STRATEGIC PRICING AND INVENTORY SOLUTIONS FOR GRADUALLY DETERIORATING PRODUCTS WITH ENVIRONMENTAL CONSIDERATIONS

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ABSTRACT:

Goals

For vendors who deal with products that gradually deteriorate time, this study investigates the relationship between pricing strategies and inventory control. In order to identify the best-selling price, replenishment schedules, and order quantities that optimize profitability under various greenness levels, the study focuses on scenarios involving partial backlog during shortages and differing levels of green investment.

Method

Deterioration behaviour, shortage policies, and the impact of greenness on demand are all incorporated into a non-linear optimization model. To determine the best strategy for action, an analytical solution approach is used. To better predict consumer behaviour and so as to capture the nonlinear relationship between price and demand, a price-dependent quadratic demand function is used.

Findings

The findings show that, in spite of the associated investment costs, higher greenness levels increase consumer demand and overall profitability. Finding the ideal price point for sustainability and competitiveness is made possible by the quadratic demand function, which shows that consumer sensitivity to price is nonlinear. The model's applicability is demonstrated through a numerical example and sensitivity analysis, which highlights the importance parameters like deterioration rate, demand sensitivity and greenness.

Novelty

In an inventory model for deteriorating items, the novelty is in combining greenness levels with a quadratic price-dependent demand function. This method offers a more thorough and realistic framework for pricing and inventory decisions, especially for products like premium, trend-driven, or eco-friendly items where consumer perception is extremely price-sensitive. The study advances both theory and application by addressing a major gap in price-driven sustainable inventory research.

KEYWORDS: Inventory; pricing; quadratic demand; partial backlogging; non-instantaneous deterioration; greenness level.

MSC Code: 90B05

RESUMEN:

Objetivos

Para los proveedores que trabajan con productos que se deterioran gradualmente con el tiempo, este estudio investiga la relación entre las estrategias de precios y el control de inventario. Para identificar el mejor precio de venta, los programas de reabastecimiento y las cantidades de pedido que optimizan la rentabilidad con diferentes niveles de sostenibilidad, el estudio se centra en escenarios que implican una acumulación parcial de pedidos durante periodos de escasez y diferentes niveles de inversión en sostenibilidad.

Métodos

El comportamiento de deterioro, las políticas de escasez y el impacto de la sostenibilidad en la demanda se incorporan en un modelo de optimización no lineal. Para determinar la mejor estrategia de acción, se utiliza un enfoque de solución analítica. Para predecir mejor el comportamiento del consumidor y captar la relación no lineal entre precio y demanda, se utiliza una función de demanda cuadrática dependiente del precio.

Resultados

Los resultados muestran que, a pesar de los costes de inversión asociados, unos niveles de sostenibilidad más elevados aumentan la demanda del consumidor y la rentabilidad general. La función de demanda cuadrática permite determinar el precio ideal para la sostenibilidad y la competitividad, lo que demuestra que la sensibilidad del consumidor al precio no es lineal. La aplicabilidad del modelo se demuestra mediante un ejemplo numérico y un análisis de sensibilidad, que destaca parámetros importantes como la tasa de deterioro, la sensibilidad a la demanda y el grado de sostenibilidad.

Novedad

En un modelo de inventario para artículos en deterioro, la novedad reside en combinar los niveles de sostenibilidad con una función de demanda cuadrática dependiente del precio. Este método ofrece un marco más completo y realista para la toma de decisiones sobre precios e inventario, especialmente para productos como artículos premium, de tendencia o ecológicos, donde la percepción del consumidor es extremadamente sensible al precio. El estudio avanza tanto en la teoría como en la aplicación al abordar una importante brecha en la investigación de inventarios sostenibles basados en precios.

PALABRAS CLAVE: Inventario; fijación de precios; demanda cuadrática; acumulación parcial; deterioro no instantáneo; grado de sostenibilidad.

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1. INTRODUCTION

Fruits, vegetables, meats, medications, chemicals, high-end clothing, and technology are all susceptible to deterioration [1]. It describes how physical changes or the course of time may reduce a product's initial quality or value. Inventory models for deteriorating products have been the focus of several studies over the last few decades [16]. Products that deteriorate gradually and retain their quality for a predetermined amount of time before deteriorating are a noteworthy subset of this category. Fresh fruits, meats, and specific food items are a few examples. In addition to an inventory control model specifically created for such items, the idea was first presented in [47].

Optimizing order quantities and replenishment schedules to increase revenue or reduce expenses is a key component of effective inventory management [23], [9], and [10]. Inventory models must take non-instantaneous deterioration into account because some inventory is affected by it. Sales price is a key driver of purchasing behaviour, and demand is a crucial component of inventory control [8], [7]. Therefore, inventory policies for non-instantaneously deteriorating products are greatly impacted by pricing strategies.

Supply chain management, including inventory planning, has changed as a result of the increased focus on sustainability and environmental impact [14], [29]. Businesses have adopted green supply chain practices due to regulations, corporate social responsibility, competitive pressures, and customer expectations [21]. Research indicates that more than 80% of consumer purchasing decisions are influenced by environmentally friendly product features [12]. Businesses benefit economically and marketing-wise from a sustainable approach [13]. Throughout the supply chain, buyers and vendors who integrate environmental concerns into their operations gain financial rewards. In order to meet market demands, inventory planning must incorporate green initiatives, especially in supply chains for non-instantaneous deteriorating items.

There are various benefits to using a price-dependent quadratic demand function. It can demonstrate how consumers respond to price changes in a nonlinear manner, exhibiting stability at moderate price levels and increased elasticity at extreme values, in contrast to linear or exponential forms. Because of this more practical framework for pricing and inventory decisions, businesses can now identify competitive price points that maximize profits. This tactic is particularly effective for products where price significantly affects consumer perception, such as luxury goods, fashion items, and eco-friendly substitutes. Also, by providing a more thorough analytical tool, the use of a quadratic demand structure fulfils a significant gap in the literature on price-driven and sustainable inventory models.

But there are some disadvantages as well. Quadratic models are more difficult to estimate and calibrate than linear demand, and they often require reliable data to avoid overfitting. They may not be appropriate for all product categories, particularly those with nearly linear demand responses.

This research seeks to address critical challenges in managing non-instantaneously deteriorating items by focusing on the following objectives:

Determining optimal pricing and replenishment schedules under deterioration, shortages, and green initiatives.

Evaluating the influence of green policies on demand, pricing, and inventory strategies.

Exploring multiple tiers of green practices with varying costs to identify the optimal greenness level that maximizes profitability.

To accomplish these objectives, this study employs a quadratic demand function, differing from the linear models utilized in prior research, such as [22]. This choice allows for a nuanced analysis of how varying levels of greenness impact optimal solutions, contributing fresh insights to the field.

The rest of this paper is structured as follows. In Section 2, the relevant literature is reviewed. The model's assumptions and notations are described in Section 3, and the detailed model formulation is given in Section 4. Section 5 provides the algorithm for the suggested solution. A numerical example illustrating the model's applicability is presented in Section 6, and Section 7 offers sensitivity analysis and important managerial insights. Section 8 wraps up the paper by summarizing the key conclusions and offering recommendations for further research directions.

2. LITERATURE REVIEW

The idea of non-instantaneous deterioration was firstly introduced by [47], various inventory control models have been developed for such kinds of items. Relevant studies are categorized using key criteria. The choice of demand function has a significant impact on inventory policies, order quantities, replenishment schedules, and overall profitability. A variety of demand functions, such as time and price dependent, stock-dependent, price-dependent, and constant demand, have been investigated.

Some studies incorporate shortages into inventory models as they are frequently observed in real-world inventory systems. These shortages fall into three categories: partially backlogged (where some customers wait while others choose alternatives), backlogged (where loyal customers wait for the next replenishment cycle), and lost sales (where customers switch to competitors due to unfulfilled demand).

A few researchers examined the inventory models for non-instantaneously deteriorating products which incorporate sustainability and green initiatives. Inflation, monetary value, and marketing tactics (such as trade credit policies, promotions, and advertisements) have all been investigated. This study focuses on creating a new inventory model for a retailer managing deteriorating products under green investment, though other factors like supply chain design, methodology, solution approaches, and inventory policies could be taken into account. Consequently, the selected criteria provide a more pertinent classification of current research in this area.

