MODIFIED CLASS OF ESTIMATORS OF POPULATION VARIANCE USING AUXILIARY ATTRIBUTE

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ABSTRACT
This article proposes a modified class of estimators of population variance under simple random sampling using information on auxiliary attribute that includes usual mean, ratio and regression estimators, Srivastava (1967) type estimators, Walsh (1970) type estimators and Kadilar and Cingi (2004) type estimators adapted by Singh and Malik (2014) for particular values of characterizing scalars. The expression of mean square error of proposed class of estimators is obtained up to first order of approximation. The proposed class of estimators is theoretically compared with the estimators existing till date and the efficiency conditions are determined. Further, the theoretical findings are numerically supported by using a real data set.

KEYWORDS: Mean square error, Efficiency, Auxiliary attribute.

MSC 2020: 62D05

RESUMEN
Este artículo propone una clase modificada de estimadores de la varianza poblacional bajo muestreo aleatorio simple, utilizando información sobre un atributo auxiliar que incluye los estimadores habituales de media, razón y regresión, así como los estimadores tipo Srivastava (1967), tipo Walsh (1970) y tipo Kadilar y Cingi (2004), adaptados por Singh y Malik (2014) para valores particulares de escalares característicos. La expresión del error cuadrático medio de la clase propuesta de estimadores se obtiene hasta el primer orden de aproximación. La clase propuesta de estimadores se compara teóricamente con los estimadores existentes hasta la fecha y se determinan las condiciones de eficiencia. Además, los hallazgos teóricos se respaldan numéricamente mediante el uso de un conjunto de datos reales.

PALABRAS CLAVE: Error cuadrático medio, Eficiencia, Atributo auxiliar.

1. INTRODUCTION

In sampling surveys, it is well known that the use of auxiliary information helps to enhance the efficiency of the estimator of population parameters of choice such as total, mean and variance of the variable of interest. In literature of survey sampling several authors have introduced a wide range of estimators of population parameters based on information about the population parameters of the auxiliary variable. Few recent relevant contributions

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in this direction include Zaman and Kadilar (2019), Bhushan and Kumar (2020a, b), Bhushan and Kumar (2021) and Bhushan *et al.* (2021). However, in several real life scenarios, instead of an auxiliary variable x there exist an auxiliary attribute (say, ϕ) that is highly correlated with the study variable y. For example:

(i). Amount of production of paddy crop (y) and a particular variety of paddy (ϕ),

(ii). Height of persons (y) and sex (ϕ),

(iii). Amount of milk produce (y) and a particular breed of buffalo (ϕ), etc.

In such conditions, considering the advantage of point bi-serial correlation (ρ) into practice several well-known authors namely Naik and Gupta (1996), Jhajj *et al.* (2006), Singh *et al.* (2008), Abd-Elfattah *et al.* (2010), Grover and Kaur (2011), Singh and Solanki (2012), Koyuncu (2012), Haq and Shabbir (2014), Zaman and Kadilar (2019), Zaman (2020) and Bhushan and Gupta (2020) investigated various improved estimators for the estimation of population mean consist of information on auxiliary attribute.
In most of the cases, one may be interested in the estimation of population variance of study variable y. When the prior information on parameters of auxiliary attribute is available, Singh and Kumar (2011) suggested conventional ratio, regression and exponential estimators of population variance utilizing auxiliary attribute. Adapting the work of Kadilar and Cingi (2004), Singh and Malik (2014) developed a new family of estimators of population variance utilizing information on auxiliary attribute. Adichwal *et al.* (2015) investigated few improved class of estimators of population variance based on auxiliary attribute whereas Adichwal et al. (2016) adapted Koyuncu (2012) estimator and investigated a generalized class of estimators for population variance using auxiliary attribute. Following Shabbir and Gupta (2006) and Singh and Solanki (2013), Singh and Pal (2018) invoked a new class of estimators for the population variance using known population proportion. This paper considers the problem of estimating population variance S_y^2 using information on auxiliary attribute.

NOTATIONS

To estimate the population variance $S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i-\bar{Y})^2$, let a sample of size n be quantified from a finite population $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_N)$ using simple random sampling without replacement ($SRSWOR$). Let y_i and ϕ_i be the total amount of units on study variable y and auxiliary attribute ϕ for unit i of the population κ. It is to be noted that the attribute ϕ_i=1 if the unit i possess the attribute ϕ and $\phi_i = 0$, otherwise. Let $A = \sum_{i=1}^{N}\phi_i$ and $a = \sum_{i=1}^{n}\phi_i$ be the total number of units in the population κ and sample respectively possessing attribute ϕ whereas $P = (A/N)$ and $p = (a/n)$ respectively denote the population proportion and sample proportion having attribute ϕ. Let $\bar{y} = n^{-1}\sum_{i=1}^{n}y_i$ and $\bar{Y} = N^{-1}\sum_{i=1}^{N}y_i$ respectively be the sample and population means of study variable y; $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i-\bar{y})^2$ and $S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i-\bar{Y})^2$ be the sample and population variances of the study variable y respectively; $s_\phi^2 = (n-1)^{-1}\sum_{i=1}^{n}(\phi_i-p)^2$ and $S_\phi^2 = (N-1)^{-1}\sum_{i=1}^{N}(\phi_i-P)^2$ be the sample and population variances of auxiliary attribute ϕ respectively.
To find out the bias and mean square error (MSE) of the suggested estimators, let us define $s_y^2 = S_y^2(1+e_0)$, $s_\phi^2 = S_\phi^2(1+e_1)$ provided $E(e_0) = E(e_1) = 0$ and

$$V_{r,s} = \frac{E[(s_\phi^2-S_\phi^2)^r(s_y^2-S_y^2)^s]}{(S_\phi^2)^r(S_y^2)^s} \tag{1.1}$$

Using (1.1), we can write

$$E(e_0^2) = \gamma(\lambda_{40} - 1) = V_{0,2} \tag{1.2}$$

$$E(e_1^2) = \gamma(\lambda_{04} - 1) = V_{2,0} \tag{1.3}$$

$$E(e_0, e_1) = \gamma(\lambda_{22} - 1) = V_{1,1} \tag{1.4}$$

where $\gamma = (1-f)/n$, $f = n/N$, $\lambda_{pq} = \mu_{pq}/\mu_{20}^{p}\mu_{02}^{q}$ and $\mu_{pq} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^p(\phi_i - P)^q$.
The rest of the article is arranged in the following sections. In Section 2., we consider a literature review of the existing estimators with their properties. In Section 3., we develop a modified class of estimators and studied their properties. The efficiency conditions are derived in Section 4. whereas a numerical study is performed in Section 5.. Lastly, the conclusion is given in Section 6..

