A NEW WEIGHTED-GENERATOR FAMILY OF STATISTICAL MECHANICS MODELS WITH SIMULATION AND ANALYSIS TO LIGHT INTENSITY AND TENSILE STRENGTH OF CARBON FIBER DATA

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ABSTRACT
Traditional distributions face limitations in adapting to the complexities of complex random phenomena. This observation prompts the exploration of creative generalizations through the application of diverse mathematical approaches. In this manuscript, we use Generalized Even Power Weighted Probability Technique, as a generator to introduce a new distribution called $2K^{th}$ Order Weighted Maxwell-Boltzmann Distribution. We derive its various structural properties including the moment generating function, moments, mean residual lifetime, mean waiting time, Renyi entropy and order statistics, among others. Additionally, we employ the maximum likelihood method for parameter estimation. A simulation study is conducted to analyse the asymptotic normality behaviour of the maximum likelihood estimators. The versatility of the new distribution is demonstrated through its application to real-life datasets and simulated data.
KEYWORDS: Weighted Maxwell-Boltzmann Distribution, Structural properties, mean residual lifetime, mean waiting time and maximum likelihood estimation.

MSC:60E05

RESUMEN
Las distribuciones tradicionales presentan limitaciones para adaptarse a la complejidad de los fenómenos aleatorios complejos. Esta observación impulsa la exploración de generalizaciones creativas mediante la aplicación de diversos enfoques matemáticos. En este manuscrito, utilizamos la técnica de probabilidad generalizada, incluso ponderada por potencia, como generador para introducir una nueva distribución denominada Distribución de Maxwell-Boltzmann Ponderada de 2Kº Orden. Derivamos sus diversas propiedades estructurales, incluyendo la función generadora de momentos, los momentos, la vida media residual, el tiempo medio de espera, la entropía de Renyi y las estadísticas de orden, entre otras. Además, empleamos el método de máxima verosimilitud para la estimación de parámetros. Se realiza un estudio de simulación para analizar el comportamiento de normalidad asintótica de los estimadores de máxima verosimilitud. La versatilidad de la nueva distribución se demuestra mediante su aplicación a conjuntos de datos reales y simulados.
PALABRAS CLAVE: Distribución de Maxwell-Boltzmann Ponderada, la vida media residual, el tiempo medio de espera Propiedades Estructurales y estimación de máxima verosimilitud.

1. INTRODUCTION

Standard probability models may be inadequate when data are obtained through mechanisms that produce uneven selection probabilities, whether due to complex survey structures, intrinsic biases or selective sampling. To enhance model flexibility under such conditions, statisticians have developed families of generalized distributions as well as generalization techniques to extend the existing models. These extended models typically incorporate up to four additional parameters, striking a practical balance between improved fit and interpretability. A particularly powerful class within these extensions is weighted distributions, which formally integrate the sampling mechanism into the probabilistic model. The theory of weighted probability distribution is a powerful concept that offers a valuable framework for addressing issues related to model specification and data

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interpretation. It provides a technique for fitting models to the unknown weight functions when samples can be taken both from the original distribution and the developed distribution. This concept was first provided by Fisher (1934), who studied how the methods of ascertainment can influence the form of the distribution of recorded observations. He showed that if the chance of observing an event with value X is proportional to $w(x)$, the resulting observed density is $f_w(x) = \frac{w(x)f(x)}{E[w(x)]}$. Later Rao (1965) introduced and formulated it in general terms in connection with modelling statistical data, when the usual practice of using standard distributions were found to be unsuitable. Building on this foundation, Patil & Rao (1978) made significant advances by applying weighted-distribution methods to human population studies and ecological sampling. Castillo & Pérez-Casany (1998) further extended the methodology by deriving new weighted exponential–Poisson families capable of modelling both over dispersion and under dispersion in count data. Contemporary research continues to expand this field, with studies such as Fatima & Ahmad (2017), Dar et al. (2018), Shakhatreh & Al-Masri (2020), Fallah & Kazemi (2022) and Ghitany and Wang (2022).

In this manuscript, we introduced a new two-parameter 2kth Order Weighted Maxwell Boltzmann Distribution (KWMBD) by utilizing a new generalization of weighted probability distribution called the Generalized Even Power Weighted Distribution to enhance the flexibility and practical utility of the classical Maxwell Boltzmann distribution, particularly in contexts where data are collected through mechanisms that produce uneven selection probabilities. This contribution advances both the theoretical understanding and the practical application of weighted distributions in modern data analysis.

Definition: Consider a random variable X with $f\ (x\ )$ as its density function and let us assume the probability of observing $X = x$ is proportional to a weight function $w(x) \geq 0$. Therefore the density function of the Generalized Even Power Weighted Distribution is given by:

$$f_{w^{2k}}(x) = \frac{\left(w(x)\right)^{2k} f(x)}{E[(w(x))^{2k}]}, \qquad -\infty < x < \infty, k \in \mathbb{R}. \tag{1.1}$$

Where,$E[(w(x))^{2k}] = \int_{-\infty}^{\infty} (w(x))^{2k} f(x) dx$,

and$x \in \mathbb{R}$, $w(x)$is weight function and $k \in \mathbb{R}$.

2. DERIVATION OF 2KTH ORDER WEIGHTED MAXWELL-BOLTZMANN DISTRIBUTION (KWMBD)

The Maxwell-Boltzmann (MB) distribution was introduced by Maxwell (1867) to describe the distribution of speeds of molecules at thermal equilibrium and nowadays is widely applied in many fields such as statistical physics, statistical mechanics and accounting theory, among others.

The Probability density function (pdf) of the Maxwell distribution is given by:

$$f(x, \alpha) = \sqrt{\frac{2}{\pi}} \alpha^{-3} x^2 e^{-\frac{x^2}{2\alpha^2}}, \qquad x > 0, \alpha > 0. \tag{2.1}$$

and the cumulative distribution function (cdf) of Maxwell Distribution is given as:

$$F(x, \alpha) = 1 - \frac{\Gamma\left(\frac{3}{2}, \frac{x^2}{2\alpha^2}\right)}{\Gamma\left(\frac{3}{2}\right)}, \qquad x > 0, \alpha > 0. \tag{2.2}$$

The MB distribution has been discussed in many works in the literature, for example, Tyagi and Bhattacharya (1989) used MB distribution as a lifetime model and discussed Bayesian and minimum variance unbiased estimation methods for its parameters and reliability function. Chaturvedi and Rani (1998) extended the MB distribution by adding another parameter and estimated both classical and Bayesian estimators. Bekker and Roux (2005) obtained empirical Bayesian estimation for MB distribution. Kazmi et al. (2012) derived the Bayesian estimation for two component mixture of Maxwell distribution, assuming censored data. Modi (2015) proposed length biased MB distribution. Saghir and Khadim (2016) derived mathematical properties of length biased MB distribution. Huang and Chen (2016) studied the tail behaviour of MB distribution. Reshi (2021) estimated parameters of weighted MB distribution using simulated and real life data sets. Some recent extensions of the MB distribution are discussed in Saghir et al (2018), Segovia et al. (2021) and Castillo et al (2023).

The pdf of KWMBD distribution is obtained by taking the weight function $w(x) = x, k > -1.5$. the Maxwell pdf (2) in the basic definition of Generalized Even Power Weighted Distribution; we have the following weighted pdf of KWMBD

$$f_{w^{2k}}(x,\alpha,k) = \frac{x^{2k}\sqrt{\frac{2}{\pi}}\alpha^{-3}x^{2}e^{-\frac{x^{2}}{2\alpha^{2}}}}{E(x^{2k})}, \qquad x > 0, \alpha > 0, k > -1.5.$$

Where, $E(x^{2k}) = \int_0^\infty x^{2k}\sqrt{\frac{2}{\pi}}\alpha^{-3}x^{2}e^{-\frac{x^{2}}{2\alpha^{2}}}dx = \frac{1}{\pi}\alpha^{2k}2^{k+1}\Gamma\left(k+\frac{3}{2}\right)$.