2.1 Non-instantaneous deterioration

Based on the foundational model by [47], subsequent studies have introduced innovative models. [4] proposed a new algorithm. [39] studied demand influenced by pricing and advertising. Other notable works include [17], which explored time and price dependent demand, [44] incorporated variable holding costs in stock dependent demand models. [2] developed multi-product inventory systems using advanced objective functions. [28] developed a mixed-integer non-linear fractional model by combining pricing and transportation in a vendor-buyer supply chain. [42] focused on pricing and replenishment strategies for non-instantaneous deteriorating products, integrating stochastic demand and promotional factors into their analysis. They also explored supply chain design, addressing inventory control and the effects of stochastic demand and disruptions. With consideration of environmental emission rates, [25] developed an inventory model for items with non-instantaneous deterioration. [27] developed an inventory model with two warehouses for non-instantaneous deteriorating items having price and time dependent demand. [45] developed an EOQ model for non-instantaneous deteriorating items under partial backlogging and time dependent demand. [5] considered profit maximization inventory model for non-instantaneous deteriorating items such as fashionable goods which are having shorter life spans achieve higher demand as they enter the market and gradually demand decreases with time. [19] studied the joint effect of preservation technology and linearly time dependent holding cost when order-size linked to advance payment for inventory models with non-instantaneous deteriorating items. [18] considered an inventory model for non-instantaneous deteriorating items with freshness, price and green efforts dependent demand with price discount.

2.2 Sustainability and green practices

Incorporating sustainability, [43] proposed an inventory model combining fuzzy approaches to reduce carbon emissions while maximizing profits. Similarly, [9] and [10] introduced sustainable supply chain models addressing environmental risks. Further, [24] and [26] developed models emphasizing investments in green practices and preservation technology, expanding them later to include shortages and carbon emission control.

[16] developed a conceptual model emphasizing sustainable procurement within the Danish supply chain. Meanwhile, [11] incorporated pricing strategies and green quality factors to align supply chain operations, considering consumer awareness of environmental concerns. [37] presented sustainable production inventory model by considering green technology investment for perishable products. [38] developed sustainable inventory model with credit and price dependent demand having carbon emission cap. [6] considered multi-phase sustainable inventory model for seasonal products with eco-friendly packaging.

2.3 Quadratic demand inventory models

[36] considered EOQ model with price sensitive quadratic demand and trade credit. [46] developed a pharmaceutical inventory model with demand rate as quadratic function of time. [30] proposed an EOQ model for items having three-parameter Weibull distribution deterioration and quadratic demand rate with quasipartial backlogging. [33] formulated a quadratic demand three layered integrated inventory model for deteriorating items with two-level trade credit financing. [34] analysed the impact of advertisement on retailer's inventory with non-Instantaneous deterioration under-price dependent quadratic demand. [35] developed an inventory control policy for substitutable deteriorating items with quadratic demand. [3] developed an inventory model having two storage facilities for non-instantaneous deteriorating items with time dependent quadratic demand with shortages having trade credit policy. [41] considered a quadratic demand EOQ model for deteriorating items. For log-gamma deteriorating items, [32] developed a partial back-ordering

inventory model with quadratic demand. [15] developed a stochastic inventory model with quadratic price sensitive demand. For inventory model with three warehouses, [20] developed a quadratic demand inventory model with time varying holding costs and backlogging over finite time horizon. [40] considered a EOQ model having quadratic demand for Pareto distributed decaying products. [31] developed three models. In the first model, Quadratic time-dependent and price-dependent demands are used. In second model quadratic-time dependent and in the third model quadratic price dependent demands are used so as to identify the optimum cycle time and the optimum quantity that minimises the total cost.

3. NOTATIONS AND ASSUMPTIONS

The study using the following notations and assumptions.

Notations

c :	The purchasing cost per unit
h :	The holding cost per unit per unit time
s:	The backorder cost per unit per unit time
l :	The cost of lost sale per unit
p :	The selling price per unit
θ :	The deterioration rate parameter
t_d :	The length of time in which product has no deterioration
t_1:	The length of time in which there is no inventory shortage
T :	The length of the inventory replenishment cycle
Q :	The order quantity
α :	The market potential
β, ξ :	Consumer price elasticity
$I_1(t)$:	The inventory level at time $t \in [0, t_d]$
$I_2(t)$:	The inventory level at time $t \in [t_d, t_1]$
$I_3(t)$:	The inventory level at time $t \in [t_1, T]$
I_0 :	The maximum inventory level
S :	The maximum amount of demand backlogged
w	The level of greenness. The retailer can select a greenness level from a pre-designed greenness level set $w \in \{w_1, w_2, \cdots, w_n\}$
η	Consumer greenness level elasticity
K	The coefficient of retailer's greenness investment, $K > 0$
$TP(p, t_1, T)$	The total profit per unit time of the inventory system
p^*	The optimal selling price per unit
t_1^*	The optimal length of time in which there is no inventory shortage
T^*	The optimal length of the inventory replenishment cycle
Q^*	The optimal order quantity
TP^*	The optimal total profit per unit time of the inventory system i.e. when $TP^* = TP(p^*, t_1^*, T^*)$

Assumptions

1) A single non-instantaneous deteriorating item is assumed.
2) The replenishment rate is infinite and lead time is zero.
3) The price and greenness level dependent demand function defined by $D(p, w) = (\alpha - \beta p - \xi p^2) + \eta w$ where $\alpha > \beta$ and $\alpha, \beta, \xi > 0$ is decreasing quadratic function of the price.

   The above defined demand function represents the products that are price sensitive. It is suitable for items whose demand decreases with rising price, it models products with declining demand, such as seasonal goods, electronics, or fashion items that lose appeal over time.

4) During each shortage cycle, only a portion of the unmet demand is backordered, depending on the waiting time until the next replenishment. This study utilizes a backlogging function $\gamma(x) = \frac{1}{1+\delta x}$, where δ is the backlogging parameter $(\delta > 0)$ and x represents the waiting time until the next inventory replenishment.
5) The length of time in which the product exhibits no deterioration is less than or equal to the length of time in which there is no inventory shortage, i.e. $t_d \le t_1$.

4. MODEL FORMULATION

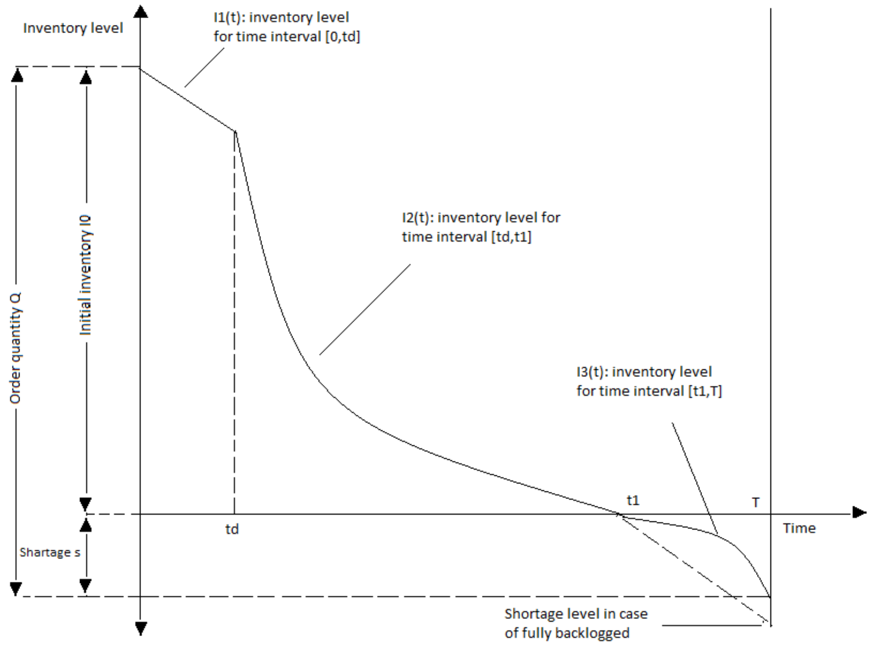

Figure 1. An inventory system for non-instantaneous deteriorating items

The inventory system is structured as follows. At the commencement of each cycle, I_0 units of the item are introduced into the inventory system and subsequently deplete to zero as a result of degradation and demand. A shortage exists until the end of the current order cycle. It is assumed that the duration of the shortage is either equal to or greater than the duration of the product's non-deterioration. During the time interval $[0, t_d]$, the inventory level decreases solely due to demand. The inventory level subsequently decreases to zero as a result of both increased demand and deterioration during the time interval $[t_d, t_1]$. Finally, a shortage occurs during the time interval $[t_1, T]$ as a consequence of partial backlogging and demand. The differential equation that represents the inventory status over the period $[0, t_d]$ is given by

$$\frac{dI_1(t)}{dt} = -D(p,w),\ \ 0 \le t \le t_d \tag{1}$$

The solution to differential equation (1) can be found by applying the initial condition $I_1(0) = I_0$.