2. LITERATURE REVIEW

In sequence of getting an estimate of the population variance of study variable y, various authors considered the known population proportion and suggested different class of estimators under simple random sampling (*SRS*).
The variance of the traditional mean estimator $s_m^2 = s_y^2$ is given by

$$V(s_m^2) = S_y^4 V_{0,2} \tag{2.1}$$

Singh and Kumar (2011) suggested the following class of estimators as

$$s_r^2 = s_y^2 \left(\frac{S_\phi^2}{s_\phi^2}\right) \tag{2.2}$$

$$s_{lr}^2 = s_y^2 + \beta_\phi(S_\phi^2 - s_\phi^2) \tag{2.3}$$

$$s_e^2 = s_y^2 \exp\left(\frac{S_\phi^2 - s_\phi^2}{S_\phi^2 + s_\phi^2}\right) \tag{2.4}$$

where β_ϕ is the regression coefficient of y on ϕ.
The *MSE* of the estimators s_r^2, s_{lr}^2 and s_e^2 are given by

$$MSE(s_r^2) = S_y^4(V_{0,2} + V_{2,0} - 2V_{1,1}) \tag{2.5}$$

$$MSE(s_{lr}^2) = (S_y^4 V_{0,2} + \beta_\phi^2 S_\phi^4 V_{2,0} - 2\beta_\phi^2 S_y^2 S_\phi^2 V_{1,1}) \tag{2.6}$$

$$MSE(s_e^2) = S_y^4\left(V_{0,2} + \frac{V_{2,0}}{4} - V_{1,1}\right) \tag{2.7}$$

The minimum *MSE* of the estimator s_{lr}^2 at optimum value of $\beta_\phi = (S_y^2 V_{1,1}/S_\phi^2 V_{2,0})$ is given by

$$minMSE(s_{lr}^2) = S_y^4\left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right] \tag{2.8}$$

Motivated by Hansen *et al.* (1954), Srivastava (1967) and Walsh (1970), one may define a class of difference and ratio type estimators as

$$s_1^2 = s_y^2 + \theta_1({s_\phi^*}^2 - {S_\phi^*}^2) \tag{2.9}$$

$$s_2^2 = s_y^2 \left(\frac{{S_\phi^*}^2}{{s_\phi^*}^2}\right)^{\theta_2} \tag{2.10}$$

$$s_3^2 = s_y^2 \left(\frac{{S_\phi^*}^2}{{S_\phi^*}^2 + \theta_3({s_\phi^*}^2 - {S_\phi^*}^2)}\right) \tag{2.11}$$

where θ_1, θ_2 and θ_3 are suitably chosen scalars. Also, ${s_\phi^*}^2 = (cs_\phi^2 + d)$ and ${S_\phi^*}^2 = (cS_\phi^2 + d)$ given that c and d are either real values or function of available parameters of auxiliary attribute ϕ such as standard deviation S_ϕ, coefficient of variation C_ϕ, coefficient of kurtosis $\beta_2(\phi)$ and coefficient of correlation ρ between study variable y and auxiliary attribute ϕ.

The *MSE* of the estimators $s_i^2,\ i = 1,2,3$ are given by

$$MSE(s_1^2) = S_y^4 V_{0,2} + \upsilon^2 \theta_1^2 S_\phi^4 V_{2,0} + 2\upsilon\theta_1 S_y^2 S_\phi^2 V_{1,1} \tag{2.12}$$

$$MSE(s_i^2) = S_y^4 (V_{0,2} + \upsilon^2 \theta_i^2 V_{2,0} - 2\upsilon\theta_i V_{1,1}),\ i = 2,3 \tag{2.13}$$

where $\upsilon = cS_\phi^2/(cS_\phi^2 + d)$. The minimum *MSE* at optimum values of $\theta_{1(opt)} = -S_y^2 V_{1,1}/(\upsilon S_\phi^2 V_{2,0})$, $\theta_{i(opt)} = V_{1,1}/(\upsilon V_{2,0}),\ i = 2,3$ is given by

$$minMSE(s_i^2) = S_y^4 \left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right],\ i = 1,2,3 \tag{2.14}$$

which attains the minimum *MSE* of the conventional regression estimator s_{lr}^2.

On the lines of Kadilar and Cingi (2004), Singh and Malik (2014) adapted the following class of estimators as

$$s_{kc_1}^2 = s_y^2 \left(\frac{S_\phi^2 + C_\phi}{s_\phi^2 + C_\phi}\right) \tag{2.15}$$

$$s_{kc_2}^2 = s_y^2 \left(\frac{S_\phi^2 + \beta_2(\phi)}{s_\phi^2 + \beta_2(\phi)}\right) \tag{2.16}$$

$$s_{kc_3}^2 = s_y^2 \left(\frac{S_\phi^2 \beta_2(\phi) + C_\phi}{s_\phi^2 \beta_2(\phi) + C_\phi}\right) \tag{2.17}$$

$$s_{kc_4}^2 = s_y^2 \left(\frac{S_\phi^2 C_\phi + \beta_2(\phi)}{s_\phi^2 C_\phi + \beta_2(\phi)}\right) \tag{2.18}$$

The *MSE* of the estimators $s_{kc_i}^2,\ i = 1,2,3,4$ are given as

$$MSE(s_{kc_i}^2) = S_y^4 (V_{0,2} + \omega_i^2 V_{2,0} - 2\omega_i V_{1,1}),\ i = 1,2,3,4 \tag{2.19}$$

where $\omega_1 = S_\phi^2/(S_\phi^2 + C_\phi)$, $\omega_2 = S_\phi^2/(S_\phi^2 + \beta_2(\phi))$, $\omega_3 = S_\phi^2\beta_2(\phi)/(S_\phi^2\beta_2(\phi) + C_\phi)$ and $\omega_4 = S_\phi^2 C_\phi/(S_\phi^2 C_\phi + \beta_2(\phi))$.