On simplifying the expression, the pdf of KWMBD is given by

$$f_{w^{2k}}(x,\alpha,k) = \frac{x^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{x^{2}}{2\alpha^{2}}}}{2^{(k+\frac{1}{2})}\Gamma(k+\frac{3}{2})}, \qquad x > 0, \alpha > 0, k > -1.5. \tag{2.3}$$

The cdf of KWMBD ($F_{w^{2k}}(x,\alpha,k)$) obtained by integrating (4) w.r.t $x$and is given by

$$F_{w^{2k}}(x,\alpha,k) = 1 - \frac{\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^{2}}{2\alpha^{2}}\right)}{\Gamma\left(k+\frac{3}{2}\right)}, \qquad x > 0, \alpha > 0, k > -1.5. \tag{2.4}$$

Where, $\Gamma(\mathrm{a},x) = \int_x^\infty t^{(a-1)}e^{-t}\,dt$ is incomplete gamma function.

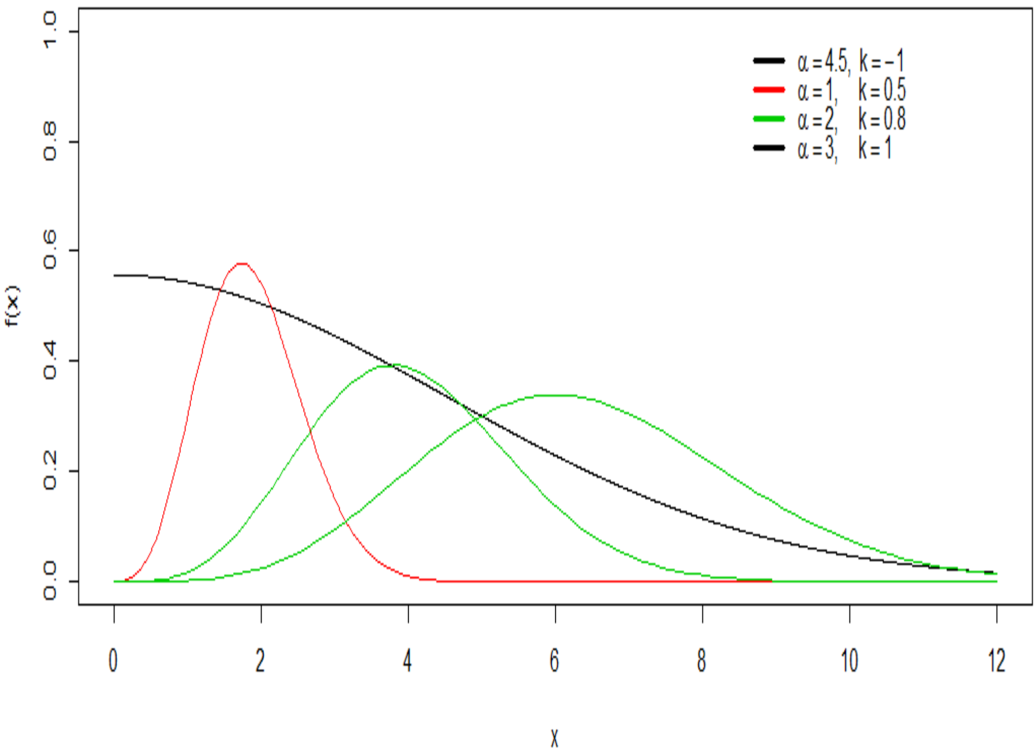

Figure 1: The PDF curve of KWMBD for different values of α and k.

2.1. Sub models of 2Kth order Weighted Maxwell-Boltzmann Distribution:

For different choices of the parameter values in (2.3), the sub-models of 2Kth order Weighted Maxwell-Boltzmann Distribution are:

Sub-Models	**Parameter restriction**	**PDF's**	**CDF's**
Half normal distribution (HND)	$k = -1$	$f(x) = \frac{\sqrt{2}e^{-\frac{x^2}{2\alpha^2}}}{\sqrt{\pi}\alpha}$	$F(x) = \text{erf}\left(\frac{x^2}{\alpha\sqrt{2}}\right)$
Rayleigh distribution (RD)	$k = -\frac{1}{2}$	$f(x) = \frac{xe^{-\frac{x^2}{2\alpha^2}}}{\propto^2}$	$F(x) = 1 - e^{-\frac{x^2}{2\alpha^2}}$
Maxwell distribution (MD)	$k = 0$	$f(x) = \sqrt{\frac{2}{\pi}}\alpha^{-3}x^2e^{-\frac{x^2}{2\alpha^2}}$	$F(x) = 1 - \frac{\Gamma\left(\frac{3}{2}, \frac{x^2}{2\alpha^2}\right)}{\Gamma\left(\frac{3}{2}\right)}$
Length biased Maxwell distribution (LBMD)	$k = \frac{1}{2}$	$f(x) = \frac{x^3e^{-\frac{x^2}{2\alpha^2}}}{2\propto^4}$	$F(x) = 1 - \left(\frac{x^2}{2\alpha^2} + 1\right)e^{-\frac{x^2}{2\alpha^2}}$
Area biased Maxwell distribution (ABMD)	$k = 1 \propto = \sqrt{\frac{1}{\alpha}}$	$f(x) = \frac{x^4\alpha^{\frac{5}{2}}e^{-\frac{\alpha x^2}{2}}}{2^{\frac{3}{2}}\Gamma\left(\frac{5}{2}\right)}$	$F(x) = 1 - \frac{\Gamma\left(\frac{5}{2}, \frac{x^2}{2\alpha^2}\right)}{\Gamma\left(\frac{5}{2}\right)}$
Length biased weighted Rayleigh distribution (LBWRD)	$k = \frac{1}{2} \propto = \frac{1}{2\beta}$	$f(x) = \frac{2x^3\beta^2e^{-\frac{\beta x^2}{\alpha}}}{\propto^2}$	$F(x) = 1 - \frac{e^{-\frac{\beta x^2}{\alpha}}(\beta x^2 + \alpha)}{\alpha}$

Table 1: Sub-models of 2Kth order Weighted Maxwell for different parameter values.

3. STRUCTURAL AND STATISTICAL PROPERTIES OF KWMBD:

3.1 Reliability function and hazard rate function of KWMBD

The reliability function ($R_{w^{2k}}(x, \alpha, k)$) and hazard rate function ($h_{w^{2k}}(x, \alpha, k)$) of KWMBD are given respectively by:

$$R_{w^{2k}}(x, \alpha, k) = 1 - F_{w^{2k}}(x, \alpha, k) = \frac{\Gamma\left(\text{k} + \frac{3}{2}, \frac{x^2}{2\alpha^2}\right)}{\Gamma\left(k + \frac{3}{2}\right)}, \qquad x > 0, \alpha > 0, k > -1.5. \qquad (3.1)$$

$$h_{w^{2k}}(x, \alpha, k) = \frac{f_{w^{2k}}(x, \alpha, k)}{R_{w^{2k}}(x, \alpha, k)} = \frac{x^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{x^2}{2\alpha^2}}}{2^{\left(k+\frac{1}{2}\right)}\left\{\Gamma\left(\text{k} + \frac{3}{2}, \frac{x^2}{2\alpha^2}\right)\right\}}, \qquad x > 0, \alpha > 0, k > -1.5. \qquad (3.2)$$

In the particular case at $= 0$, where the proposed distribution reduces to the Maxwell distribution, the reliability function and hazard rate function of KWMBD simplifies to the well-known form of the Maxwell distribution, and are given by

$$R(x, \alpha) = \frac{\Gamma(\frac{3}{2}, \frac{x^2}{2\alpha^2})}{\Gamma(\frac{3}{2})}, \qquad x > 0, \alpha > 0.$$

$$h(x,\alpha) = \frac{x^2\alpha^{-3}e^{-\frac{x^2}{2\alpha^2}}}{\sqrt{2}\left\{\Gamma\left(\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right\}}, \qquad x > 0, \alpha > 0.$$