$$I_1(t) = -t\,D(p,w) + I_0,\ \ 0 \le t \le t_d \tag{2}$$

Both product deterioration and demand fulfilment cause the inventory level to decrease over time interval $[t_d, t_1]$. As a result, the inventory status during this time is represented by differential equation (3).

$$\frac{dI_2(t)}{dt} = -D(p,w) - \theta I_2(t),\ \ t_d \le t \le t_1 \tag{3}$$

where θ denotes the constant deterioration parameter. When the inventory level falls to zero, the condition $I_2(t_1)=0$ occurs. Under these conditions, equation (3) can be solved, and the solution is

$$I_2(t)=\frac{\left(-1+e^{(t_1-t)\theta}\right)D(p,w)}{\theta},\ t_d\le t\le t_1 \tag{4}$$

The inventory level t_d is the same for both $I_1(t)$ and $I_2(t)$, as Figure 1 illustrates. As a result $I_1(t_d)=I_2(t_d)$. This suggests that the following expression can be used to determine the initial inventory level, which also serves as the maximum inventory level I_0 in each cycle.

$$I_0=\frac{\left(-1+e^{(t_1-t_d)\theta}\right)D(p,w)}{\theta}+t_dD(p,w) \tag{5}$$

The following expression for the inventory level over time $[0,t_d]$ can be obtained by substituting equation (5) into equation (2).

$$I_1(t)=D(p,w)\left[-t+\frac{\left(-1+e^{(t_1-t_d)\theta}\right)}{\theta}+t_d\right] \tag{6}$$

A shortage occurred during the third interval $[t_1,T]$, and the fraction $\gamma(T-t)$ indicates that the demand is partially backlogged. Then the following differential equation governs the inventory level at time t:

$$\frac{dI_3(t)}{dt}=-D(p,w)\gamma(T-t)=\frac{-D(p,w)}{1+\delta(T-t)},\ t_1\le t\le T \tag{7}$$

The solution of equation (7) with the condition $I_3(t_1)=0$ is given by

$$I_3(t)=-\frac{D(p,w)}{\delta}\left(\log\left(1+\delta(T-t_1)\right)-\log\left(1+\delta(T-t)\right)\right),\ t_1\le t\le T \tag{8}$$

The maximum amount of demand backlog will be as follows if we put $t=T$ into equation (8)

$$S=-I_3(T)=\frac{D(p,w)}{\delta}\log\left(1+\delta(T-t_1)\right) \tag{9}$$

The order quantity $Q=S+I_0$

$$Q=S+I_0=D(p,w)t_d+\frac{\left(-1+e^{(t_1-t_d)\theta}\right)D(p,w)}{\theta}+\frac{D(p,w)}{\delta}\log\left(1+(T-t_1)\delta\right) \tag{10}$$

The following seven elements can now be used to calculate sales revenue and inventory costs per cycle:

Ordering Cost: $C_O=A$

Holding Cost: C_H

$$C_H=h\left[\int_0^{t_d}I_1(t)dt+\int_{t_d}^{t_1}I_2(t)dt\right]=\frac{hD(p,w)t_d\left(-2+2e^{(t_1-t_d)\theta}+t_d\theta\right)}{2\theta}+\frac{hD(p,w)\left(-1+e^{(t_1-t_d)\theta}-t_1\theta+t_d\theta\right)}{\theta^2} \tag{11}$$

Greening Cost: C_G

$$C_G=Kw^2 \tag{12}$$

The investment coefficient $K>0$ and the retailer's intrinsic greenness level w determine the greening cost, which is represented by Kw^2 in equation (12). Because of their complexity and limited availability, advanced green initiatives come with much higher costs than basic sustainability efforts, as this quadratic form illustrates.

Backlog Shortage Cost: C_{BS}

$$C_{BS}=s\left[\int_{t_1}^{T}-I_3(t)dt\right]=-\frac{sD(p,w)}{\delta}\left(t_1-T+\frac{\log\left(1+(T-t_1)\delta\right)}{\delta}\right) \tag{13}$$

Lost sale Shortage Cost: C_{LS}

$$C_{LS}=l\left[\int_{t_1}^{T}D(p,w)\big(1-\gamma(T-t)\big)dt\right]=lD(p,w)\left(T-t_1-\frac{\log\big(1+\delta(T-t_1)\big)}{\delta}\right) \tag{14}$$

Purchase Cost: C_p

$$C_P=cQ\;=c\left(\frac{D(p,w)\log\big(1+(T-t_1)\delta\big)}{\delta}+\frac{\left(-1+e^{(t_1-t_d)\theta}\right)D(p,w)}{\theta}+t_dD(p,w)\right) \tag{15}$$

Sales Revenue: R_S

$$R_S=p\left[\int_0^{t_1}D(p,w)dt+S\right]=p\left(D(p,w)t_1+\frac{D(p,w)}{\delta}\log\big(1+(T-t_1)\delta\big)\right) \tag{16}$$

Accounting for all inventory-related expenses and income produces the total profit function. It shows the average profit made by the retailer for each inventory cycle. In an inventory cycle, $TP(p,w,t_1,T)$ stands for the retailer 's average total profit.

$$TP(p,w,t_1,T)=\frac{R_S-C_O-C_H-C_{BS}-C_{LS}-C_P-C_G}{T}$$

$$=\frac{D(p,w)}{T}\left(\begin{array}{l} pt_1-\dfrac{ht_d\left(-2+2e^{(t_1-t_d)\theta}+t_d\theta\right)}{2\theta}-\dfrac{h\left(-1+e^{(t_1-t_d)\theta}-t_1\theta+t_d\theta\right)}{\theta^2}\\ +\dfrac{s}{\delta}(t_1-T)-l(T-t_1)-\left(\dfrac{c\left(-1+e^{(t_1-t_d)\theta}\right)}{\theta}+ct_d\right)\\ +\log\big(1+(T-t_1)\delta\big)\left(\dfrac{s+\delta(l+p-c)}{\delta^2}\right)-\dfrac{\left(A+Kw^2\right)}{D(p,w)} \end{array}\right) \tag{17}$$

Four factors affect the total profit function: p,w,t_1,T. It is necessary to determine the optimal values of the remaining three variables while choosing w from the ordinary set. For any given value of p, we can maximize $TP(p,w,t_1,T)$, thus satisfying the necessary condition $\frac{\partial TP(p,w,t_1,T)}{\partial T}=0$ and $\frac{\partial TP(p,w,t_1,T)}{\partial t_1}=0$ simultaneously. That is

$$\frac{\partial TP}{\partial t_1}=-\frac{\left(-e^{\theta t_1}h\left(e^{-\theta t_1}\right)^2+\left(e^{\theta t_1}he^{-t_d\theta}-\big((l+p)(T-t_1)\delta+Ts-st_1+c\big)\dfrac{\theta}{\big(1+\delta(T-t_1)\big)}\right)e^{-\theta t_1}+(ht_d+c)\theta e^{-t_d\theta}\right)\left(\alpha-\beta p-\xi p^2+\eta w\right)}{T\theta e^{-\theta t_1}}=0 \tag{18}$$

$$\frac{\partial TP}{\partial T}=\frac{\begin{aligned}&2\big(-1+(t_1-T)\delta\big)e^{-\theta t_1}\big((c-l-p)\delta-s\big)\theta^2\ln\big(1+\delta(T-t_1)\big)\\ &+2\delta\left(\alpha-\beta p-\xi p^2+\eta w\right)\left(\begin{array}{l} -\big(1+(t_1-t_d)\theta\big)\big(-1+(t_1-T)\delta\big)h\delta e^{\theta t_1}\left(e^{-\theta t_1}\right)^2\\ +\left(\begin{array}{l}\big(-1+(t_1-T)\delta\big)h\delta e^{\theta t_1}e^{-t_d\theta}\\ +\left(\begin{array}{l}(t_1-T)\left(\left(\begin{array}{l}t_1\xi p^3+\left((\xi l+\beta)t_1-\frac{1}{2}t_d\xi(ht_d+2c)\right)p^2+\left((\beta l-\eta w-\alpha)t_1-\frac{1}{2}t_d\beta(ht_d+2c)\right)p-l(\eta w+\alpha)t_1\\ +\frac{1}{2}t_d^2(\eta w+\alpha)h+c(\eta w+\alpha)t_d+Kw^2+A\end{array}\right)\dfrac{\theta}{\left(\alpha-\beta p-\xi p^2+\eta w\right)}-(ht_d+c)\right)\delta^2\\ +\left(\left(\begin{array}{l}-\xi(t_1-T)p^3+\left(\begin{array}{l}t_1^2\xi s+(-T\xi s-\xi l-\beta)t_1\\ +\big((-c+l)\xi+\beta\big)T+\frac{1}{2}t_d\xi(ht_d+2c)\end{array}\right)p^2\\ +\left(\begin{array}{l}t_1^2\beta s+(-T\beta s-\beta l+\eta w+\alpha)t_1\\ +\big(-\eta w-\alpha+(-c+l)\beta\big)T+\frac{1}{2}t_d\beta(ht_d+2c)\end{array}\right)p-s(\eta w+\alpha)t_1^2\\ +(Ts+l)(\eta w+\alpha)t_1+(c-l)(\eta w+\alpha)T\\ -\frac{1}{2}t_d^2(\eta w+\alpha)h-c(\eta w+\alpha)t_d-Kw^2-A\end{array}\right)\dfrac{\theta}{\left(\alpha-\beta p-\xi p^2+\eta w\right)}+(ht_d+c)\right)\delta+\theta s(t_1-T)\end{array}\right)\theta\end{array}\right)e^{-\theta t_1}\\ +\big(-1+(t_1-T)\delta\big)e^{-td\theta}(ht_d+c)\delta\theta\end{array}\right)\end{aligned}}{2\big(-1+(t_1-T)\delta\big)e^{-\theta t_1}\delta^2T^2\theta^2}=0 \tag{19}$$