Following Singh *et al.* (2008), Singh and Malik (2014) also adapted the undermentioned class of estimators for the population variance using information on auxiliary attribute as

$$s_s^2 = \{s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)\} \left(\frac{aS_\phi^2 + b}{as_\phi^2 + b} \right) \tag{2.20}$$

where a and b are either real numbers or the functions of known parameters of auxiliary attribute ϕ such as, C_ϕ, ρ, $\beta_2(\phi)$ etc. Few members of the estimator s_s^2 are discussed in Table 1 for ready reference.
The MSE of the estimator s_s^2 is given by

$$MSE(s_s^2) = S_y^4 V_{0,2} + V_{2,0} \left\{\beta_\phi^2 S_\phi^4 + A_1^2 S_y^4 + 2A_1 \beta_\phi S_y^2 S_\phi^2\right\} - 2S_y^2 V_{1,1} \left\{\beta_\phi S_\phi^2 + A_1 S_y^2\right\} \tag{2.21}$$

where $A_1 = aS_\phi^2/(aS_\phi^2 + b)$.
Singh and Malik (2014) suggested the following class of estimators given as

Table 1: Some members of Singh *et al.* (2008) estimators s_s^2

Members of estimator $s_{s(j)}^2$	Values of	
$j = 1, 2, ..., 9$	a	b
$s_{s_{(1)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 + \beta_2(\phi)}{s_\phi^2 + \beta_2(\phi)}\right]$	1	$\beta_2(\phi)$
$s_{s_{(2)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 + C_\phi}{s_\phi^2 + C_\phi}\right]$	1	C_ϕ
$s_{s_{(3)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 \beta_2(\phi) + C_\phi}{s_\phi^2 \beta_2(\phi) + C_\phi}\right]$	$\beta_2(\phi)$	C_ϕ
$s_{s_{(4)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 C_\phi + \beta_2(\phi)}{s_\phi^2 C_\phi + \beta_2(\phi)}\right]$	C_ϕ	$\beta_2(\phi)$
$s_{s_{(5)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 + \rho}{s_\phi^2 + \rho}\right]$	1	ρ
$s_{s_{(6)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 C_\phi + \rho}{s_\phi^2 C_\phi + \rho}\right]$	C_ϕ	ρ
$s_{s_{(7)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 \rho + C_\phi}{s_\phi^2 \rho + C_\phi}\right]$	ρ	C_ϕ
$s_{s_{(8)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 \beta_2(\phi) + \rho}{s_\phi^2 \beta_2(\phi) + \rho}\right]$	$\beta_2(\phi)$	ρ
$s_{s_{(9)}}^2 = [s_y^2 + \beta_\phi (S_\phi^2 - s_\phi^2)] \left[\frac{S_\phi^2 \rho + \beta_2(\phi)}{s_\phi^2 \rho + \beta_2(\phi)}\right]$	ρ	$\beta_2(\phi)$

$$s_{sm}^2 = s_y^2 [m_1 + m_2 (S_\phi^2 - s_\phi^2)] \exp\left\{\delta \frac{(cS_\phi^2 + d) - (cs_\phi^2 + d)}{(cS_\phi^2 + d) + (cs_\phi^2 + d)}\right\} \tag{2.22}$$

where m_1 and m_2 are suitably chosen scalars and δ assumes values +1 and -1 to design different estimators. Few unknown members of the estimator s_{sm}^2 are discussed in Table 2 for ready reference.
The MSE of the estimator s_{sm}^2 up to the first order of approximation is given by

$$MSE(s_{sm}^2) = S_y^4 \left[1 + m_1^2 R_1 + m_2^2 R_2 + 2m_1 m_2 R_3 - 2m_1 R_4 - 2m_2 R_5\right] \tag{2.23}$$

where $R_1 = 1 + V_{0,2} + \delta^2 \nu^2 V_{2,0} + 2\delta (1 + \delta/2) \nu^2 V_{2,0} - 4\delta \nu V_{1,1}$, $R_2 = S_\phi^4 V_{2,0}$; $R_3 = S_\phi^2 [2V_{1,1} + 2\delta \nu V_{2,0}]$; $R_4 = 1 + \delta (1 + \delta/2) \nu^2 V_{2,0} - \delta \nu V_{1,1}$; $R_5 = S_\phi^2 [\delta \nu V_{2,0} - V_{1,1}]$ and $\nu = cS_\phi^2 / 2(cS_\phi^2 + d)$.

The minimum MSE at optimum values of $m_{1(opt)} = (R_2R_4 - R_3R_5)/(R_1R_2 - R_3^2)$ and $m_{2(opt)} = (R_1R_5 - R_3R_4)/(R_1R_2 - R_3^2)$ is expressed by

$$minMSE(s^2_{sm}) = S_y^4\left[1 - \frac{(R_1R_5^2 + R_2R_4^2 - 2R_3R_4R_5)}{(R_1R_2 - R_3^2)}\right] \tag{2.24}$$

On the lines of Bhushan and Gupta (2016, 2020), we define a log type estimator for population variance using attribute as

$$s^2_{bg} = s_y^2\left[1 + \log\left(\frac{s_\phi^{*^2}}{S_\phi^{*^2}}\right)\right]^{\theta_4} \tag{2.25}$$

The minimum MSE at optimum value of $\theta_{4(opt)} = -V_{1,1}/(\upsilon V_{2,0})$ is given by

$$minMSE(s^2_{bg}) = S_y^4\left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right] \tag{2.26}$$

which is similar to the minimum MSE of the conventional regression estimator s^2_{lr}.

3. PROPOSED ESTIMATORS

Following the procedure of Kadilar and Cingi (2006), we have developed the following classes of estimators for population variance by combining respectively the class of difference, Srivastava and Walsh type estimators given in (2.9), (2.10) and (2.11) with the log type estimator defined on the lines of Bhushan and Gupta (2016, 2020) given in (2.25) as

$$s^2_{bk_1} = \zeta_1\{s_y^2 + \theta_1(s_\phi^{*^2} - S_\phi^{*^2})\} + \psi_1 s_y^2\left[1 + \log\left(\frac{s_\phi^{*^2}}{S_\phi^{*^2}}\right)\right]^{\theta_4} \tag{3.1}$$

$$s^2_{bk_2} = \zeta_2 s_y^2\left(\frac{S_\phi^{*^2}}{s_\phi^{*^2}}\right)^{\theta_2} + \psi_2 s_y^2\left[1 + \log\left(\frac{s_\phi^{*^2}}{S_\phi^{*^2}}\right)\right]^{\theta_4} \tag{3.2}$$

$$s^2_{bk_3} = \zeta_3 s_y^2\left(\frac{S_\phi^{*^2}}{S_\phi^{*^2} + \theta_3(s_\phi^{*^2} - S_\phi^{*^2})}\right) + \psi_3 s_y^2\left[1 + \log\left(\frac{s_\phi^{*^2}}{S_\phi^{*^2}}\right)\right]^{\theta_4} \tag{3.3}$$

where ζ_i and ψ_i, $i = 1,2,3$ are suitably chosen characterizing scalars. Some unknown members of these classes of estimators are given in Table 2 for ready reference. The MSE of all these estimators is given in appendix A for ready reference.

