - \frac{\beta - 1}{1 - T(x_i; \boldsymbol{\mu})} \right) \frac{\partial T(x_i; \boldsymbol{\mu})}{\partial \boldsymbol{\mu}} \right]\\
&\quad + \sum_{i=1}^{n} \left[\frac{1}{J(x_i; \boldsymbol{\mu})} \frac{\partial J(x_i; \boldsymbol{\mu})}{\partial \boldsymbol{\mu}} + \frac{1}{f(x_i; \boldsymbol{\mu})} \frac{\partial f(x_i; \boldsymbol{\mu})}{\partial \boldsymbol{\mu}} \right], && (7.5)
\end{aligned}
$$

where $\psi(\cdot)$ is the digamma function, i.e., $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$.
The MLE $\hat{\boldsymbol{\Theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\boldsymbol{\mu}})^T$ is obtained by solving the system $\boldsymbol{\Omega}(\boldsymbol{\Theta}) = 0$. Due to the nonlinearity of the equations, numerical methods are typically employed.

Fisher Information Matrix

To provide a concise representation of the asymptotic variances and covariances of the MLEs in the Beta – X family, the fisher information matrix $\mathcal{I}(\boldsymbol{\Theta})$ for the parameter vector is essential. It has the form

$$
\mathcal{I}(\boldsymbol{\Theta}) = -\mathbb{E}[\nabla^2 \ell(\boldsymbol{\Theta})] = \begin{pmatrix} \Omega_{\alpha\alpha} & \Omega_{\alpha\beta} & | & \Omega_{\alpha\boldsymbol{\mu}}^T \\ \Omega_{\beta\alpha} & \Omega_{\beta\beta} & | & \Omega_{\beta\boldsymbol{\mu}}^T \\ -- & -- & | & -- \\ \Omega_{\alpha\boldsymbol{\mu}} & \Omega_{\beta\boldsymbol{\mu}} & | & \Omega_{\boldsymbol{\mu}\boldsymbol{\mu}} \end{pmatrix}, \qquad (7.6)
$$

with components:

$$\begin{aligned}
\Omega_{\alpha\alpha} &= n[\psi'(\alpha) - \psi'(\alpha+\beta)],\\
\Omega_{\beta\beta} &= n[\psi'(\beta) - \psi'(\alpha+\beta)],\\
\Omega_{\alpha\beta} &= -n\psi'(\alpha+\beta),\\
\Omega_{\alpha\bar{\mu}\mu} &= -n\,\mathbb{E}\left[\sum_{i=1}^{n}\frac{1}{T(x_i;\bar{\mu}\mu)}\frac{\partial T(x_i;\bar{\mu}\mu)}{\partial\bar{\mu}\mu}\right] \qquad\qquad \equiv -n\mathbb{E}\left[\frac{1}{T(X;\bar{\mu}\mu)}\nabla_{\bar{\mu}\mu}T(X;\bar{\mu}\mu)\right],\\
\Omega_{\beta\bar{\mu}\mu} &= -n\,\mathbb{E}\left[\sum_{i=1}^{n}\frac{1}{1-T(x_i;\bar{\mu}\mu)}\frac{\partial T(x_i;\bar{\mu}\mu)}{\partial\bar{\mu}\mu}\right] \qquad \equiv -n\mathbb{E}\left[\frac{1}{1-T(X;\bar{\mu}\mu)}\nabla_{\bar{\mu}\mu}T(X;\bar{\mu}\mu)\right],
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{\bar{\mu}\mu\mu} = n\mathbb{E}\Bigg\{ &\left[\frac{\alpha-1}{T(X;\bar{\mu}\mu)} - \frac{\beta-1}{1-T(X;\bar{\mu}\mu)}\right]^2 \frac{\partial^2 T(X;\bar{\mu}\mu)}{\partial\bar{\mu}\mu\partial\bar{\mu}\mu^T}\\
&+ \frac{1}{J(X;\bar{\mu}\mu)^2}\frac{\partial^2 J(X;\bar{\mu}\mu)}{\partial\bar{\mu}\mu\partial\bar{\mu}\mu^T} + \frac{1}{f(X;\bar{\mu}\mu)^2}\frac{\partial^2 f(X;\bar{\mu}\mu)}{\partial\bar{\mu}\mu\partial\bar{\mu}\mu^T}\\
&- \left[\frac{\alpha-1}{T(X;\bar{\mu}\mu)}\frac{\partial^2 T(X;\bar{\mu}\mu)}{\partial\bar{\mu}\mu\partial\bar{\mu}\mu^T} + \frac{\beta-1}{1-T(X;\bar{\mu}\mu)}\frac{\partial^2 T(X;\bar{\mu}\mu)}{\partial\bar{\mu}\mu\partial\bar{\mu}\mu^T}\right]\\
&- \left[\frac{1}{J(X;\bar{\mu}\mu)}\frac{\partial^2 J(X;\bar{\mu}\mu)}{\partial\bar{\mu}\mu\partial\bar{\mu}\mu^T} - \frac{1}{f(X;\bar{\mu}\mu)}\frac{\partial^2 f(X;\bar{\mu}\mu)}{\partial\bar{\mu}\mu\partial\bar{\mu}\mu^T}\right]\Bigg\}.
\end{aligned}$$