T and t_1 are the results of solving the previously mentioned simultaneous equation systems.

Theorem
For any given value of p the system of equations (18) and (19) admits a unique solution that satisfies the second-order conditions for a maximum of the total profit function.

Proof. Refer to Appendix.
Theorem is formulated based on the structure of the total profit function $TP(p,w,t_1,T)$. This is a multivariable function where p is treated as constant and w is chosen from a predefined set. As a result, the function reduces to a bivariate form depending only on the decision variables T and t_1. The function $TP(p,w,t_1,T)$ attains a maximum at a critical point if and only if the following second-order conditions are satisfied: $\frac{\partial^2 TP(p,w,t_1,T)}{\partial t_1^2} < 0$ and $\frac{\partial^2 TP(p,w,t_1,T)}{\partial t_1^2} \cdot \frac{\partial^2 TP(p,w,t_1,T)}{\partial T^2} - \frac{\partial^2 TP(p,w,t_1,T)}{\partial t_1 \partial T} > 0$
Further details and derivations are provided in the Appendix.
Next, we examine the condition under which the optimal selling price exists and is unique. For any given values T^* and t_1^* the first-order derivative condition with respect to p for maximizing $TP(p,w,t_1^*,T^*)$ is $\frac{\partial TP(p,w,t_1^*,T^*)}{\partial p} = 0$. That is,

$$\frac{\partial TP}{\partial p} = \frac{\begin{array}{l} 2\left(\left(-3\xi p^2 + \left((2c-2l)\xi - 2\beta\right)p + (c-l)\beta + \eta w + \alpha\right)\delta - 2\left(\xi p + \frac{\beta}{2}\right)s\right)\theta^2 e^{-\theta t_1^*}\ln\left(1+\delta\left(T^*-t_1^*\right)\right) \\ +4\left(\begin{array}{l} -\left(\xi p + \frac{\beta}{2}\right)e^{\theta t_1^*}h\delta\left(1+\left(t_1^*-t_d\right)\theta\right)\left(e^{-\theta t_1^*}\right)^2 + \left(\delta e^{\theta t_1^*}h\left(\xi p + \frac{\beta}{2}\right)e^{-t_d\theta} + \theta\left(\left(\left(\begin{array}{l} -\frac{3}{2}\xi p^2 t_1^* + \left(\left(T^* l + \frac{1}{2}t_d^2 h + c t_d - t_1^* l\right)\xi - \beta t_1^*\right)p \\ +\left(\frac{1}{4}t_d^2 h + \frac{1}{2}T^* l + \frac{1}{2}c t_d - \frac{1}{2}t_1^* l\right)\beta + \frac{1}{2}t_1^*(\eta w + \alpha) \end{array}\right)\theta - \left(\xi p + \frac{\beta}{2}\right)(h t_d + c)\right)\delta + \left(\xi p + \frac{\beta}{2}\right)s\theta\left(T^*-t_1^*\right)\right)\right)e^{-\theta t_1^*} \\ +\left(\xi p + \frac{\beta}{2}\right)(h t_d + c)\theta\delta e^{-t_d\theta} \end{array}\right)\delta \end{array}}{2\delta^2\theta^2 e^{-\theta t_1^*}T^*} = 0 \tag{20}$$

The second derivative of $TP(p,w,t_1^*,T^*)$ with respect to p is $\frac{\partial^2 TP(p,w,t_1^*,T^*)}{\partial p^2}$

$$\frac{\partial^2 TP}{\partial p^2} = \frac{\begin{array}{l} -2e^{-\theta t_1^*}\left(\left((-c+l+3p)\xi + \beta\right)\delta + s\xi\right)\theta^2 \ln\left(1+\delta\left(T^*-t_1^*\right)\right) \\ +2\left(\begin{array}{l} -\left(1+\left(t_1^*-t_d\right)\theta\right)\xi\delta e^{\theta t_1^*}h\left(e^{-\theta t_1^*}\right)^2 + \left(\delta e^{\theta t_1^*}e^{-t_d\theta}h\xi + \left(\left(\begin{array}{l} \left(\left(\left(\frac{1}{2}t_d^2 h + (-l-3p)t_1^* + T^* l + c t_d\right)\xi - \beta t_1^*\right)\theta - \xi(h t_d + c)\right)\delta \\ +\xi s\theta\left(T^*-t_1^*\right) \end{array}\right)\theta\right)e^{-\theta t_1^*}\right) \\ +\delta\theta e^{-t_d\theta}\xi(h t_d + c) \end{array}\right)\delta \end{array}}{\delta^2\theta^2 e^{-\theta t_1^*}T^*} < 0 \tag{21}$$

Thus, for the given values T^* and t_1^* the function $TP(p,w,t_1^*,T^*)$ is concave with respect to p. Therefore, there exists a unique value p i.e. p^* that maximizes the total profit $TP(p,w,t_1^*,T^*)$. The optimal selling price p^* is obtained by solving equation (20).

5. ALGORITHM

A simple algorithm is proposed to obtain the optimal solution of the problem.

Step 1. Start with $w = w_i$ (initial value $i = 1$)

Step 2. Consider a starting value for the selling price, p_1 and assume $j = 0$. Then, $p_j = p_1$.

Step 3. For a given price p_j, compute t_1^* and T^* according to equations (18) and (19).

Step 4. Put t_1^* and T^* in eq. (20) and determine the next selling price value p_{j+1}.

Step 5. If the difference between two consecutive selling prices (p_j and p_{j+1}) is insignificant (for example, $\left|p_{j+1}-p_j\right| \le 10^{-4}$), then consider that $p^* = p_{j+1}$, $\left(w_i, p^*, t_1^*, T^*\right)$ will be the optimal solution and the algorithm stops. Otherwise, $j = j+1$ and go back to step 3.

Step 6. Determine the optimal order quantity and total profit using equations (10) and (17), respectively.

Step 7. If $i \ge n$ (n is the number of the greenness level), stop. Else, set $i = i+1$ and go back to step 1.

6. NUMERICAL EXAMPLE

To demonstrate the practical implementation of the proposed model and solution method, the following numerical example is presented. This example outlines the solution process by analysing an inventory scenario with the following given data.

$A = \$\ 500000/\text{order}$, $K = 2000000$, $\alpha = 1300000$, $\beta = 1250$, $\xi = 0.1$, $c = \$\ 100/\text{unit}$, $\delta = 0.1$,
$\eta = 50000$, $h = \$\ 20/\text{unit}$, $l = \$\ 150/\text{unit}$, $s = \$\ 100/\text{unit/unit time}$, $t_d = 0.2\ \text{year}$, $\theta = 0.08$

The optimal solutions, with $w = 2$, include an optimal sales price $p^* = \$587.44/\text{unit}$, duration without inventory shortage $t_1^* = 0.6488$ year, replenishment cycle time $T^* = 0.9537$ year, optimal total profit $TP^* = \$\ 293255173.37$ and optimal order quantity $Q^* = 604246.73$ units per cycle. These values represent the key outcomes of the model.