The objective of this paper is to investigate an efficient choice to the survey practitioners. The proposed estimator furnish a better alternative to the existing estimators of this study. We propose the following modified estimator for the estimation of population variance using auxiliary attribute as

$$s_a^2 = \left[w_1 s_y^2 + w_2 s_y^2\left(\frac{cS_\phi^2 + d}{\alpha(cs_\phi^2 + d) + (1-\alpha)(cS_\phi^2 + d)}\right)^g\right]\left[1 + \log\left(\frac{cs_\phi^2 + d}{cS_\phi^2 + d}\right)\right]^{\theta_4} \tag{3.4}$$

where α, g are scalars that assumes real values to construct several estimators. However, w_1, w_2 and θ_4 are suitably chosen scalars to be determined. The proposed class of estimator s_a^2 disfigure into:

(i). the usual mean estimator s_m^2 for $(w_1, w_2, \theta_4) = (1,0,0)$,

(ii). the classical ratio estimator s_r^2 for $(w_1, w_2, \alpha, g, c, d, \theta_4) = (0,1,1,1,1,0,0)$,

(iii). Srivastava (1967) type estimator s_2^2 for $(w_1, w_2, \alpha, g, c, d, \theta_4) = (0,1,1,\theta_2,1,0,0)$,

(iv). Walsh (1978) type estimator s_3^2 for $(w_1, w_2, \alpha, g, c, d, \theta_4) = (0,1,\theta_3,1,1,0,0)$,

(v). Kadilar and Cingi (2004) type estimator $s_{kc_1}^2$ for $(w_1, w_2, \alpha, g, c, d, \theta_4) = (0,1,1,1,1,C_x,0)$,

(vi). Kadilar and Cingi (2004) type estimator $s_{kc_2}^2$ for $(w_1, w_2, \alpha, g, c, d, \theta_4)$=$(0,1,1,1,1,\beta_2(x),0)$,

(vii). Kadilar and Cingi (2004) type estimator $s_{kc_3}^2$ for $(w_1, w_2, \alpha, g, c, d, \theta_4) = (0,1,1,1,\beta_2(x),C_x,0)$,

(viii). Kadilar and Cingi (2004) type estimator $s_{kc_4}^2$ for $(w_1, w_2, \alpha, g, c, d, \theta_4) = (0,1,1,1,C_x,\beta_2(x),0)$.

Several other estimators can be generated from the suggested estimators for different values of auxiliary attribute. Furthermore, few unknown members of the proposed estimators are reported in Table 2 for ready reference. Now, considering the notations defined in earlier section, the proposed estimator s_a^2 is converted in terms of $e's$ as

$$s_a^2 = \left[w_1 S_y^2(1+e_0) + w_2 S_y^2(1+e_0)(1+\alpha\upsilon e_1)^{-g}\right]\left[1+\upsilon e_1 - \frac{\upsilon^2 e_1^2}{2}\right]^{\theta_4}$$

$$s_a^2 - S_y^2 = S_y^2 \begin{bmatrix} w_1 + w_2 - 1 + w_1 e_0 + w_2 e_0 + w_1\theta_4\upsilon e_1 + w_2\theta_4\upsilon e_1 - w_2 g\alpha\upsilon e_1 - \\ w_2 g\alpha\upsilon e_0 e_1 + w_2\frac{g(g+1)}{2}\alpha^2\upsilon^2 e_1^2 + w_1\theta_4\upsilon e_0 e_1 + w_2\theta_4\upsilon e_0 e_1 \\ -w_2\theta_4 g\alpha\upsilon^2 e_1^2 - w_1\theta_4\upsilon^2 e_1^2 - w_2\theta_4\upsilon^2 e_1^2 + w_1\frac{\theta_4^2}{2}\upsilon^2 e_1^2 + w_2\frac{\theta_4^2}{2}\upsilon^2 e_1^2 \end{bmatrix} \tag{3.5}$$

Squaring and taking expectation both sides of (3.5), we obtain the *MSE* of the proposed estimator up to first order of approximation as

$$MSE(s_a^2) = S_y^4\left[1 + w_1^2 E_1 + w_2^2 M_2 + 2w_1 w_2 M_3 - 2w_1 M_4 - 2w_2 M_5\right] \tag{3.6}$$

where

$$M_1 = 1 + V_{0,2} + (2\theta_4^2\upsilon^2 - 2\theta_4\upsilon^2)V_{2,0} + 4\theta_4\upsilon V_{1,1}$$
$$M_2 = 1 + V_{0,2} + \left\{2\theta_4^2\upsilon^2 - 2\theta_4\upsilon^2 + g^2\alpha^2\upsilon^2 - 4\theta_4 g\alpha\upsilon^2 + g(g+1)\alpha^2\upsilon^2\right\}V_{2,0} + 4\upsilon(\theta_4 - g\alpha)V_{1,1}$$
$$M_3 = 1 + V_{0,2} + \left\{2\theta_4^2\upsilon^2 - 2\theta_4 g\alpha\upsilon^2 - 2\theta_4\upsilon^2 + \frac{g(g+1)}{2}\alpha^2\upsilon^2\right\}V_{2,0} + 2\upsilon(2\theta_4 - g\alpha)V_{1,1}$$
$$M_4 = 1 + \left(\frac{\theta_4^2}{2}\upsilon^2 - \theta_4\upsilon^2\right)V_{2,0} + \theta_4\upsilon V_{1,1}$$
$$M_5 = 1 + \left\{\frac{\theta_4^2}{2}\upsilon^2 - \theta_4\upsilon^2 - \theta_4 g\alpha\upsilon^2 + \frac{g(g+1)}{2}\alpha^2\upsilon^2\right\}V_{2,0} + (\theta_4\upsilon - g\alpha\upsilon)V_{1,1}$$

Now, minimizing (3.6) with respect to scalars, we get

$$w_{1(opt)} = \frac{(M_2 M_4 - M_3 M_5)}{(M_1 M_2 - M_3^2)} \quad \text{and} \quad w_{2(opt)} = \frac{(M_1 M_5 - M_3 M_4)}{(M_1 M_2 - M_3^2)} \tag{3.7}$$

Now, the minimum MSE can be obtained by putting the optimum values of w_1 and w_2 in (3.6) as

$$minMSE(s_a^2) = S_y^4 \left\{ 1 - \frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} \right\} \tag{3.8}$$

Hence, we develop the following theorem.