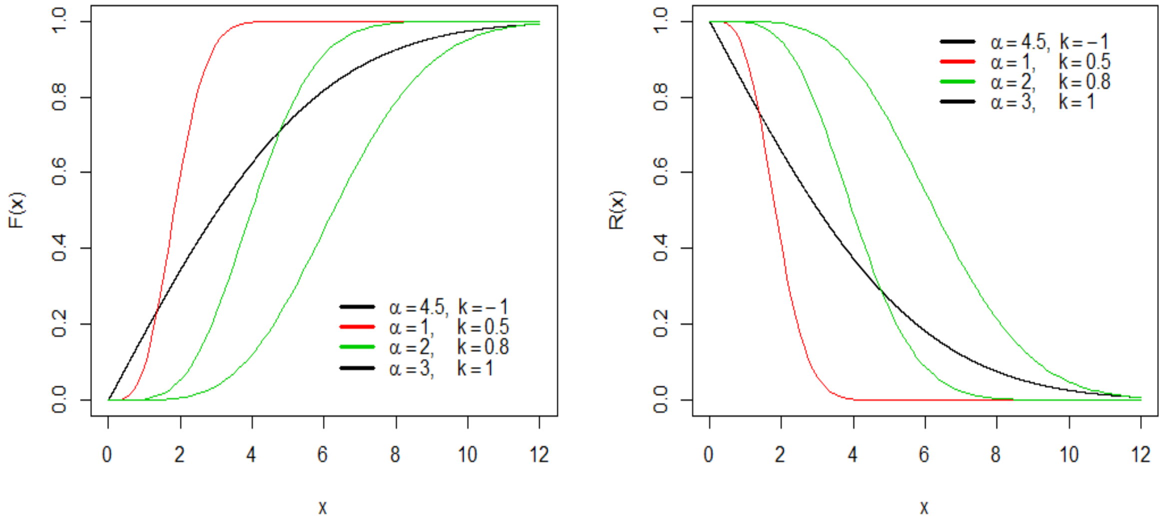

Figure 2: The cumulative distribution function and reliability function of KWMBD for different choices of α and k.

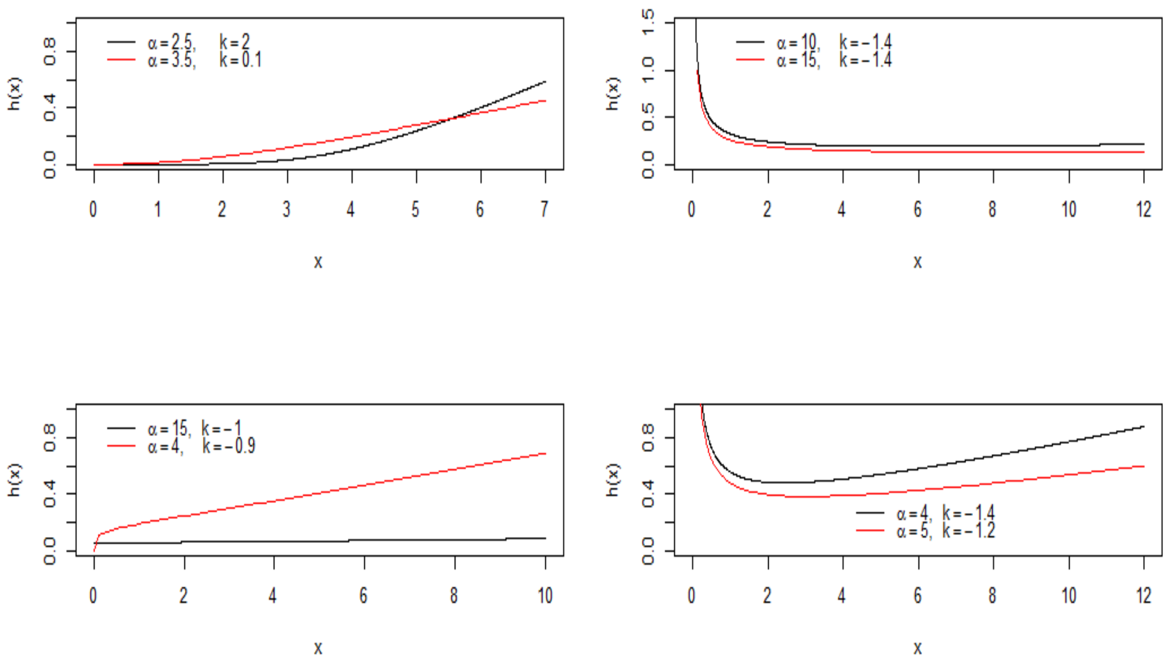

Figure 3: The hazard rate function curve of KWMBD for different value of parameters.

The figure 3 describes the behaviour of hazard rate function of KWMBD for different choice of parameters and it depicts that the hazard rate function of KWMBD is increasing, decreasing, linearly increasing, constant and decreasing- increasing.

3.2 Reverse hazard rate function

The reverse hazard rate function ($(r_f)_{w^{2k}}(x,\propto,k)$) of KWMBD distribution is given by:

$$(r_f)_{w^{2k}}(x,\propto,k)=\frac{f_{w^{2k}}(x,\alpha,k)}{F_{w^{2k}}(x,\alpha,k)}=\frac{x^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{x^2}{2\alpha^2}}}{2^{\left(k+\frac{1}{2}\right)}\left\{\Gamma\left(k+\frac{3}{2}\right)-\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right\}},\qquad x>0,\alpha>0,k>-1.5. \quad (3.3)$$

At $k=0$, the reverse hazard rate reduces to the form of Maxwell distribution reverse hazard rate function, which is given by

$$r_f(x,\propto)=\frac{x^2\alpha^{-3}e^{-\frac{x^2}{2\alpha^2}}}{\sqrt{2}\left\{\Gamma\left(\frac{3}{2}\right)-\Gamma\left(\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right\}},\qquad x>0,\alpha>0.$$

3.3 Moments of KWMBD

In this we compute the r^{th} moment, mean, variance, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of KWMBD.

If a random variable X follows KWMBD, then the r^{th} moment about origin is given by

$$E(X^r)=\int_0^\infty x^r f_{w^{2k}}(x,\alpha,k)dx$$

$$E(X^r)=\int_0^\infty x^r\frac{x^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{x^2}{2\alpha^2}}}{2^{\left(k+\frac{1}{2}\right)}\Gamma\left(k+\frac{3}{2}\right)}dx$$

After simplifying the integral, we get

$$E(X^r)=\left(\sqrt{2}\right)^r\propto^r\frac{\Gamma\left(\frac{2k+r+3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} \quad (3.4)$$

Substituting $r=1$, we obtain the mean of KWMBD and is given by

$$E(X)=\sqrt{2}\alpha\frac{\Gamma\left(\frac{2k+4}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}. \quad (3.5)$$

Similarly, by substituting $r=2,3,4\,...$, we obtain the expressions of higher order moments about origin ($E(X^2),E(X^3),E(X^4)\,...$) of KWMBD and then variance, CV,CS and CK are calculated by basic definition in terms of moments about origin.