The concavity behaviour of the objective function can be seen through Hessian matrix H as below,

$$H = \begin{pmatrix} \dfrac{\partial^2 TP(.)}{\partial t_1^2} & \dfrac{\partial^2 TP(.)}{\partial t_1 \partial T} & \dfrac{\partial^2 TP(.)}{\partial t_1 \partial p} \\ \dfrac{\partial^2 TP(.)}{\partial T \partial t_1} & \dfrac{\partial^2 TP(.)}{\partial T^2} & \dfrac{\partial^2 TP(.)}{\partial T \partial p} \\ \dfrac{\partial^2 TP(.)}{\partial p \partial t_1} & \dfrac{\partial^2 TP(.)}{\partial p \partial T} & \dfrac{\partial^2 TP(.)}{\partial p^2} \end{pmatrix}$$

where, $TP(.) = TP(p, t_1, T)$

The Hessian matrix at point $p^* = \$587.44,\ t_1^* = 0.6488\ \text{year},\ T^* = 0.9537\ \text{year}$ is

$$H = \begin{pmatrix} -1.2148\times10^8 & 8.0188\times10^7 & -25595.3629 \\ 8.0188\times10^7 & -7.2661\times10^7 & 34191.9364 \\ -25595.3629 & 34191.9364 & -2816.7682 \end{pmatrix}$$

The first principal minor of H is, $\left|H_{11}\right| = -1.2148\times10^8 < 0$,

The second principal minor of H is, $\left|H_{22}\right| = 2.3968\times10^{15} > 0$,

The third principal minor of H is, $\left|H_{33}\right| = -6.7019\times10^{18} < 0$

As the sign of principal minor is alternate, the Hessian matrix H is negative definite at point $\left(p^*, t_1^*, T^*\right)$. Hence, the total profit function yields global maximum at that point.

The concavity of the profit function with respect to the decision variables is confirmed, as illustrated in Figure 2, which visually depicts the behaviour of the objective function. The optimal solution is indicated by a solid dot. As shown in Figure 2, the optimal total profit is strictly concave with respect to p . This implies that the proposed algorithm ensures a total profit that is not only a local maximum but also the global maximum solution.

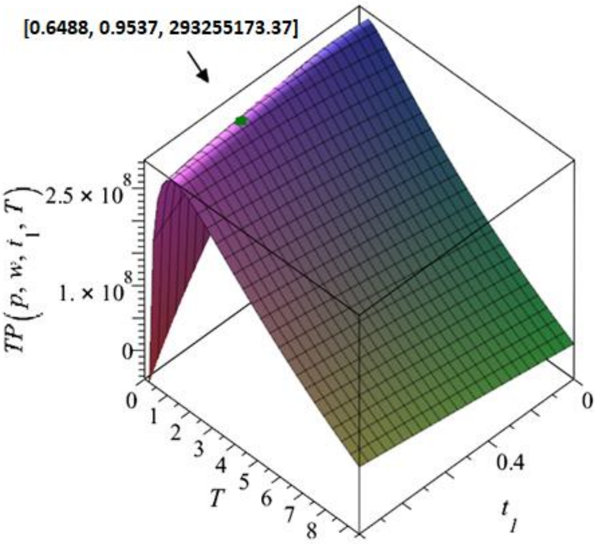

Figure 2. Profit function with respect to t_1 and T for fixed p

7. SENSITIVITY ANALYSIS AND MANAGERIAL INSIGHTS

7.1 Effect of Greenness level

Utilizing the data from the above example, the analysis explores the sensitivity of optimal decision variables and the associated total expected profit across various greenness levels.

The optimal outcomes for various values of w are presented in Table 1.

w	p^* (in $)	t_1^* (year)	T^* (year)	TP^* (in $)	Q^* (units)
1	575.97	0.1812	0.4008	278574395.26	237829.69
2	587.44	0.6488	0.9537	293255173.37	604246.73
3	606.10	1.0421	1.4304	308934230.58	952002.51
4	625.65	1.4081	1.8773	325224895.21	1307992.74
5	645.43	1.7532	2.3012	342078172.01	1674809.51
6	665.26	2.0804	2.7049	359481225.98	2053028.40
7	685.05	2.3915	3.0906	377430179.84	2442745.70
8	704.77	2.6882	3.4599	395924152.12	2843904.45
9	724.42	2.9716	3.8139	414963413.73	3256389.34
10	743.98	3.2428	4.1540	434548693.08	3680051.56

Table 1 Computational results for different values of w

Table 1 highlights key insights. As the greenness level increases, total profit also rises, underscoring the advantages of investing in sustainable practices. However, achieving higher greenness levels may necessitate advanced technology and equipment. The findings suggest businesses should aim for the highest feasible greenness level within their technological and financial limits. Furthermore, the results show that greater greenness levels are associated with higher sales prices and increased order quantities a well-documented benefit of green initiatives recognized by multiple researchers.

The sensitivity of the optimal decision variables and total profit was examined by varying different model parameters while maintaining the same data as in numerical example section. This analysis involved individually adjusting each parameter within a range of -20% to +20%, keeping all other parameters unchanged. The results of this sensitivity analysis are presented and discussed in Sections 7.2–7.5.

7.2 The effect of demand parameters

Parameters	**Values**	p^* (in \$)	t_1^* (year)	T^* (year)	TP^* (in \$)	Q^* (units)
	1040000	491.16	0.8281	1.1325	182531484.92	574214.81
	1170000	539.19	0.7379	1.0411	234745074.12	594342.28
α	1300000	587.44	0.6488	0.9537	293255173.37	604246.73
	1430000	636.18	0.5558	0.8647	358066361.89	600647.79
	1560000	686.03	0.4520	0.7665	429224285.16	577315.15
	1000	724.09	0.2776	0.6585	379751687.54	406328.88
	1125	643.87	0.5400	0.8746	331758699.34	554122.89
β	1250	587.44	0.6488	0.9537	293255173.37	604246.73
	1375	541.76	0.7140	0.9976	261441439.48	628448.18
	1500	503.53	0.7578	1.0257	234686491.95	640918.41
	0.08	591.38	0.7088	0.9872	296075068.38	628927.33
	0.09	589.30	0.6796	0.9714	294638374.45	617299.07
ξ	0.1	587.44	0.6488	0.9537	293255173.37	604246.73
	0.11	585.81	0.6162	0.9339	291925158.72	589636.96
	0.12	584.43	0.5815	0.9116	290648718.51	573289.30

Table 2 Computational results for demand parameters

The sensitive analysis shown in Table 2 indicates the following observations and managerial insights.
(i) As market potential α increases, total profit TP also increases indicating that a higher market potential α expands the baseline demand, allowing firms to capture greater sales and profit even when price sensitivity and time effects are present. Managers should prioritize strategies that boost α, such as market expansion, customer acquisition, and product innovation, as this parameter provides a strong, scalable lever for profitability growth.
(ii) When market potential α grows, customers are more willing to pay, enabling firms to set a higher optimal price. Managers should align pricing strategies with market expansion, leveraging greater demand to improve per-unit revenue.
(iii) As consumer price elasticity β and ξ increases, demand becomes more responsive to price changes forcing firms to lower prices and resulting in reduced total profit. This highlights the need for value differentiation and efficiency improvements to mitigate sensitivity effects and sustain profitability.
The trends for selling price, order quantity and total profit related to changing parameters α, β and ξ are depicted in Figure 3, Figure 4 and Figure 5 respectively.