Theorem 3.1. *To the first degree of approximation*

$$MSE(s_a^2) \geq S_y^4 \left\{ 1 - \frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} \right\} \tag{3.9}$$

with equality holding if $w_1 = w_{1(opt)}$ *and* $w_2 = w_{2(opt)}$ *which are given in (3.7).*

Table 2: Few members of Singh and Malik (2014) estimators s_{sm}^2, classes of estimators $s_{bk_i}^2$, $i = 1,2,3$ and proposed estimators s_a^2

Members of $s_{sm(j)}^2$ $(\delta = 1)$	Members of $s_{b_{1(j)}}^2$	Members of $s_{b_{2(j)}}^2$	Members of $s_{b_{3(j)}}^2$	Members of $s_{a_j}^2 (g = 1,\ \alpha = 1)$	Values of	
$j = 1,2,\ldots,9$	$j = 1,2,\ldots,9$	$j = 1,2,\ldots,9$	$j = 1,2,\ldots,9$	$j = 1,2,\ldots,9$	c	d
$s_{sm(1)}^2$	$s_{bk_{1(1)}}^2$	$s_{bk_{2(1)}}^2$	$s_{bk_{3(1)}}^2$	$s_{a_{(1)}}^2$	N	1
$s_{sm(2)}^2$	$s_{bk_{1(2)}}^2$	$s_{bk_{2(2)}}^2$	$s_{bk_{3(2)}}^2$	$s_{a_{(2)}}^2$	N	f
$s_{sm(3)}^2$	$s_{bk_{1(3)}}^2$	$s_{bk_{2(3)}}^2$	$s_{bk_{3(3)}}^2$	$s_{a_{(3)}}^2$	N	$1-f$
$s_{sm(4)}^2$	$s_{bk_{1(4)}}^2$	$s_{bk_{2(4)}}^2$	$s_{bk_{3(4)}}^2$	$s_{a_{(4)}}^2$	$\beta_2(\phi)$	$\rho(C_y/C_\phi)$
$s_{sm(5)}^2$	$s_{bk_{1(5)}}^2$	$s_{bk_{2(5)}}^2$	$s_{bk_{3(5)}}^2$	$s_{a_{(5)}}^2$	N	ρ
$s_{sm(6)}^2$	$s_{bk_{1(6)}}^2$	$s_{bk_{2(6)}}^2$	$s_{bk_{3(6)}}^2$	$s_{a_{(6)}}^2$	$\beta_2(\phi)$	C_ϕ
$s_{sm(7)}^2$	$s_{bk_{1(7)}}^2$	$s_{bk_{2(7)}}^2$	$s_{bk_{3(7)}}^2$	$s_{a_{(7)}}^2$	C_ϕ	$\beta_2(\phi)$
$s_{sm(8)}^2$	$s_{bk_{1(8)}}^2$	$s_{bk_{2(8)}}^2$	$s_{bk_{3(8)}}^2$	$s_{a_{(8)}}^2$	N	$\rho(C_y/C_\phi)$
$s_{sm(9)}^2$	$s_{bk_{1(9)}}^2$	$s_{bk_{2(9)}}^2$	$s_{bk_{3(9)}}^2$	$s_{a_{(9)}}^2$	n	f

4. ANALYTICAL COMPARISON

On comparing the minimum MSE of the proposed classes of estimators s_a^2 from (3.8) with the minimum MSE of existing estimators from (2.1), (2.5), (2.8), (2.14), (2.26), (2.7), (2.19), (2.21), (2.24) and (A.8), we get the following efficiency conditions.

$$MSE(s_m^2) > MSE(s_a^2)$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > 1 - V_{0,2} \tag{4.1}$$

$$MSE(s_r^2) > MSE(s_a^2)$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > 1 - V_{0,2} - V_{2,0} + 2V_{1,1} \tag{4.2}$$

$$MSE(s_*^2) > MSE(s_a^2) \text{ where } s_*^2 = s_{lr}^2, s_i^2,\ i = 1,2,3 \text{ and } s_{bg}^2$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > 1 - V_{0,2} + \frac{V_{1,1}^2}{V_{2,0}} \tag{4.3}$$

$$MSE(s_e^2) > MSE(s_a^2)$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > 1 - V_{0,2} - \frac{V_{2,0}}{4} + V_{1,1} \tag{4.4}$$

$$MSE(s_{kc_i}^2) > MSE(s_a^2)$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > 1 - V_{0,2} - \omega_i^2 V_{2,0} + 2\omega_i V_{1,1} \tag{4.5}$$

$$MSE(s_s^2) > MSE(s_a^2)$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > 1 - \frac{1}{S_y^4}\left[\begin{array}{l} S_y^4 V_{0,2} + V_{2,0}\left\{\beta_\phi^2 S_\phi^4 + A_1^2 S_y^4 + 2A_1\beta_\phi S_y^2 S_\phi^2\right\} \\ -2S_y^2 V_{1,1}\left\{\beta_\phi S_\phi^2 + A_1 S_y^2\right\} \end{array}\right] \tag{4.6}$$

$$MSE(s_{sm}^2) > MSE(s_a^2)$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > \frac{(R_1R_5^2 + R_2R_4^2 - 2R_3R_4R_5)}{(R_1R_2 - R_3^2)} \tag{4.7}$$

$$MSE(s_{bk_i}^2) > MSE(s_a^2)$$

$$\frac{(M_1M_5^2 + M_2M_4^2 - 2M_3M_4M_5)}{(M_1M_2 - M_3^2)} > \frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)},\ i = 1,2,3 \tag{4.8}$$

Under the above conditions, the proposed class of estimators s_a^2 perform better than the traditional mean estimator, classical ratio, regression and exponential estimators suggested by Singh and Kumar (2011), Singh and Malik (2014) estimators, log type estimators defined on the lines of Bhushan and Kumar (2016, 2020) and the classes of estimators $s_{bk_i}^2,\ i = 1,2,3$ defined on the lines of Kadilar and Cingi (2006). Further, these conditions are verified by a numerical study using real populations.

5. NUMERICAL STUDY

In the present section, we have performed a numerical study over a real population taken from Sukhatme and Sukhatme (1970, pp. 256). The description about the population is given below.
y: Number of villages in the circles, ϕ: A circle consisting of more than five villages,

N=89, n=23, S_y^2=4.074, S_ϕ^2=0.11, C_y=0.601, C_ϕ=2.678, ρ=0.766, $\beta_2(\phi)$=6.162, $\lambda_{22} = 3.996$, $\lambda_{40} = 3.811$ and $\lambda_{04} = 6.162$.