Variance of random variable X following KWMBD is calculated by

$$var(X)=E(X^2)-\left(E(X)\right)^2$$

On substituting the mean and $E(X^2)$, we get variance of KWMBD and is given by

$$var(X)=\frac{2\propto^2}{\left(\Gamma\left(k+\frac{3}{2}\right)\right)^2}\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right)-\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right] \quad (3.6)$$

CV of random variable X following KWMBD is calculated by

$$CV = \frac{\sqrt{E(X^2) - \left(E(X)\right)^2}}{E(X)}$$

On substituting the variance and $E(X)$, we get CV of KWMBD and is given by

$$CV = \frac{\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right) - \left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right]^{\frac{1}{2}}}{\Gamma\left(\frac{2k+4}{2}\right)} \quad 3.7)$$

CS of random variable X following KWMBD is calculated by

$$CS = \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{[V(X)]^{\frac{3}{2}}}$$

On substituting the variance, $E(X^3), E(X^2)$ and $E(X)$, we obtained CS of KWMBD and is given by

$$CS = \frac{\left[\Gamma\left(\frac{2k+6}{2}\right)\left(\Gamma\left(k+\frac{3}{2}\right)\right)^2 - 3\Gamma\left(\frac{2k+4}{2}\right)\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right) + 2\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^3\right]}{\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right) - \left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right]^{\frac{3}{2}}} \quad (3.8)$$

CK of random variable X following KWMBD is calculated by

$$CK = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4}{[V(X)]^2}$$

On substituting the variance, $E(X^4), E(X^3), E(X^2)$and $E(X)$, we obtained CK of KWMBD and is given by

$$CK = \frac{\left[\Gamma\left(\frac{2k+7}{2}\right)\left(\Gamma\left(k+\frac{3}{2}\right)\right)^3 - 4\Gamma\left(\frac{2k+4}{2}\right)\Gamma\left(\frac{2k+6}{2}\right)\left(\Gamma\left(k+\frac{3}{2}\right)\right)^2 + 6\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right) - 3\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^4\right]}{\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right) - \left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right]^2} \quad (3.9)$$

3.4 Mode of KWMBD

The logarithm of the pdf (4) is:

$$\log f_{w^{2k}}(x, \alpha, k) = -\left(k + \frac{1}{2}\right)\log(2) + 2(k+1)\log(x) - (3+2k)\log(\alpha) - \frac{x^2}{2\alpha^2} - \log\left(\Gamma\left(k + \frac{3}{2}\right)\right)$$

$$\frac{\partial}{\partial x}\log f_{w^{2k}}(x, \alpha, k) = \frac{2(k+1)}{x} - \frac{x}{\alpha^2}, \qquad \frac{\partial^2}{\partial^2 x}\log f_{w^{2k}}(x, \alpha, k) = \frac{-2(k+1)}{x^2} - \frac{1}{\alpha^2} < 0$$

The first derivative has positive root $x = \alpha\sqrt{2(k+1)}$

α	k	Mean	Variance	CV	CS	CK	Mode
1	**-1**	0.7979	0.3634	0.7555	0.9952	2.9232	0
2	**-1**	1.5958	1.4535				0
3	**-1**	2.3936	3.2704				0
1	**-0.5**	1.2533	0.4292	0.5227	0.6311	1.5033	1
2	**-0.5**	2.5066	1.7168				2
3	**-0.5**	3.7599	3.8628				3
1	**1**	2.1277	0.4729	0.3232	0.3542	1.9632	2

2	**1**	4.2554	1.8917				4
3	**1**	6.3831	4.2563				6
1	**2**	2.5532	0.4810				2.4494
2	**2**	5.1065	1.9241	0.2716	0.2906	4.9190	4.8990
3	**2**	7.6597	4.3291				7.3484

Table 2: Descriptive statistics (Mean, Variance, CV, CS, CK, and Mode) of KWMBD for different choices of parameters.

Table 2 presents the numerically evaluated values of the Mean, Variance, CV, CS, CK and Mode for selected values of parameters α and k. The results indicate that as we increase the value of α and weight parameter$k$, the distribution becomes more concentrated, with less variability, as evidenced by decreased coefficient of variance and coefficient of skewness and accompanied by an increased mean, variance , and mode.

3.5 Incomplete moments and Conditional moments

The incomplete moments display the graphical structure of a distribution's moments, which is helpful in various fields such as econometrics, finance and reliability. The n^{th} incomplete moments of KWMBD ($I_{w^{2k}}(t,n)$) is given by:

$$I_{w^{2k}}(t,n) = \int_0^t x^n f_{w^{2k}}(x,\alpha,k)dx$$

$$I_{w^{2k}}(t,n) = \frac{\left(\sqrt{2}\alpha\right)^n \gamma\left(\frac{2k+4}{2}, \frac{t^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}. \tag{3.10}$$

The n^{th}conditional momentof KWMBD is given by:

$$E(T^n|T>t) = \frac{\int_t^\infty x^n f_{w^{2k}}(x,\alpha,k)dx}{R(t)},$$

where, $\int_t^\infty x^n f_{w^{2k}}(x,\alpha,k)dx = \frac{(\sqrt{2}\alpha)^n \Gamma\left(\frac{n+2k+1}{2}, \frac{t^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}$,

$$E(T^n|T>t) = \frac{(\sqrt{2}\alpha)^n \Gamma\left(\frac{n+2k+1}{2}, \frac{t^2}{2\alpha^2}\right)}{\Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{x^2}{2\alpha^2}\right)} \tag{3.11}$$

3.6 Mean residual life and mean waiting time

Mean residual life $\mu_{w^{2k}}(t)$ of KWMBD is given by:

$$\mu_{w^{2k}}(t) = \frac{1}{R_{w^{2k}}(x,\alpha,k)}\left[E(t) - \int_0^t x f_{w^{2k}}(x,\alpha,k)dx\right] - t$$

where,$\int_0^t x f_{w^{2k}}(x,\alpha,k)dx = \frac{\sqrt{2}\alpha\gamma\left(\frac{2k+4}{2}, \frac{t^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}$

$$\mu_{w^{2k}}(t) = \frac{\sqrt{2}\alpha}{\Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{t^2}{2\alpha^2}\right)}\left\{\Gamma\left(\frac{k+2}{2}\right) - \gamma\left(\mathrm{k}+2, \frac{t^2}{2\alpha^2}\right)\right\} - t \tag{3.12}$$

The mean waiting time $\widehat{\mu_{w^{2k}}}$ of KWMBD is given by:

$$\widehat{\mu_{w^{2k}}} = t - \frac{1}{F_{w^{2k}}(x,\alpha,k)}\int_0^t x f_{w^{2k}}(x,\alpha,k)dx$$

$$\widehat{\mu_{w^{2k}}} = t - \frac{\sqrt{2}\alpha\gamma\left(\frac{2k+4}{2}, \frac{t^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(k+\frac{3}{2}, \frac{t^2}{2\alpha^2}\right)} \quad (3.13)$$

3.7 Entropy measures of KWMBD

Entropy measures provide a quantification of uncertainty or randomness within the system. We have calculated the expressions for Renyi entropy, Arimoto's entropy and Havrda and Charvat entropy.