8. REAL DATA APPLICATIONS

Two data sets have been illustrated to examine the performance of the new Beta–X distributions in fitting data appropriately. This performance can be measured via Akaike Information Criterion (AIC), which mainly depends on the log-likelihood function such that

$$\mathrm{AIC} = 2k - 2\ell(\hat{\Theta}),$$

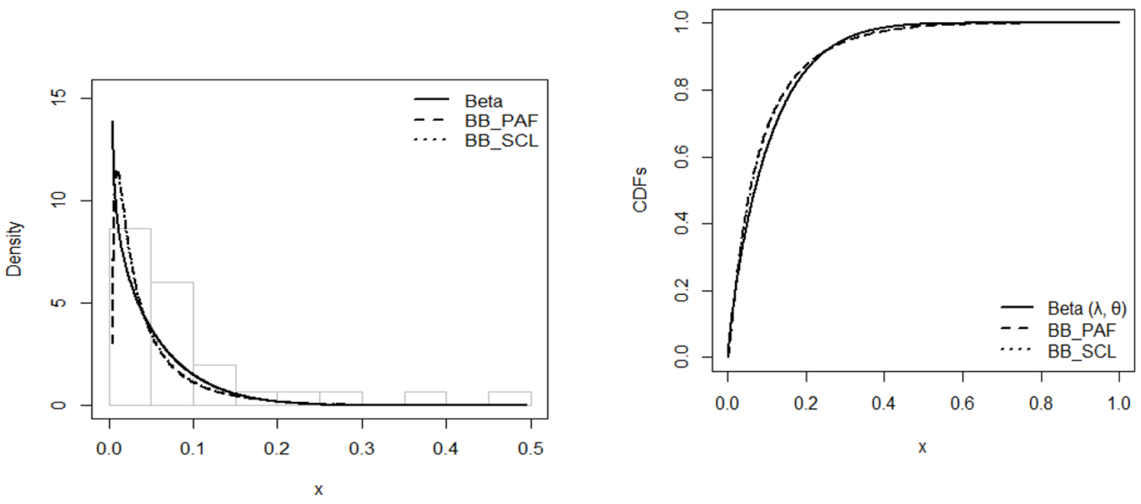
where k is the number of estimated parameters in the model, and $\ell(\hat{\Theta})$ is the maximized value of the likelihood function at the MLEs $\hat{\Theta}$.

First Data Set: Snowfall Amounts

The first data set contains 30 observations of daily snowfall amounts, recorded in inches of water, collected during a study carried out in the vicinity of Climax, Colorado [2]. Appendix C includes the full data set. For these data, we fit the Beta–Beta distribution as defined in Section 3.4. Both transformations have been used to produce the corresponding pdfs. For shortening, we call these two densities BBPAF (for $0 \leq x \leq 1$ and $\alpha, \beta, a, b > 0; n = 1$) and BBSCL (for $0 \leq x \leq 1$ and $\alpha, \beta, a, b, \gamma > 0$) respectively.

Second Data Set: Water Storage Levels

The second data set includes the monthly water storage level of the Shasta reservoir, spanning the months of January 1999 till December 2016 [14]. All observations have been normalized by dividing by the total



(a) (b)

Figure 5: Estimated Beta–Beta (PAF and SCL transformers) pdfs (a) and the CDFs (b) for 30 days of snowfall amounts in Climax, Colorado.

capacity of the Shasta reservoir, which is 4,552,000 acre-feet. Appendix C includes the full data set.
For this data set, we fit the Beta–Kumaraswamy distribution as defined in Section 3.3, using the PAF and SCL transformations to construct the associated distributions. For abbreviation, we call these two densities BKPAF (for $0 \leq x \leq 1$ and $\alpha, \beta, a, b > 0; n \in \mathbb{N}$) and BKSCL (for $0 \leq x \leq 1$ and $\alpha, \beta, a, b, \gamma > 0$) respectively.
Table 4 presents the MLEs of the parameters with standard errors beneath, along with AIC values for the Beta–Beta and Beta–Kumaraswamy distributions for the two data sets. These results demonstrate the distributional flexibility of the Beta–Beta and Beta–Kumaraswamy forms and highlight the significance of incorporating additional parameters to achieve improved fits.
Although the Beta–X distribution with the SCL transformation is comparatively less effective in fitting the data than the other two distributions, it still performs reasonably well. In contrast, the Beta–X distribution with the PAF transformation emerges as the best-fitting model. A notable aspect of the estimated distributions for BK$_{PAF}$ and BK$_{SCL}$, as shown in Figure 6, is their heavy-tailed nature, which indicates a degree of robustness to potential outliers.