The percent relative efficiency (PRE) of several estimators s_t^2 regarding the traditional mean estimator s_m^2 for the above population is calculated using the following expression.

$$PRE = \frac{MSE(s_m^2)}{MSE(s_t^2)} \times 100$$

The numerical results are disclosed by means of PRE in Table 3. It has been observed from Table 3 that the members $s^2_{a_{(j)}},\ j = 1,2,...,9$ of the proposed class of estimators s_a^2 dominate:

(i). the traditional mean estimator s_m^2, ratio estimator s_r^2, regression estimator s_{lr}^2 and exponential estimator s_e^2 envisaged by Singh and Kumar (2011), class of difference, Srivastava (1967) and Walsh (1970) type estimators, the estimators $s^2_{kc_i},\ i = 1,2,3,4$ adapted by Singh and Malik (2014) and log type estimators s_{bg}^2 defined on the lines of Bhushan and Gupta (2016, 2020).

(ii). the members $s^2_{s_{(j)}}$, j=1,2,...,9; of the class of estimators s_s^2 adapted by Singh and Malik (2014).

(iii). the corresponding members $s^2_{sm_{(j)}}$, j=1,2...,9; of the class of estimators s^2_{sm} suggested by Singh and Malik (2014).

(iv). the corresponding members $s^2_{bk_{i(j)}}$, i=1,2,3, j=1,2,...,9; of the classes of estimators $s^2_{bk_i},\ i = 1,2,3$ developed on the lines of Kadilar and Cingi (2006).

It is observed from Table 3 that the members $s^2_{a_{(j)}},\ j = 1,2,...,9$; of the proposed class of estimators s_a^2 show their superiority over the existing estimators. Moreover, it is also observed that the member $s^2_{a_{(8)}}$ of the suggested estimator s_a^2 based on the information $(N, \rho(Cy/C_\phi))$ is found to be the most efficient among the proposed class of estimators.

These results are expected because the conditions (4.1) to (4.8) are satisfied for the above data set.

6. CONCLUSION

In this study, we have investigated a modified class of estimators of population variance utilizing known population proportion under simple random sampling. The usual mean, classical ratio and regression estimators, Srivastava (1967) type estimator, Walsh (1970) type estimator and Kadilar and Cingi (2004) type estimators adapted by Singh and Malik (2014) are identified as the members of the suggested class of estimators for suitably chosen values of characterizing scalars. The mean square error expression of the proposed class of estimators is obtained up to first order of approximation using Taylor series method. The efficiency conditions are obtained under which the proposed class of estimators dominate the existing estimators which are further verified numerically using a real data set. The numerical results reported in Table 3 show the superiority of the members of proposed class of estimators in terms of minimum MSE and maximum PRE over the estimators existing till date such as usual mean estimator, classical ratio, regression and exponential estimators, Kadilar and Cingi (2004) and Singh *et al.* (2008) types of estimators adapted by Singh and Malik (2014), the estimators suggested by Singh and Malik (2014), log type estimators defined on the lines of Bhushan and Gupta (2016, 2020) and the classes of estimators developed on the lines of Kadilar and Cingi (2006). Thus, the proposed class of estimators is highly justified for the estimation

Table 3: MSE and PRE of different estimators

Estimators	MSE	PRE	Estimators	MSE	PRE
s_m^2	1.5042	100	$s_{bk_{1(6)}}^2$	0.5626	267.3434
s_r^2	1.0601	141.898	$s_{bk_{1(7)}}^2$	0.5717	263.1100
s_*^2	0.5737	262.1869	$s_{bk_{1(8)}}^2$	0.4274	351.8887
s_e^2	0.5915	254.2741	$s_{bk_{1(9)}}^2$	0.4461	337.1808
$s_{kc_1}^2$	1.3820	108.8429	$s_{bk_{2(1)}}^2$	0.4459	337.3115
$s_{kc_2}^2$	1.4488	103.8228	$s_{bk_{2(2)}}^2$	0.4291	350.5108
$s_{kc_3}^2$	0.9693	155.1915	$s_{bk_{2(3)}}^2$	0.4404	341.5039
$s_{kc_4}^2$	1.3637	110.3062	$s_{bk_{2(4)}}^2$	0.4713	319.1473
$s_{s_{(1)}}^2$	0.5745	261.7992	$s_{bk_{2(5)}}^2$	0.4410	341.0906
$s_{s_{(2)}}^2$	0.5780	260.2364	$s_{bk_{2(6)}}^2$	0.5623	267.5066
$s_{s_{(3)}}^2$	0.6864	219.1417	$s_{bk_{2(7)}}^2$	0.5716	263.1560
$s_{s_{(4)}}^2$	0.5794	259.5852	$s_{bk_{2(8)}}^2$	0.4269	352.3106
$s_{s_{(5)}}^2$	0.6173	243.6867	$s_{bk_{2(9)}}^2$	0.4459	337.3115
$s_{s_{(6)}}^2$	0.7868	191.1765	$s_{bk_{3(1)}}^2$	0.4447	338.2140
$s_{s_{(7)}}^2$	0.5763	261.0175	$s_{bk_{3(2)}}^2$	0.4284	351.0928
$s_{s_{(8)}}^2$	1.1825	127.2051	$s_{bk_{3(3)}}^2$	0.4394	342.3204
$s_{s_{(9)}}^2$	0.5742	261.9574	$s_{bk_{3(4)}}^2$	0.4697	320.2522
$s_{sm_{(1)}}^2$	0.5489	274.0323	$s_{bk_{3(5)}}^2$	0.4399	341.9162
$s_{sm_{(2)}}^2$	0.5414	277.8316	$s_{bk_{3(6)}}^2$	0.5619	267.7022
$s_{sm_{(3)}}^2$	0.5465	275.2199	$s_{bk_{3(7)}}^2$	0.5715	263.2020
$s_{sm_{(4)}}^2$	0.5588	269.1781	$s_{bk_{3(8)}}^2$	0.4263	352.8376
$s_{sm_{(5)}}^2$	0.5468	275.1019	$s_{bk_{3(9)}}^2$	0.4447	338.2140
$s_{sm_{(6)}}^2$	0.5715	263.1846	$\mathbf{s_{a_{(1)}}^2}$	**0.4289**	**350.6870**
$s_{sm_{(7)}}^2$	0.5767	260.8288	$\mathbf{s_{a_{(2)}}^2}$	**0.4070**	**369.5435**
$s_{sm_{(8)}}^2$	0.5404	278.3619	$\mathbf{s_{a_{(3)}}^2}$	**0.4218**	**356.5575**
$s_{sm_{(9)}}^2$	0.5489	274.0323	$\mathbf{s_{a_{(4)}}^2}$	**0.4607**	**326.4594**
$s_{bk_{1(1)}}^2$	0.4461	337.1724	$\mathbf{s_{a_{(5)}}^2}$	**0.4225**	**355.9740**
$s_{bk_{1(2)}}^2$	0.4296	350.1316	$\mathbf{s_{a_{(6)}}^2}$	**0.5616**	**267.8418**
$s_{bk_{1(3)}}^2$	0.4407	341.3021	$\mathbf{s_{a_{(7)}}^2}$	**0.5714**	**263.2481**
$s_{bk_{1(4)}}^2$	0.4712	319.2021	$\mathbf{s_{a_{(8)}}^2}$	**0.4041**	**372.2014**
$s_{bk_{1(5)}}^2$	0.4412	340.8956	$\mathbf{s_{a_{(9)}}^2}$	**0.4289**	**350.6870**