The Renyi entropy for KWMBD is defined by

$$H_R(\delta) = \frac{1}{1-\delta} log\left\{\int_0^\infty f^\delta{}_{w^{2k}}(x,\alpha,k)dx\right\} \qquad \delta > 0, \delta \neq 0$$

Incorporating equation (4), integration $f^\delta{}_{w^{2k}}(x,\alpha,k)$ gives

$$\int_0^\infty f^\delta{}_{w^{2k}}(x,\alpha,k)dx = \frac{\sqrt{2}^{(\delta-1)}\alpha^{(1-\delta)}}{\Gamma\left(k+\frac{3}{2}\right)^\delta \delta^{\left(\delta(k+1)+\frac{1}{2}\right)}} \Gamma\left(\frac{2\delta(k+1)+1}{2}\right)$$

Hence, the Renyi entropy reduces to

$$H_R(\delta) = \frac{1}{1-\delta} log\left\{\frac{\sqrt{2}^{(\delta-1)}\alpha^{(1-\delta)}}{\Gamma\left(k+\frac{3}{2}\right)^\delta \delta^{\left(\delta(k+1)+\frac{1}{2}\right)}} \Gamma\left(\frac{2\delta(k+1)+1}{2}\right)\right\} \qquad \delta > 0, \delta \neq 0 \quad (3.14)$$

The Arimoto entropy for KWMBD is defined by

$$H_A(\delta) = \frac{1}{2^{(\delta-1)}-1}\left\{\left[\int_0^\infty f^\delta{}_{w^{2k}}(x,\alpha,k)dx\right]^{\frac{1}{\delta}} - 1\right\} \qquad \delta > 0, \delta \neq 0$$

Incorporating integration $f^\delta{}_{w^{2k}}(x,\alpha,k)$, the Arimoto entropy is given by

$$H_A(\delta) = \frac{1}{2^{(\delta-1)}-1}\left\{\left[\frac{\sqrt{2}^{\left(\frac{1-\delta}{\delta}\right)}\alpha^{\left(\frac{\delta-1}{\delta}\right)}\delta^{\left(\frac{2(k+1)+\delta}{2\delta}\right)}}{\Gamma\left(k+\frac{3}{2}\right)^{\frac{1}{\delta}}} \Gamma\left(\frac{2(k+1)+\delta}{2\delta}\right)\right]^{\frac{1}{\delta}} - 1\right\} \qquad \delta > 0, \delta \neq 0 \quad (3.15)$$

The Havrda and Charvat entropy for KWMBD is defined by

$$H_R(\delta) = \frac{1}{1-\delta}\left\{\int_0^\infty f^\delta{}_{w^{2k}}(x,\alpha,k)dx - 1\right\} \qquad \delta > 0, \delta \neq 0$$

Incorporating integration $f^\delta{}_{w^{2k}}(x,\alpha,k)$, the Havrda and Charvat entropy is given by

$$H_H(\delta) = \frac{1}{1-\delta}\left\{\left|\frac{\sqrt{2}^{(\delta-1)}\alpha^{(1-\delta)}}{\Gamma\left(k+\frac{3}{2}\right)^\delta \delta^{\left(\delta(k+1)+\frac{1}{2}\right)}} \Gamma\left(\frac{2\delta(k+1)+1}{2}\right)\right| - 1\right\} \qquad \delta > 0, \delta \neq 0 \quad (3.16)$$

Parameters	Entropy measures

α	k	δ	Renyi entropy	Arimoto entropy	Havrda and Charvat entropy
1	-1	1.5	0.6313	0.7737	0.5413
		2.0	0.5724	0.5832	0.4358
		2.5	0.5312	0.4433	0.3662
		3.0	0.5004	0.3378	0.3162
2	-1	1.5	1.3244	1.6023	0.9686
		2.0	1.2655	1.2390	0.7179
		2.5	1.2244	0.9540	0.5604
		3.0	1.1936	0.7320	0.4541
1	1	1.5	0.9438	1.1050	0.7524
		2.0	0.8882	0.8282	0.5886
		2.5	0.8491	0.6239	0.4801
		3.0	0.8197	0.4709	0.4029
2	1	1.5	1.6369	2.0197	1.1178
		2.0	1.5814	1.5854	0.7943
		2.5	1.5422	1.2278	0.6007
		3.0	1.5128	0.9433	0.4757

Table 3: Numerical analysis for entropy measures (Renyi entropy, Arimoto's entropy and Havrda and Charvat entropy).

The table 3 presented numerical analysis of Renyi entropy, Arimoto's entropy and Havrda and Charvat entropy for different value α , k and level of generalization δ. The results demonstrate that as we increase the parameters α and k value the entropy/uncertainty increases, means introducing more randomness and allows it for wide range of events. Also it is evident that for a fixed combination of parameters α and k as we increase level of generalization δ the entropy decreases, makes it more predictable and deterministic.

3.8 Odds ratio and Mills ratio

The odds ratio ($O_{w^{2k}}(x,\propto,k)$) and Mills ratio ($m_{w^{2k}}(x,\propto,k)$) for KWMBD distribution are given respectively by

$$O_{w^{2k}}(x,\propto,k) = \frac{F_{w^{2k}}(x,\propto,k)}{f_{w^{2k}}(x,\propto,k)} = \frac{2^{\left(k+\frac{1}{2}\right)}\left\{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right\}}{x^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{x^2}{2\alpha^2}}}, \quad x>0, \alpha>0, k>-1.5. \tag{3.17}$$

$$m_{w^{2k}}(x,\alpha,k) = \frac{R_{w^{2k}}(x,\propto,k)}{f_{w^{2k}}(x,\propto,k)} = \frac{2^{\left(k+\frac{1}{2}\right)}\left\{\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right\}}{x^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{x^2}{2\alpha^2}}}, \quad x>0, \alpha>0, k>-1.5. \tag{3.18}$$

In the particular case at $k=0$, where the proposed KWMBD reduces to the Maxwell distribution, the odds and mills ratio simplifies to the well-known form of the Maxwell distribution, and are given by

$$O(x,\alpha) = \frac{\sqrt{2}\left\{\Gamma\left(\frac{3}{2}\right) - \Gamma\left(\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right\}}{x^2\alpha^{-3}e^{-\frac{x^2}{2\alpha^2}}}, \quad x>0, \alpha>0.$$

$$m(x,\alpha) = \frac{\sqrt{2}\left\{\Gamma\left(\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right\}}{x^2\alpha^{-3}e^{-\frac{x^2}{2\alpha^2}}}, \qquad x > 0, \alpha > 0.$$

3.9 Lorenz inequality and Bonferoni inequality

Bonferroni and Lorenz curves were first presented by Bonferroni (1961) to measure the inequality of the distribution. The Bonferroni and Lorenz curve for a random variable X following KWMBD are respectively given by:

$$L(t) = \frac{\int_0^t x f_{w^{2k}}(x,\alpha,k)dx}{E(x)} = \frac{\gamma\left(\frac{2k+4}{2},\frac{t^2}{2\alpha^2}\right)}{\Gamma\left(\frac{2k+4}{2}\right)} \tag{3.19}$$

$$B(t) = \frac{L(t)}{F(t)} = \frac{\Gamma\left(k+\frac{3}{2}\right)\gamma\left(\frac{2k+4}{2},\frac{t^2}{2\alpha^2}\right)}{\Gamma\left(\frac{2k+4}{2}\right)\left[\Gamma\left(k+\frac{3}{2}\right)-\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{t^2}{2\alpha^2}\right)\right]} \tag{3.20}$$

3.10 Order statistics

Let us assume that the random sample $X_1, X_2, X_3 \ldots, X_n$ come from the KWMBD, with pdf $f_{w^{2k}}(x,\alpha,k)$ and cdf $F_{w^{2k}}(x,\alpha,k)$. Let $X_{(1)} \le X_{(2)} \le, \ldots, \le X_{(r)} \le, \ldots, \le X_{(n)}$denote the corresponding order statistics, then the pdf and cdf of the $r^{th}(r = 1,2,3,\ldots,n)$ order statistic are respectively, given by:

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!}\left[F_{w^{2k}}(x,\alpha,k)\right]^{r-1} f_{w^{2k}}(x,\alpha,k)\left[1-F_{w^{2k}}(x,\alpha,k)\right]^{n-r}$$

and, $F_r(x) = \sum_{j=r}^{n}\binom{n}{j} f_{w^{2k}}{}^{j}(x,\alpha,k)\left[1-F_{w^{2k}}(x,\alpha,k)\right]^{n-j}$.

$$f_r(x) = \frac{n!\,\alpha^{-(3+2k)}x^{2(k+1)}e^{-\frac{x^2}{2\alpha^2}}}{(r-1)!\,(n-r)!\,2^{\left(k+\frac{1}{2}\right)}\left(\Gamma\left(k+\frac{3}{2}\right)\right)^n}\left|\Gamma\left(k+\frac{3}{2}\right)-\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right|^{(r-1)}\left|\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right|^{(n-r)} \tag{3.21}$$

and,

$$F_r(x) = \frac{1}{\left(\Gamma\left(k+\frac{3}{2}\right)\right)^n}\sum_{j=r}^{n}\binom{n}{j}\left(\Gamma\left(k+\frac{3}{2}\right)-\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right)^j\left[\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right]^{n-j} \tag{3.21}$$

4. PARAMETRIC ESTIMATION OF KWMBD USING MAXIMUM LIKELIHOOD ESTIMATION TECHNIQUE

Let $X_1, X_2, X_3 \ldots, X_n$ be an observed sample taken from the KWMBD(k,α) with unknown parameters $\alpha$and k, then the log-likelihood function can be written as

$$\log L(x,\alpha,k) = -n\left(k+\frac{1}{2}\right)log2 + 2(k+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\frac{x_i{}^2}{2\alpha^2} - n(2k+3)\log(\alpha) - nlog\,\Gamma\left(k+\frac{3}{2}\right)$$

The values of $\widehat{\alpha}$ and $\widehat{k}$ that maximize the log-likelihood function are called the maximum likelihood estimates of the parameters α and k.