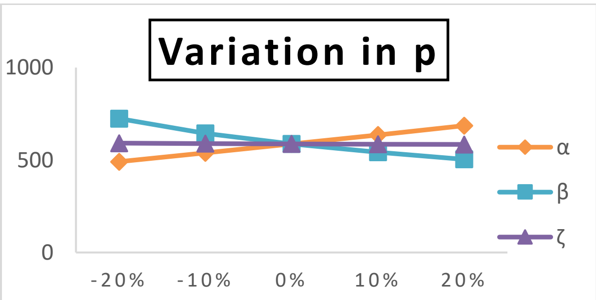

Figure 3. Variation in p with respect to α, β and ξ

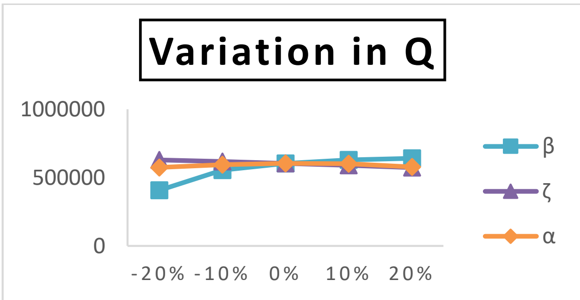

Figure 4. Variation in Q with respect to α, β and ξ

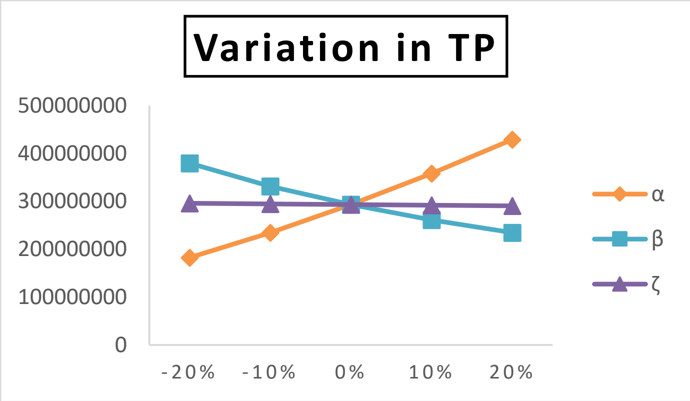

Figure 5. Variation in $TP(.)$ with respect to α , β and ξ

7.3 The effect of greenness parameters

Parameters	**Values**	p^* (in \$)	t_1^* (year)	T^* (year)	TP^* (in \$)	Q^* (units)
η	40000	579.99	0.6626	0.9671	283843951.81	603500.34
	45000	583.71	0.6557	0.9604	288530933.45	603910.69
	50000	587.44	0.6488	0.9537	293255173.37	604246.73
	55000	591.17	0.6418	0.9470	298016679.49	604504.22
	60000	594.89	0.6349	0.9403	302815461.01	604680.84
K	1600000	587.79	0.5531	0.8405	295038513.57	530738.57
	1800000	587.57	0.6026	0.8990	294118748.47	568704.1
	2000000	587.44	0.6488	0.9537	293255173.37	604246.73
	2200000	587.36	0.6922	1.0053	292438428.62	637788.88
	2400000	587.32	0.7333	1.0541	291661512.44	669641.97

Table 3 Computational results for greenness parameters

The sensitive analysis shown in Table 3 indicates the following observations and managerial insights.

(i) As consumer greenness elasticity η increases, demand rises because customers place more value on environmentally friendly products. This allows businesses to set higher prices, increase order quantities to meet growing demand, and achieve greater total profit. Investing in green features and targeting eco-conscious markets becomes more rewarding, as these consumers are willing to pay premiums for sustainable products. To fully capitalize on this trend, managers should focus on controlling costs and improving turnover speed, ensuring that the gains from higher greenness sensitivity translate into sustained profitability.

(ii) As the coefficient of the retailer's greenness investment K increases, both total profit and price decrease because higher investment costs put downward pressure on pricing decisions and reduce overall margins. Although sustainability efforts can boost demand, excessive spending on greenness diminishes profitability and limits pricing flexibility. Managers should optimize greenness investments to achieve environmental goals while maintaining competitive prices and healthy profits.

(iii) As the coefficient of the retailer's greenness investment K increases, the order quantity Q increases because greener products attract more demand despite the higher investment costs. Retailers respond to this stronger demand by stocking and ordering larger quantities to meet customer preferences for sustainable options. Managers should ensure that inventory planning aligns with this growth in demand while keeping investment levels efficient to protect profitability.

The trends for selling price, order quantity and total profit related to changing parameters K and η are depicted in Figure 6, Figure 7, Figure 8 respectively.

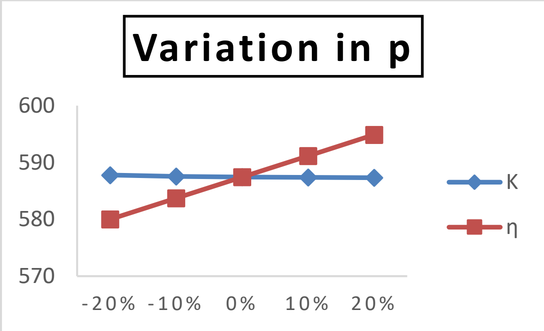

Figure 6. Variation in p with respect to K and η

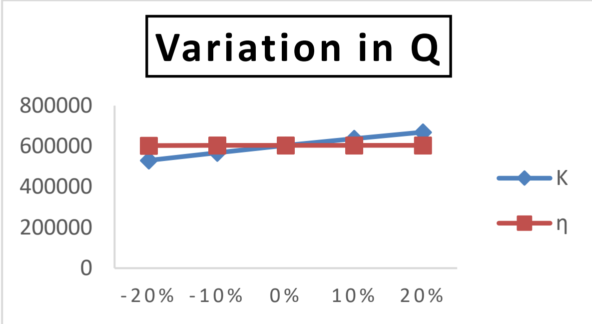

Figure 7. Variation in Q with respect to K and η

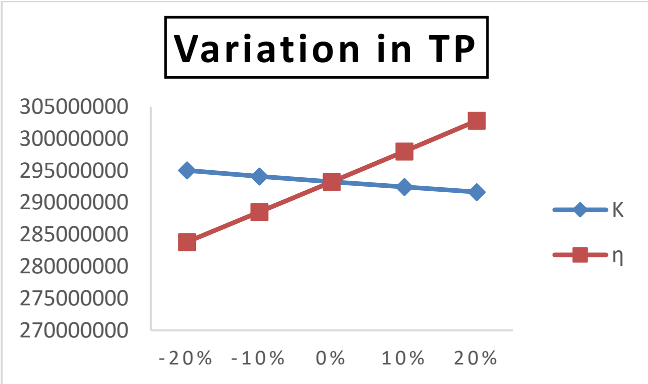

Figure 8. Variation in $TP(.)$ with respect to K and η

7.4 The effect of non-instantaneous deterioration on the outcomes

The impact of non-instantaneous deterioration on optimal solutions is analysed. As shown in Table 4, increasing non-instantaneous deterioration enhances total profit by preserving product quality for extended periods and reducing spoilage-related losses. With a lower risk of deterioration, vendors can increase order quantities and lower sales prices to attract more customers, ultimately driving higher demand. A longer inventory cycle without shortages improves stock availability, improving service levels and customer satisfaction. However, an extended replenishment cycle necessitates efficient supply chain coordination to prevent stock outs. To sustain profitability, managers should invest in advanced storage solutions and preservation technologies that extend product lifespan.

Parameters	**Values**	p^* (in \$)	t_1^* (year)	T^* (year)	TP^* (in \$)	Q^* (units)
t_d	0	588.50	0.6232	0.9346	292721111.22	590122.18
	0.1	587.95	0.6345	0.9424	293012729.52	596054.34
	0.2	587.44	0.6488	0.9537	293255173.37	604246.73
	0.3	586.96	0.6661	0.9685	293448496.94	614703.38
	0.4	586.51	0.6863	0.9869	293593361.72	627404.05

Table 4 The effect of non-instantaneous deterioration on the outcomes

7.5 Effect of other parameters

Parameters	**Values**	p^* (in \$)	t_1^* (year)	T^* (year)	TP^* (in \$)	Q^* (units)
c	80	577.37	0.6855	0.9760	306757925.71	632944.00
	90	582.39	0.6671	0.9647	299971964.04	618450.78
	100	587.44	0.6488	0.9537	293255173.37	604246.73
	110	592.52	0.6305	0.9429	286607799.65	590284.40
	120	597.63	0.6122	0.9322	280030139.90	576518.42

h	16	586.09	0.7174	1.0147	293881203.12	646450.24
	18	586.78	0.6812	0.9824	293557174.79	624069.20
	20	587.44	0.6488	0.9537	293255173.37	604246.73
	22	588.07	0.6197	0.9281	292972706.47	586545.62
	24	588.68	0.5933	0.9050	292707687.91	570622.89
s	80	588.67	0.6111	0.9525	293929171.68	600308.23
	90	588.02	0.6311	0.9532	293573479.03	602428.52
	100	587.44	0.6488	0.9537	293255173.37	604246.73
	110	586.91	0.6647	0.9541	292968683.09	605820.71
	120	586.43	0.6789	0.9543	292709486.83	607195.62
l	120	587.63	0.6434	0.9535	293352209.42	603682.75
	135	587.53	0.6461	0.9536	293303282.96	603969.15
	150	587.44	0.6488	0.9537	293255173.37	604246.73
	165	587.35	0.6514	0.9538	293207860.41	604518.81
	180	587.26	0.6540	0.9539	293161324.50	604790.58
θ	0.064	587.07	0.6724	0.9757	293383131.37	618058.42
	0.072	587.26	0.6603	0.9644	293317904.03	610974.58
	0.080	587.44	0.6488	0.9537	293255173.37	604246.73
	0.088	587.62	0.6377	0.9436	293194798.30	597849.55
	0.096	587.79	0.6275	0.9339	293136643.37	591755.71