where $s_*^2 = s_{lr}^2,\ s_i^2,\ i = 1,2,3$ and s_{bg}^2

of population variance when the information is available in the form of auxiliary attribute.
In forthcoming studies, we hope to extend the proposed class of estimators for the estimation of population variance using two-phase sampling.

ACKNOWLEDGMENT

The authors are thankful to the editor-in-chief Professor Carlos N. Bouza and learned reviewers for their valuable comments towards the article.

RECEIVED: FEBRUARY, 2021.
REVISED: JANUARY, 2022.

REFERENCES

[1] ABD-ELFATTAH, A.M., EL-SHERPIENY, E.A., MOHAMED, S.M. and ABDOU, O.F. (2010): Improvement in estimating the population mean in simple random sampling using information on auxiliary attribute. **Applied Mathematics and Computation**, 215, 4198-4202.

[2] ADICHWAL, N.K., SHARMA, P., VERMA, H.K. and SINGH, R. (2015): Generalized class of estimators for estimating population variances using auxiliary attribute. **International Journal of Applied and Computational Mathematics**, 1(3), 1-10.

[3] ADICHWAL, N.K., SHARMA, P., VERMA, H.K. and SINGH, R. (2016): Generalized class of estimators for population variances using auxiliary attribute. International Journal of Applied and Computational Mathematics, 1(3), 1-10.

[4] BHUSHAN, S. and GUPTA, R. (2016): Efficient class of estimators for population mean using attribute. **Recent Advances in Applied Statistics and its Applications**, 225-229.

[5] BHUSHAN, S. and GUPTA, R. (2020): An improved log-type family of estimators using attribute. **Journal of Statistics and Management Systems**, DOI:10.1080/09720510.2019.1661604

[6] BHUSHAN, S. and KUMAR, A. (2020a): On optimal classes of estimators under ranked set sampling. **Communications in Statistics - Theory and Methods**, DOI:10.1080/03610926.2020.1777431.

[7] BHUSHAN, S. and KUMAR, A. (2020b): Log type estimators of population mean under ranked set sampling. **Predictive Analytics using Statistics and Big Data: Concepts and Modelling**, 28, 47-74. DOI:10.2174/9789811490491120010007.

[8] BHUSHAN, S. and KUMAR, A. (2021): Novel log type class of estimators under ranked set sampling. **Sankhya B: The Indian Journal of Statistics**, 1-27. https://doi.org/10.1007/s13571-021-00265-y

[9] BHUSHAN, S., KUMAR, A. and SINGH, S. (2021): Some efficient classes of estimators under stratified sampling. **Communications in Statistics - Theory and Methods**, DOI: 10.1080/03610926.2021.1939052

[10] GROVER, L.K. and KAUR, P. (2011): An improved exponential estimator of finite population mean in simple random sampling using an auxiliary attribute. **Applied Mathematics and Computation**, 218 (7), 3093-3099.

[11] HANSEN, M.H. HURWITZ, W.N. and MADOW, W.G. (1953): **Sample Survey Methods and Theory**. John Wiley and Sons, New York, U.S.A.

[12] HAQ, A. and SHABBIR, J. (2014): An improved estimator of finite population mean when using two auxiliary attributes. **Applied Mathematics and Computation**, 241, 14-24.

[13] JHAJJ, H.S., SHARMA, M.K., GROVER, L.K. (2006): A family of estimators of population mean using information on auxiliary attribute. **Pakistan Journal of Statistics**, 22, 43-50.

[14] KADILAR, C. and CINGI, H. (2004): Ratio estimators in simple random sampling. **Applied Mathematics and Computation**, 151, 893-902.

[15] KADILAR, C. and CINGI, H. (2006): Improvement in estimating the population mean in simple random sampling. **Applied Mathematics Letters**, 19, 75-79.

[16] KOYUNCU, N. (2012): Efficient estimators of population mean using auxiliary attributes. **Applied Mathematics and Computation** 218 (22), 10900-10905.

[17] NAIK, V.D. and GUPTA, P.C (1996): A note on estimation of mean with known population proportion of an auxiliary character. **Journal of Indian Society of Agricultural Statistics**, 48(2), 151-158.

[18] SINGH, R., CHAUHAN, P., SAWAN, N. and SMARANDACHE, F. (2008): Ratio estimators in simple random sampling using information on auxiliary attribute. **Pakistan Journal of Statistics and Operation Research**, IV (1), 47-53.

[19] SINGH, R. and KUMAR, M. (2011): A family of estimators of population variance using information on auxiliary attribute. **Studies in Sampling Techniques and Time Series Analysis**. Zip Publishing, Columbus.

[20] SINGH, H.P. and SOLANKI, R.S. (2012): Improved estimation of population mean in simple random sampling using information on auxiliary attribute. **Applied Mathematics and Computation**, 218(15), 7798-7812.

[21] SINGH, H.P. and SOLANKI, R.S. (2013): A new procedure for variance estimation in simple random sampling using auxiliary information. **Stat Papers**, 54, 479-497.

[22] SINGH, R. and MALIK, S. (2014): Improved estimation of population variance using information on auxiliary attribute in simple random sampling. **Applied Mathematics and Computation**, 235, 43-49.

[23] SINGH, H.P. and PAL, S.K. (2018): An efficient new class of estimators of population variance using information on auxiliary attribute in sample surveys. **Hacettepe Journal of Mathematics and Statistics**, 47(1), 267-277.

[24] SHABBIR, J. and GUPTA, S. (2006): On improvement in variance estimation using auxiliary information. **Communications in Statistics - Theory and Methods**, 36(12), 2177-2185.

[25] SRIVASTAVA, S.K. (1967): An estimator using auxiliary information. **Calcutta Statistical Association Bulletin**, 16, 121-132.