The equations obtained on equating the first-order partial derivatives of $\log L(x,\alpha,k)$ with respect to α and k to zero, are given as

$$\sum_{i=1}^{n}\frac{x^2}{\alpha^3} - \frac{n(2k+3)}{\alpha} = 0 \tag{4.1}$$

$$-nlog2 + 2\sum_{i=1}^{n} \log(x_i) - 2nlog(\alpha) - n\Psi\left(k+\frac{3}{2}\right) = 0 \quad (4.2)$$

Where, $\Psi\left(k+\frac{3}{2}\right) = \frac{\partial}{\partial x}\log\Gamma\left(k+\frac{3}{2}\right) = \frac{\acute{\Gamma}\left(k+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}$

The equations (28) and (29) must be solved simultaneously to obtain MLEs of KWMBD parameters. Since a closed form solution is not known, an iterative technique is required to compute the estimators $\hat{\alpha}$ and $\hat{k}$. The system of equations is solved by Newton-Raphson iteration method.

5. SIMULATION STUDY AND COMPUTATIONS OF KWMBD

In this subsection, we executed a simulation study to evaluate the accuracy of our estimated parameters of the KWMBD. Random datasets were generated from KWMBD using the inverse cdf method. In this method, a sample of size n from a particular distribution is obtained by solving the equation $F(x;\Theta) = p \sim U(0,1)$ for x, at preassigned values of Θ and at n independent values of p. Following the same procedure, the equation for generating random numbers from KWMBD is

$$F_{w^{2k}}(x,\alpha,k) = 1 - \frac{\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} = p$$

Solving this equation for x at n independent values of $p \sim U(0,1)$ and at fixed values of α and k, yields the required sample of size n from the KWMBD. Since solving this equation manually is tedious, the uniroot function from the R package stats is employed to numerically find the root. By applying the uniroot function to the equation above for each value of p, we can generate the desired sample from the KWMBD. Herein, we generated multiple random dataset from KWMBD, each with sizes of $25, 50, 100$ and 200. These datasets were replicated 100 times, considering various combinations of parameter values for $\alpha = (1,2)$ and $k = (0.5, 1, 1.5, 2)$. For each case, we computed the average estimates along with their corresponding mean square errors (MSEs) and bias. The results are present in Table 4.

Also, a simulated dataset comprising 51 different observations from KWMBD characterized by parameter values α=1 and k=2 is generated by solving the equation (5) using R software. This simulated data set is used for modelling comparison and represents quartiles of KWMBD. The R-code for resulted simulated dataset is as follows

```
> Data<-function(n,r,alpha,k)
+ {set.seed(1)
+ U=runif(n,0,1)
+ library(zipfR)
+ cdf<-function(x,alpha,k)
+ {fn<-1-Igamma(k+3/2,(x^2)/(2*alpha^2), lower=FALSE)/gamma(k+3/2)}
+ data=c() #Create an empty vector
+ for(i in 1:length(U)){
+ fn<-function(x){cdf(x,alpha,k)-U[i]}
 + uni<-uniroot(fn,c(0,100000))
+ data=c(data,uni$root)}
+ return(data)}
>Simulateddata<-Data(51,1,1,2)
>Simulateddata
>cat(round(Simulateddata,4))
2.0941 2.2942 2.6489 3.5039 1.9589 3.4595 3.7115 2.8152 2.7536 1.5318 1.9686 1.900 2.8683 2.3156 3.0537
2.515 2.9332 4.3624 2.3083 3.0725 3.6459 1.9823 2.7972 1.7641 2.0975 2.3191 1.1685 2.3125 3.346 2.2366
2.4877 2.6979 2.5078 1.9231 3.2081 2.8305 3.1157 1.7096 2.9465 2.3637 3.1893 2.7882 3.0864 2.6131 2.5715
3.1029 1.2844 2.4792 2.9657 2.8802 2.4799
```

n	Parameters		MLE		MSE		BIAS	
	α	K	$\hat{\alpha}$	$\hat{k}$	$\hat{\alpha}$	$\hat{k}$	$\hat{\alpha}$	$\hat{k}$
25	1	0.5	1.009443	1.114015	0.000581	0.411303	0.009443	0.614015
50			1.006308	1.083338	0.000260	0.359678	0.006308	0.583338
100			1.003223	1.060062	0.000045	0.322014	0.003223	0.560062
200			1.001965	1.043841	0.000019	0.298972	0.001965	0.543841
25	1	1	1.035333	1.444696	0.004062	0.353786	0.035333	0.444696
50			1.031529	1.402255	0.003174	0.275075	0.031529	0.402255
100			1.027932	1.388184	0.002170	0.225602	0.027932	0.388184
200			1.027186	1.356266	0.001941	0.175490	0.027186	0.356266
25	1	1.5	1.042998	1.835612	0.007009	0.455788	0.042998	0.335612
50			1.042728	1.825221	0.006382	0.371971	0.042728	0.325221
100			1.036248	1.807491	0.003979	0.237899	0.036248	0.307491
200			1.033539	1.790357	0.003361	0.199572	0.033539	0.290357
25	1	2	1.046555	2.338572	0.010094	0.719172	0.046555	0.338572
50			1.043798	2.329445	0.008459	0.591237	0.043798	0.329445
100			1.042290	2.295191	0.006747	0.425754	0.042290	0.295191
200			1.041246	2.267568	0.006300	0.283974	0.041246	0.267568
25	2	0.5	2.117254	0.860687	0.019129	0.243612	0.117254	0.360687
50			2.110143	0.846802	0.015213	0.166326	0.110143	0.346802
100			2.103364	0.832683	0.012019	0.133773	0.103364	0.332683
200			2.097136	0.829511	0.010047	0.125636	0.097136	0.329511
25	2	1	2.134382	1.276969	0.031200	0.287898	0.134382	0.276969
50			2.122661	1.248040	0.022329	0.165287	0.122661	0.248040
100			2.114878	1.238515	0.020267	0.112391	0.114878	0.238515
200			2.108847	1.231972	0.014906	0.085827	0.108847	0.231972
25	2	1.5	2.167331	1.694842	0.053833	0.393048	0.167331	0.194842
50			2.14118	1.67663	0.034150	0.246105	0.14118	0.17663
100			2.130761	1.651060	0.026514	0.133897	0.130761	0.151060
200			2.129223	1.648105	0.025962	0.096800	0.129223	0.148105
25	2	2	2.203508	2.145473	0.110299	0.838411	0.203508	0.145473
50			2.189253	2.120873	0.081828	0.492189	0.189253	0.120873
100			2.149048	2.111927	0.040105	0.255040	0.149048	0.111927
200			2.134574	2.100532	0.026184	0.146768	0.134574	0.100532

Table 4: Average values of MLEs and the corresponding MSEs and Bias values**.**

The results presented in Table 4 reveals that as we increase the size of the dataset, the precision of our parameter estimates improves. Additionally, we observed a decrease in both MSE and bias with an increase in sample size. This suggests that the estimators are consistent and maximum likelihood (ML) estimation performs effectively.