Table 5 Sensitivity analysis of model parameters

The sensitive analysis shown in Table 5 indicates the following observations and managerial insights.
(i) A rise in purchase costs c reduces total profit, which indicates that the vendor to decrease order quantities to control expenses. Higher sales prices are introduced to compensate, which may affect customer demand. Implementing efficient inventory management strategies can help to maintain cost stability and profitability.
A shorter inventory cycle and frequent replenishments increase logistical complexities which needs careful planning. Managers should negotiate better with supplier contracts, should explore alternative sourcing options and leverage bulk-purchasing discounts to manage costs effectively.
(ii) Increased holding costs h lowers total profit, indicating that the vendor should reduce order quantities to cut storage expenses. To balance these costs, sales prices rise which potentially affects the market competitiveness. With a shorter inventory cycle and replenishment time, efficient inventory control becomes essential to prevent unnecessary expenses. Investing in improved storage solutions or refining order frequency can reduce the financial strain of rising holding costs. Also, by adopting cost-effective warehousing strategies vendor can maintain operational efficiency.
(iii) Higher backlogged shortage costs s reduces total profit, which indicates that the vendor should increase order quantity. Also, the vendor should increase both the inventory cycle and replenishment cycle time. Decrease in sales price suggests that efforts should be taken so as to stimulate demand and reduce shortages. Managers should optimize inventory planning so as to balance stock levels and backlog costs while maintaining profitability. Strengthening supplier partnerships will also help to maintain replenishment reliability and minimize risks associated with extended cycles. Also, strategic pricing and customer retention initiatives should be practiced so as to recover revenue losses and potential fulfilment delays.
(iv) Rising lost sale costs l lead to declining total profit, indicating that the vendor should increase order quantities and extend the inventory cycle to prevent lost sales. A slight reduction in sales price indicates an attempt to attract more customers and minimize revenue losses. Managers should enhance demand forecasting and inventory management to maintain product availability while controlling costs.
(v) A higher deterioration rate θ results in more significant product wastage, reducing total profit and requiring lower order quantities to prevent excessive losses. A moderate increase in sales price helps mitigate cost impacts, but pricing

must remain competitive to maintain demand. Shorter inventory cycles and faster replenishments require efficient supply chain coordination to ensure product availability. Investing in better preservation technologies or better storage conditions can slow down the deterioration so as to minimize losses. Additionally, adopting just-in-time inventory strategies can optimize stock levels and reduce the adverse effects of product decay.

The trends for selling price, order quantity and total profit related to changing parameters c, h, s, l and θ are depicted in Figure 9, Figure 10 and Figure 11 respectively.

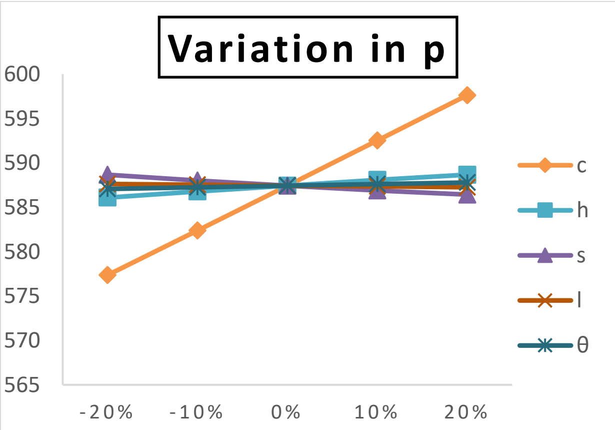

Figure 9. Variation in p with respect to c, h, s, l and θ

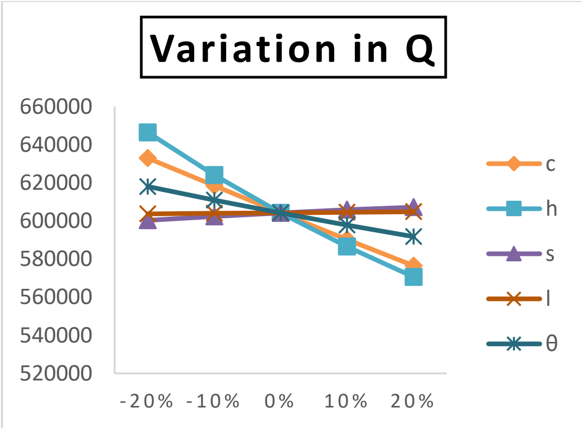

Figure 10. Variation in Q with respect to c, h, s, l and θ

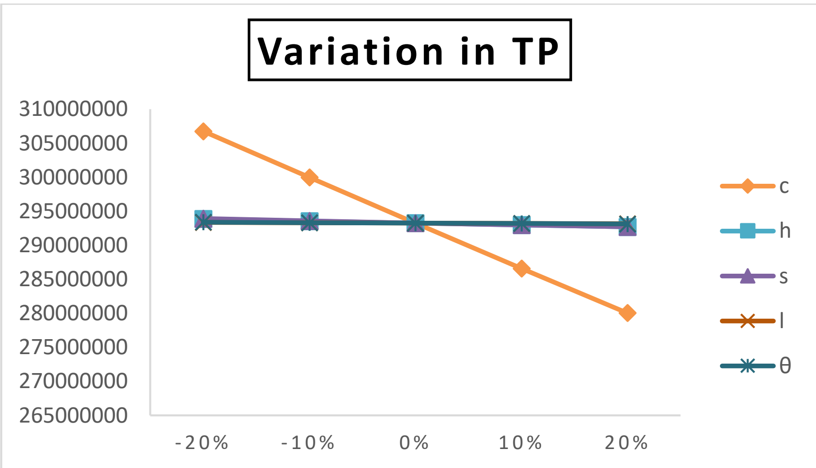

Figure 11. Variation in $TP(.)$ with respect to c, h, s, l and θ

8. CONCLUSION

Researchers and practitioners are becoming more interested in cutting-edge developments in operations and supply chain management, especially those that highlight sustainability and green initiatives. By creating a non-linear optimization model that incorporates greenness levels with inventory and pricing decisions for non-instantaneously deteriorating items, this study advances this expanding field. In order to better represent the nonlinear behaviour of consumers, this study uses a quadratic price-dependent demand function, compared to traditional models that rely on linear or exponential demand forms. Finding the ideal pricing strategy that strikes a balance between profitability and competition is made possible by the evidence that demand is highly elastic at extreme prices but remains relatively stable at moderate price levels.

The results show that, in spite of the costs involved, increased greenness investments increase overall profit by promoting sustainable practices and increasing consumer demand. These findings are supported by the sensitivity analysis and numerical example, which show how important factors like price sensitivity, deterioration rates, and greenness levels affect the best times for replenishments, order quantities, and pricing policies. Crucially, compared to linear or exponential forms, a quadratic demand function offers a more adaptable and practical framework that enables companies to better understand complex customer reactions to price changes and make more precise decisions about inventory and pricing.

These contributions could be expanded upon in future studies by looking at stochastic green inventory models, especially in situations where demand and deterioration rates are unknown. A more comprehensive and practical view of sustainable supply chain optimization would be offered by extending the framework to multi-item or multi-vendor systems. Further understanding of how government regulations, incentive programs, and consumer behaviour dynamics affect greenness investments may also help develop robust and flexible inventory strategies that support global sustainability objectives.

APPENDIX

Proof of Theorem

Let the unique solution be denoted by $t_1 = t_1^*, T = T^*$ and after simplification, we obtain:

$$\frac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial t_1^{*2}} = -\frac{\left(-\xi p^2 - \beta p + \eta w + \alpha\right)\left(\left(1+\delta\left(T^*-t_1^*\right)\right)^2\left(\mathrm{e}^{\theta t_1^*} h \mathrm{e}^{-\theta t_1^*} + \theta\left(ht_d + c\right)\right)\mathrm{e}^{-t_d\theta} - \mathrm{e}^{-\theta t_1^*}\left(-\delta\left(p - c + \left(\frac{s+l\delta}{\delta}\right)\right)\right)\right)}{\left(1+\delta\left(T^*-t_1^*\right)\right)^2 T^* \mathrm{e}^{-\theta t_1^*}}$$