[26] SUKHATME, P.V. and SUKHATME, B.V (1970): **Sampling Theory of Surveys with Applications**. Iowa State University Press, Ames, U.S.A.

[27] WALSH, J.E. (1970): Generalization of ratio estimator for population total. **Sankhya**, A, 32, 99-106.

[28] ZAMAN, T. and KADILAR, C. (2019): Novel family of exponential estimators using information of auxiliary attribute. **Journal of Statistics and Management Systems**, DOI:10.1080/09720510.2019.1621488

[29] ZAMAN, T. (2020): Generalized exponential estimators for the finite population mean. **Statistics in Transition new series**, 21 (1), 159-168.

APPENDIX A

We consider the first suggested estimator as

$$s_{bk_1}^2 = \zeta_1\{s_y^2 + \theta_1(s_\phi^{*^2} - S_\phi^{*^2})\} + \psi_1 s_y^2 \left[1 + \log\left(\frac{s_\phi^{*^2}}{S_\phi^{*^2}}\right)\right]^{\theta_4} \tag{A.1}$$

Express the above estimator in terms of $e's$ by using the notations defined earlier as

$$s_{bk_1}^2 - S_y^2 = S_y^2\left[\zeta_1\left\{1 + e_0 + \left(\frac{\theta_1}{R}\right)ce_1\right\} + \psi_1\left\{\begin{array}{c} 1 + e_0 + \theta_4\upsilon e_1 - \theta_4\upsilon^2 e_1^2 \\ + \left(\frac{\theta_4^2}{2}\right)\upsilon^2 e_1^2 + \theta_4\upsilon e_0 e_1 \end{array}\right\} - 1\right] \tag{A.2}$$

where $R = S_y^2/S_\phi^2$.
Squaring and taking expectation each side of (A.2), we obtain the MSE of the estimator $s_{bk_1}^2$ to the first order of approximation as

$$MSE(s_{bk_1}^2) = S_y^4\left[1 + \zeta_1^2 A_1 + \psi_1^2 B_1 + 2\zeta_1\psi_1 C_1 - 2\zeta_1 D_1 - 2\psi_1 E_1\right] \tag{A.3}$$

On differentiating the $MSE(s_{bk_1}^2)$ w.r.t ζ_1 and ψ_1, we get the optimum values as

$$\zeta_{1(opt)} = \frac{(B_1D_1 - C_1E_1)}{(A_1B_1 - C_1^2)} \quad \text{and} \quad \psi_{1(opt)} = \frac{(A_1E_1 - C_1D_1)}{(A_1B_1 - C_1^2)} \tag{A.4}$$

The minimum MSE at $\zeta_{1(opt)}$ and $\psi_{1(opt)}$ is expressed as

$$minMSE(s_{bk_1}^2) = S_y^4\left[1 - \frac{(A_1E_1^2 + B_1D_1^2 - 2C_1D_1E_1)}{(A_1B_1 - C_1^2)}\right] \tag{A.5}$$

Similarly, we can obtain the MSE of other proposed estimators. In general, we can write

$$MSE(s_{bk_i}^2) = S_y^4\left[1 + \zeta_i^2 A_i + \psi_i^2 B_i + 2\zeta_i\psi_i C_i - 2\zeta_i D_i - 2\psi_i E_i\right],\ i = 1,2,3 \tag{A.6}$$

The MSE of above estimator is minimized for

$$\zeta_{i(opt)} = \frac{(B_iD_i - C_iE_i)}{(A_iB_i - C_i^2)} \quad \text{and} \quad \psi_{i(opt)} = \frac{(A_iE_i - C_iD_i)}{(A_iB_i - C_i^2)} \tag{A.7}$$

The minimum MSE at $\zeta_{i(opt)}$ and $\psi_{i(opt)}$ is expressed as

$$minMSE(s_{bk_i}^2) = S_y^4\left[1 - \frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)}\right] \tag{A.8}$$

where $A_1 = 1+V_{0,2}+(\theta_1/R)^2 c^2 V_{2,0}+2(\theta_1/R)cV_{1,1};\ B_1 = 1+V_{0,2}+(2\theta_4^2\upsilon^2-2\theta_4\upsilon^2)V_{2,0}+4\theta_4\upsilon V_{1,1};\ C_1 = 1+ V_{0,2}+\left\{(\theta_4^2/2)\upsilon^2-\theta_4\upsilon^2+(\theta_1\theta_4/R)c\upsilon\right\}V_{2,0}+\{(\theta_1/R)c+2\theta_4\upsilon\}V_{1,1};\ D_1 = 1;\ E_1 = 1+\theta_4\upsilon V_{1,1}+\left\{(\theta_4^2/2)\upsilon^2-\theta_4\upsilon^2\right\}V_{2,0};\ A_2 = 1+V_{0,2}+\left(2\theta_2^2+\theta_2\right)\upsilon^2 V_{2,0}-4\theta_2\upsilon V_{1,1};\quad B_2 = 1+V_{0,2}+(2\theta_4^2-2\theta_4)\upsilon^2 V_{2,0}+4\theta_4\upsilon V_{1,1};\quad C_2 = 1+V_{0,2}+(2\theta_4\upsilon-2\theta_2\upsilon)V_{1,1}+$

$\left\{(\theta_4^2/2)\upsilon^2-\theta_4\upsilon^2+\{\theta_2(\theta_2+1)/2\}\upsilon^2-\theta_2\theta_4\upsilon^2\right\}V_{2,0}; D_2 = 1+\{\theta_2(\theta_2+1)/2\}\upsilon^2 V_{2,0}-\theta_2\upsilon V_{1,1}; E_2 = 1+\theta_4\upsilon V_{1,1}+ \left\{(\theta_4^2/2)-\theta_4\right\}\upsilon^2 V_{2,0};\ A_3 = 1+V_{0,2}+3\theta_3^2\upsilon^2 V_{2,0}-4\theta_3\upsilon V_{1,1};\ B_3 = 1+V_{0,2}+(2\theta_4^2-2\theta_4)\upsilon^2 V_{2,0}+4\theta_4\upsilon V_{1,1}; C_3 = 1+V_{0,2}+\left\{\theta_3^2+(\theta_4^2/2)-\theta_4-\theta_3\theta_4\right\}\upsilon^2 V_{2,0}+(2\theta_4\upsilon-2\theta_3\upsilon)V_{1,1}; D_3 = 1+\theta_3^2\upsilon^2 V_{2,0}-\theta_3\upsilon V_{1,1}; E_3 = 1+ \theta_4\upsilon V_{1,1}+\left\{(\theta_4^2/2)-\theta_4\right\}\upsilon^2 V_{2,0}.$