6. APPLICATIONS OF 2KTH ORDER WEIGHTED MAXWELL-BOLTZMANN DISTRIBUTION

This section illustrates the practical applicability of the KWMBD through analysing three real-life datasets and one simulated data. The objective is to access the versatility and compatibility of the proposed model in comparison to its sub-models. For that propose, we are using the maximum likelihood estimation technique for parameter estimation and various model selection tools. Generally, a superior distribution is indicated by smaller values of these model selection tools.

The data set I represent the tensile strength measures in GPA of 69 carbon fibres tested under tension at gauge lengths of 20mm, reported first by Bader and Priest (1982). While as, data set II is related to the logarithm of light intensity of 47 stars in the star cluster CYG OB1, reported first by Rousseeuw and Leroy (1987) and Data set III represents the resistance of 48 semiconductor devices, reported first by R. C., Milliken, Stroup and Wolfinger (1996).

Tensile Strength Data								
Model	**Estimates**			**Model selection tools**				
	$\hat{\alpha}$	$\hat{k}$	$\hat{\beta}$	**-2logl**	**AIC**	**BIC**	**AICC**	**HQIC**
KWMBD	0.709	4.723	-	98.463	102.463	106.931	102.645	104.236
ABMD	0.800	-	-	121.735	123.735	125.969	123.795	124.622
LBMD	1.250	-	-	132.569	134.569	136.803	134.628	135.455
LBWMD	1.541	-	0.493	132.569	136.569	141.037	136.750	138.341
MD	1.443	-	-	148.586	150.586	152.820	150.646	151.472
RD	1.768	-	-	174.493	176.493	178.727	176.553	177.379
HND	2.500	-	-	226.614	228.614	230.849	228.674	229.501
Light Intensity Data								
Model	**Estimates**			**Model selection tools**				
	$\hat{\alpha}$	$\hat{k}$	$\hat{\beta}$	**-2logl**	**AIC**	**BIC**	**AICC**	**HQIC**
KWMBD	0.802	18.277	-	79.650	83.650	87.351	83.923	85.043
ABMD	0.197	-	-	138.177	140.177	142.027	140.266	140.874
LBMD	2.522	-	-	148.236	150.236	152.086	150.325	150.932
LBWMD	1.589	-	0.125	148.236	152.236	155.936	152.508	153.628
MD	2.912	-	-	161.825	163.825	165.675	163.914	164.521
RD	3.567	-	-	182.151	184.151	186.001	184.240	184.847
HND	5.044	-	-	220.333	222.333	224.183	222.422	223.029
Semiconductor Resistance Data								
Model	**Estimates**			**Model selection tools**				
	$\hat{\alpha}$	$\hat{k}$	$\hat{\beta}$	**-2logl**	**AIC**	**BIC**	**AICC**	**HQIC**
KWMBD	0.713	34.146	-	70.290	74.290	78.033	74.557	75.704
ABMD	0.138	-	-	155.970	157.970	159.841	158.057	158.677
LBMD	4.259	-	-	166.790	168.790	170.661	168.877	169.497
LBWMD	1.591	-	0.088	166.790	170.790	174.532	171.056	172.204
MD	3.478	-	-	181.216	183.216	185.087	183.303	183.923
RD	1.184	-	-	202.521	204.521	206.393	204.608	205.229
HND	6.024	-	-	242.063	244.063	245.934	244.150	244.770
Simulated Data								
Model	Estimates			**Model selection tools**				
	$\hat{\alpha}$	$\hat{k}$	$\hat{\beta}$	**-2logl**	**AIC**	**BIC**	**AICC**	**HQIC**
KWMBD	0.906	2.882	-	97.096	101.096	104.959	101.346	102.572
ABMD	0.696	-	-	104.432	106.432	108.364	106.514	107.170
LBMD	1.341	-	-	110.607	112.607	114.539	112.689	113.346
LBWMD	1.529	-	0.425	110.607	114.607	118.471	114.857	116.084
MD	1.548	-	-	120.614	122.614	124.546	122.696	123.353
RD	1.896	-	-	137.931	139.931	141.863	140.013	140.669
HND	2.681	-	-	174.624	176.624	178.555	176.705	177.362

Table 5: MLEs and Model selection tools (-2logl, AIC, BIC, AICC, HQIC) for all the dataset**.**

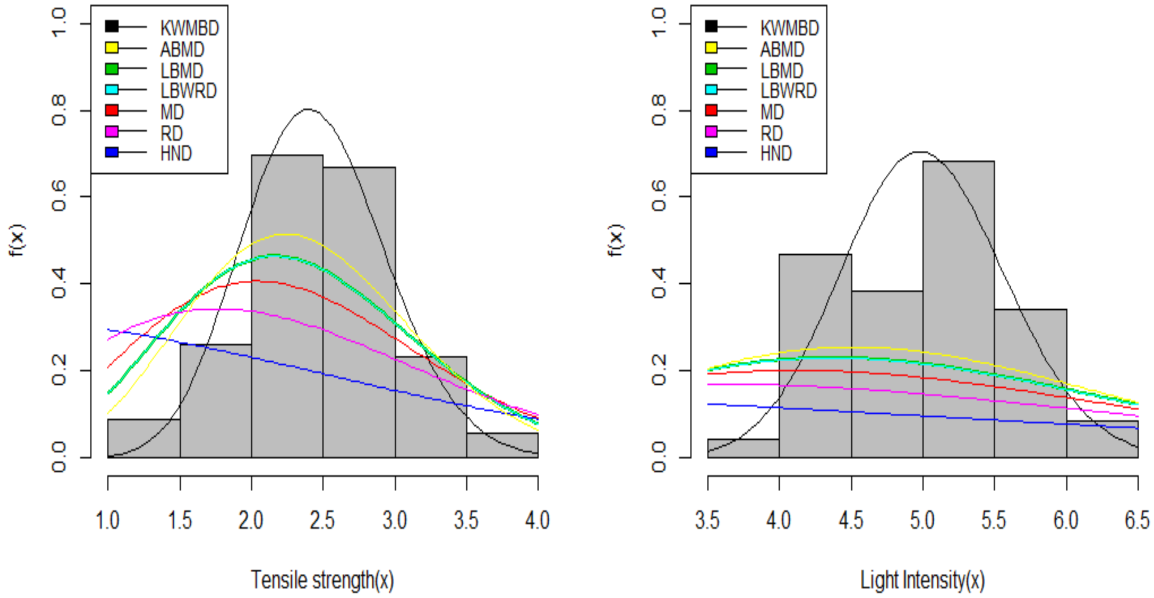

Figure 3: The histogram and fitted density functions for dataset I and II.