9. CONCLUSIONS

Building on the framework of extended families of distributions as discussed by Merovci et al. (2016), we derive the general mathematical properties of a newly proposed wider beta family of distributions by introducing novel transformer functions. These transformers can extend several well-known distributions, including the uniform, exponential, and Kumaraswamy distributions. The resulting Beta–X density can be represented as a weighted mixture of Beta–X density functions—a formulation that facilitates the derivation of key structural properties in a general form. These properties include, but are not limited

Application	Model	Estimates					AIC
First data set	$\mathbf{BB}_{PAF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}, \boldsymbol{b}, \mathbf{1})$	92.5699	59.3941	0.0747	0.0862	–	-75.56
		195.4300	84.4323	0.0961	0.1142	–	
	$\mathrm{BB}_{SCL}(\alpha, \beta, a, b, \gamma)$	118.3685	56.5390	0.0415	0.1284	0.0100	-73.31
		93.2016	82.8779	0.0341	0.1457	0.0078	
	$\mathrm{Beta}(\lambda, \theta)$	0.8576	7.8057	–	–	–	-75.12
		0.1924	2.2431	–	–	–	
Second data set	$\mathbf{BK}_{PAF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$	5.021	0.2372	2.2901	5.8017	5.4870	-126.79
		0.7973	0.0181	0.0568	0.0451	0.0454	
	$\mathrm{BK}_{SCL}(\alpha, \beta, a, b, \gamma)$	18.2223	0.6954	0.3439	1.8063	0.0016	-124.04
		23.1195	0.4170	0.4392	0.9061	0.0000	
	Kumaraswamy(a, b)	2.9571	1.6028	–	–	–	-126.09
		0.2331	0.1547	–	–	–	

Table 4: The MLEs of the parameters with standard errors beneath and AIC of the Beta–Beta and Beta–Kumaraswamy distributions for the two data sets.

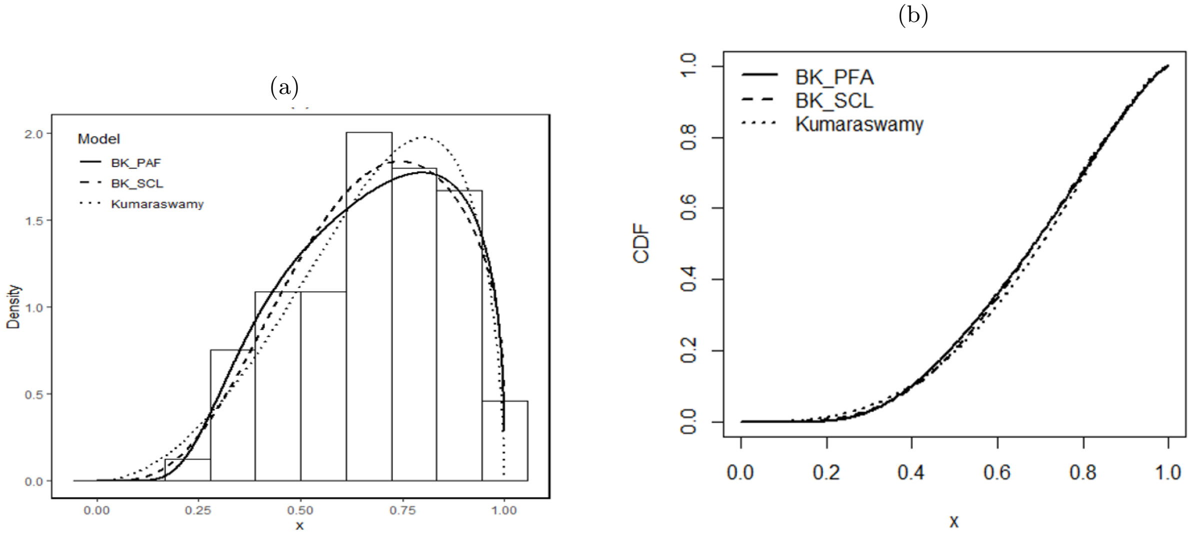


Figure 6: Estimated Beta–Kumaraswamy (PAF and SCL transformers) pdfs (a) and CDFs (b) for the monthly water storage levels in the Shasta reservoir (1999–2016).

to, the ordinary moments, the quantile function, order statistics, and entropy.
For any given baseline distribution, the corresponding beta-generated form can be instantly analyzed using our results. Parameter estimation is conducted via the method of maximum likelihood, and the observed information matrix is derived to support inference. To illustrate the practical applicability of the proposed family, we fit selected Beta–X distributions to two real-world data sets. This generalization is expected to broaden the range of statistical applications and analysis.

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