$$\begin{aligned}
\frac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial T^{*2}} &= \frac{\left(\alpha - \beta p - \xi p^2 + \eta w\right)}{T^*\left(1+\delta\left(T^*-t_1^*\right)\right)^2}\left(\left(-\delta\left(p-c+\left(\frac{s+l\delta}{\delta}\right)\right)\right) - 2\frac{1}{T^{*2}\left(1+\delta\left(T^*-t_1^*\right)\right)}\left(p-c+\frac{s}{\delta^2} - l\delta\left(T^*-t_1^*\right)\right)\right) \\
&+ \frac{2\left(\alpha-\beta p-\xi p^2+\eta w\right)}{T^{*3}}\left(\begin{array}{l}
\left(pt_1^* + \dfrac{\ln\left(1+\delta\left(T^*-t_1^*\right)\right)}{\delta}\right) - \dfrac{\left(A+Kw^2\right)}{\left(\alpha-\beta p-\xi p^2+\eta w\right)} \\
-h\left(-\dfrac{1}{2}td^2 + \dfrac{\left(t_d\theta \mathrm{e}^{-\theta t_1^*} + \mathrm{e}^{-td\theta} - \mathrm{e}^{-\theta t_1^*}\right)t_d}{\theta \mathrm{e}^{-\theta t_1^*}} - \dfrac{\left(\begin{array}{l} +\mathrm{e}^{-\theta t_1^*} t_1^* \theta - \mathrm{e}^{-\theta t_1^*} t_d \theta \\ -\mathrm{e}^{-t_d\theta} + \mathrm{e}^{-\theta t_1^*}\end{array}\right)\mathrm{e}^{\theta t_1^*}}{\theta^2}\right) - \dfrac{s\left(\delta\left(T^*-t_1^*\right) - \ln\left(1+\delta\left(T^*-t_1^*\right)\right)\right)}{\delta^2} \\
-l\left(T^* - t_1^* - \dfrac{\ln\left(1+\delta\left(T^*-t_1^*\right)\right)}{\delta}\right) - c\left(\dfrac{\ln\left(1+\delta\left(T^*-t_1^*\right)\right)}{\delta} + \dfrac{\left(t_d\theta \mathrm{e}^{-\theta t_1^*} + \mathrm{e}^{-t_d\theta} - \mathrm{e}^{-\theta t_1^*}\right)}{\theta \mathrm{e}^{-\theta t_1^*}}\right)
\end{array}\right)
\end{aligned}$$

$$\frac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial t_1^*\partial T^*} = \frac{\left(\begin{array}{l}
-\left(1+\delta\left(T^*-t_1^*\right)\right)^2 \mathrm{e}^{\theta t_1^*} h\left(\mathrm{e}^{-\theta t_1^*}\right)^2 + \left(\begin{array}{l}
\left(1+\delta\left(T^*-t_1^*\right)\right)^2 \mathrm{e}^{\theta t_1^*} h \mathrm{e}^{-t_d\theta} \\
-\delta\left(\left(T^*-t_1^*\right)^2 (l+p)\delta^2 + \left(\begin{array}{l} st_1^{*2} - 2T^* t_1^* s - \left(p-c+\left(\frac{s+l\delta}{\delta}\right)\right)t_1^* \\ +T^*\left(T^* s + 2c\right) + 2ct_1^*\end{array}\right) + c\right)\theta
\end{array}\right)\mathrm{e}^{-\theta t_1^*} \\
+\left(1+\delta\left(T^*-t_1^*\right)\right)^2\left(ht_d + c\right)\theta \mathrm{e}^{-t_d\theta}
\end{array}\right)\left(\alpha-\beta p-\xi p^2+\eta w\right)}{\left(1+\delta\left(T^*-t_1^*\right)\right)^2 T^{*2}\theta \mathrm{e}^{-\theta t_1^*} t_1^*}$$

$$H = \begin{bmatrix} \dfrac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial t_1^{*2}} & \dfrac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial t_1^*\partial T^*} \\ \dfrac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial t_1^*\partial T^*} & \dfrac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial T^{*2}} \end{bmatrix}$$

$$
\begin{aligned}
\det(H) = {} & \frac{\left(\left(\begin{array}{l} h\mathrm{e}^{\theta t_1^*}\left(\delta\left(T^*-t_1^*\right)+1\right)^2\mathrm{e}^{-t_d\theta} \\ -\delta c+(l+p)\delta+s \end{array}\right)\mathrm{e}^{-\theta t_1^*} + \theta\mathrm{e}^{-t_d\theta}\left(\delta\left(T^*-t_1^*\right)+1\right)^2\left(ht_d+c\right)\right)\left(\alpha-\beta p-\xi p^2+\eta w\right)^2}{2\delta^2\theta^2T^{*4}\left(\mathrm{e}^{-\theta t_1^*}\right)^2\left(1+\delta\left(T^*-t_1^*\right)\right)^4} \\
& \times\Bigg(\mathrm{e}^{-\theta t_1^*}\left(\delta\left(p-c+\left(\frac{s+l\delta}{\delta}\right)\right)\right)\left(-\delta\left(T^*-t_1^*\right)+1\right)^2\theta^2\ln\left(1+\delta\left(T^*-t_1^*\right)\right) \\
& -\delta\Bigg(h\delta\mathrm{e}^{\theta t_1^*}\left(\delta\left(T^*-t_1^*\right)+1\right)^2\left(\theta t_1^*-t_d\theta+1\right)\left(\mathrm{e}^{-\theta t_1^*}\right)^2 \\
& +\Bigg(h\delta\mathrm{e}^{\theta t_1^*}\left(\delta\left(T^*-t_1^*\right)+1\right)^2\mathrm{e}^{-t_d\theta} \\
& +\Bigg(\Bigg(+\frac{1}{2}h\delta^2\theta t_d^{\,2}+\delta^2\left(c\theta-h\right)t_d-\left(\delta+\frac{3}{2}\theta\right)\delta c+\left(\begin{array}{l}-\left(\left(l+p\right)\delta+s\right)\delta t_1^* \\ +\dfrac{\left(Kw^2+A\right)}{\left(\alpha-\beta p-\xi p^2+\eta w\right)}\delta^2+\dfrac{3}{2}\left(l+p\right)\delta+\dfrac{3}{2}s\end{array}\right)\theta\Bigg)\delta T^{*2} \\
& -2\left(\delta t_1^*-1\right)\left(\begin{array}{l}-\dfrac{1}{2}h\delta^2\theta t_d^{\,2}-\delta^2\left(c\theta-h\right)t_d+\delta\left(\delta+\dfrac{\theta}{2}\right)c \\ +\left(\left(\left(l+p\right)\delta+s\right)\delta t_1^*+\dfrac{\left(Kw^2+A\right)}{\left(\alpha-\beta p-\xi p^2+\eta w\right)}\delta^2-\dfrac{1}{2}\left(l+p\right)\delta-\dfrac{1}{2}s\right)\theta\end{array}\right)T^* \\
& +\left(\begin{array}{l}-\dfrac{1}{2}h\delta\theta t_d^{\,2}-\delta\left(c\theta-h\right)t_d+\delta c \\ +\left(\begin{array}{l}\left(\left(l+p\right)\delta+s\right)t_1^* \\ +\delta\dfrac{\left(Kw^2+A\right)}{\left(\alpha-\beta p-\xi p^2+\eta w\right)}\end{array}\right)\theta\end{array}\right)\left(\delta t_1^*-1\right)^2\Bigg)\theta\Bigg)\mathrm{e}^{-\theta t_1^*} \\
& -\delta\theta\mathrm{e}^{-t_d\theta}\left(1+\delta\left(T^*-t_1^*\right)\right)^2\left(ht_d+c\right)\Bigg)\Bigg) \\
& -\frac{\left(\alpha-\beta p-\xi p^2+\eta w\right)^2\mathrm{e}^{2\theta t_1^*}}{\theta^2T^{*4}\left(1+\delta\left(T^*-t_1^*\right)\right)^4}\Bigg(-\left(1+\delta\left(T^*-t_1^*\right)\right)^2h\mathrm{e}^{-\theta t_1^*}+\Big(\left(1+\delta\left(T^*-t_1^*\right)\right)^2h\mathrm{e}^{\left(t_1^*-t_d\right)\theta} \\
& -\left(\left(T^*-t_1^*\right)^2\left(l+p\right)\delta^2+\left(-t_1^*p+st_1^{*2}+\left(-2T^*s-c-l\right)t_1^*+T^*\left(T^*s+2c\right)\right)\delta-st_1^*+c\right)\theta\Big)\mathrm{e}^{-\theta t_1^*}+\left(1+\delta\left(T^*-t_1^*\right)\right)^2\left(ht_d+c\right)\theta\mathrm{e}^{-t_d\theta}\Bigg)^2
\end{aligned}
$$

Considering $\left(\alpha-\beta p-\xi p^2+\eta w\right)>0,\ p>c,\ \left(ht_d+c\right)>0,\ \left(\frac{s+l\delta}{\delta}\right)>0$, it can be concluded that

$$\frac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial t_1^{*2}}<0$$

$$\frac{\partial^2 TP\left(p,w,t_1^*,T^*\right)}{\partial T^{*2}}<0$$

$\det\left(H\right)>0$.

Hence, the Hessian matrix H at t_1^*, T^* is negative definite, confirming the required conditions. This concludes the proof.

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