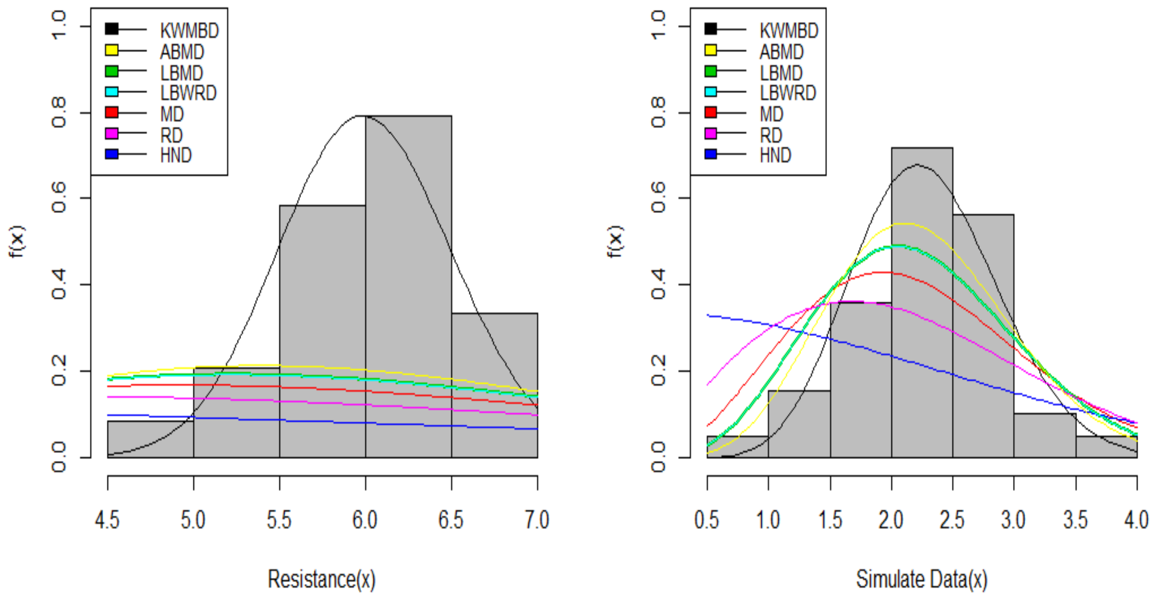

Figure 4: The histogram and fitted density functions for dataset III and simulated dataset.

The results in Table 5 clearly demonstrate that the proposed model consistently yields the smallest values across the model selection tools. Therefore, we conclude that the KWMBD distribution provides a better fit than the other compared models. Also the results are validated by figure 3 and 4.

7. CONCLUSION

To enhance the modelling of complex real world data, researchers are focusing on developing more adaptable models. Consequently, there have been significant efforts to generalize the classical Maxwell distribution. In this manuscript, a two-parametric $2K^{th}$ Order Weighted Maxwell-Boltzmann Distribution (KWMBD) is formulated using a Generalized Even Power Weighted Probability Technique. We investigate some statistical properties and estimate the KWMBD parameters using maximum likelihood estimation method. Numerical analysis of structural properties reveals that as the parameter (α and k) values increases, the distribution becomes more concentrated with reduced variability. Additionally, the entropy measures analysis shows that increasing parameter (α and k) values lead to higher entropy, indicating a shift toward greater randomness and allowing the distribution to model a broader range of events. The simulation study reveals that as the sample size increases, the maximum likelihood estimators (MLEs) tend to converge to the true parameter values. Moreover, a decrease in bias and mean squared error (MSE) of MLEs is observed with larger sample sizes. For practical validation, we apply the proposed distribution to three real world datasets related to mechanical physics, along with a simulated dataset. It is observed from results that the proposed distribution offers better fits than its sub-models when applied to these datasets. Further research can focus on both theory and applications. On the theoretical side, the model may be extended through new parameterizations, Bayesian methods and asymptotic studies. Practically, it can be applied in fields like reliability, biomedical sciences, finance and environmental studies to assess its usefulness with real data.

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REFERENCES

1. BADER, M. G., PRIEST, A. M. (1982). Statistical aspects of fibre and bundle strength in hybrid composites**. Progress in Science in Engineering Composites**. Tokyo: ICCM-IV , 1129–1136.
2. BEKKER, A., ROUX, J. (2005). Reliability characteristics of the Maxwell distribution: A Bayes estimation study. Communications in Statistics Theory and Methods 34(11), 2169–2178.
3. BONFERRONI, C. E. (1961). **Elementi di statistica generale: anno accademico** 1940-41-XIX. Universita commerciale L. Bocconi.
4. CASTILLO, J.D., CASANY, M.P. (1998). Weighted Poisson Distributions and under dispersion Situations. **Annals of the institute of statistical mathematics**, 5, 567-585.
5. CASTILLO, J.S., GAETE, K.P., MUNOZ, A.H. (2023). Scale mixture of Maxwell Boltzmann distribution. **Mathematics**, 11(3), 529.
6. CHATFIELD C. (1995) **Problem Solving: A Statistician's Guide**, 2ed Chapman Hall.
7. CHATURVEDI, A., RANI, U. (1998). Classical and Bayesian reliability estimation of the Generalised Maxwell failure distribution. **Journal of Statistical Research**, 32, 113-120.
8. DAR, A., A. AHMED, J. RESHI.(2018). Characterization and estimation of weighted maxwell-Boltzmann distribution. **Applied Mathematics & Information Sciences** 12 (1):193–202.
9. FALLAH, A.,R. KAZEMI. (2022). Statistical inference for the Generalised weighted exponential distribution**. Communications in Statistics - Simulation and Computation** 51 (8),4296–313.
10. FATIMA, K., AHMED, S.P. (2017). Weighted Inverse Raleigh Distribution**. International Journal of Statistics and Systems,** 12(1), 119-137.
11. FISHER, R.A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. **The Annals of Eugenics,** 6, 13-25.
12. GHITANY, M., S. WANG.(2022). A note on parameter asymptotic for weighted Lindley distribution. Communications in Statistics - **Simulation and Computation** 51 (4),1513–24.
13. HUANG, J., CHEN, S. (2016). Tail behaviour of the Generalised Maxwell distribution. **Communications in Statistics-Theory and Methods** 45(14), 4230–4236.
14. KAZMI, S., ASLAM, M., ALI, S. (2012). On the Bayesian estimation for two component mixture of Maxwell distribution, assuming type I censored data**. International Journal of Applied Science and Technology** 2(1), 197–218.
15. MAXWELL, J. (1867). On the dynamical theory of gases. Philos. Trans. R. Soc. Lond. 35, 185–217.
16. MODI, K. (2015). Length-Biased Weighted Maxwell Distribution. **Pakistan Journal of Statistics and Operation Research**, 11, 465-472.
17. PATIL, G.P., RAO, G.R. (1978). Weighted distributions and size biased sampling with applications to wildlife populations and human families. **Biometrics,** 34, 179-189.

18. RAO, C.R. (1965). On discrete distributions arising out of methods of ascertainment, in **Classical and Contagious Discrete Distributions**, G.P. Patil, ed., Pergamon Press and Statistical Publishing Society, Calcutta, pp. 320-332.
19. RESHI, J.A. (2021). Parameter estimation of weighted Maxwell Boltzmann distribution. **Intech open publisher** (1),1-21.
20. ROUSSEEUW, P., A. LEROY (1987). Robust Regression and Outlier Detection. New York: Wiley.
21. SAGHIR, A., HAMEDANI, G.G., TAZEEM, S., KHADIM, A. (2017). Weighted Distributions: A Brief Review, Perspective and Characterizations**. International Journal of Statistics and Probability,** 6(3), 109-131.
22. SAGHIR, A., HAMEDANI, G.G. (2018). A generalised weighted Maxwell distribution: properties and estimation**. Pakistan journal of statistics**, 34(3), 195-209.
23. SEGOVIA, F.; GÓMEZ, Y.M.; GALLARDO, D.I. (2021). Exponentiated power Maxwell distribution with quantile regression and applications. **Statistics and operations research transactions**, 45, 181–200.
24. SHAKHATREH, M., A. Q. AL-MASRI. (2020). On the weighted burrxii distribution: Theory and practice. **Electronic Journal of Applied Statistical Analysis,** 13 (1) ,229–55.
25. SHARMA, V.K.; DEY, S.; SINGH, S.K.; MANZOOR, U. (2017). On length and area-biased Maxwell distributions. **Communication in Statistics- Simulation and Computation**, 47, 1506–1528.
26. TYAGI, R.K., BHATTACHARYA, S.K. (1989). A note on the MVU estimation of reliability for the Maxwell failure distribution. **Estadistica**, 41, 73-79.