COMPARING ONE-PARAMETRIC SCHEMES FOR SOLVING MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS: LIMIT POINTS, CONTINUATION AND NUMERICAL EXPERIENCES

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ABSTRACT
Parametric problems have been used for solving mathematical programs with complementarity constraints (MPCC). In this paper, seven parametric approaches are considered. We study if the limits of the stationary points of the parametric problems satisfy the stationary conditions for MPCC when the parameter tends to 0. We also analyze the types of solutions the parametric problems have for a generic MPCC. Some numerical examples are also displayed. This work continues the comparative study of these approaches started in a homonyms article.

KEYWORDS: mathematical program with complementarity constraints, smoothing scheme, regularization scheme, stationarity, genericity.

MSC: 90C30.

RESUMEN
Un forma de resolver los modelos de programación matemática con restricciones de complementariedad (MPCC, por sus siglas en inglés) es utilizando parametrizaciones. En este trabajo se consideran siete enfoques paramétricos. Se estudian las propiedades de los puntos límite de la sucesión de puntos estacionarios cuando el parámetro tiende a 0. También se analizan los tipos de soluciones de los problemas paramétricos y se comparan los esquemas numéricamente. Este trabajo culmina el estudio comparativo iniciado en el artículo homónimo.

PALABRAS CLAVE: programación matemática con restricciones de complementariedad, esquema de suavización, esquema de regularización, estacionariedad, genericidad.

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1. INTRODUCTION

Mathematical Programs with Complementarity Constraints (MPCC for short) are specific non-linear programs defined as

$$(\mathcal{P}): \quad minf(x) \quad s.t. \quad x \in \mathcal{M} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_j(x) \leq 0, & j = 1, \dots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \dots, m, \\ r_i(x)s_i(x) = 0, & i = 1, \dots, m. \end{array} \right\} \tag{1.1}$$

where $f, g_1, \dots, g_q, r_1, \dots, r_m, s_1, \dots, s_m : \mathbb{R}^n \to \mathbb{R}$.

As pointed out in many articles of [6], an MPCC model with a particular structure is the keystone of one of the main solution approaches for bilevel problems. Bilevel problems are important to describe situations appearing, for instance, in electricity markets, optimal design of industrial parks, and data analysis, see [6, 20]. On the other hand, [17, 18], consider optimal control problems given by a piecewise smooth dynamical system. This type of optimal control model appears in mechanical, electrical and biological problems. The solution approaches proposed in these papers compute the solution of an auxiliary MPCC.

Algorithms based on active index strategy, sequential programming and augmented Lagrangian have been considered, see [10, 15, 8, 14] and references therein. An important class of algorithms is based on substituting the complementarity constraints with parametric constraints. The parametric program and (1.1) are equivalent if the values of the parameter tend to 0. See [17, 4] for more details on smoothing and regularization approaches. In this paper we consider the following two smoother programs:

$$(\mathcal{Q}_\tau): \quad minf(x) \quad s.t. \quad x \in \mathcal{M}^{\mathcal{Q}}{}_\tau = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_j(x) \leq 0, & j = 1, \dots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \dots, m, \\ r_i(x)s_i(x) = \tau, & i = 1, \dots, m. \end{array} \right\} \tag{1.2}$$

$$(\mathcal{P}_\tau): \quad minf(x) \quad s.t. \quad x \in \mathcal{M}^{\mathcal{P}}{}_\tau = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_j(x) \leq 0, & j = 1, \dots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \dots, m, \\ r^T(x)s(x) = \tau. & \end{array} \right\} \tag{1.3}$$

and the following regularization of MPCC

$$(\mathcal{R}^{\mathcal{S}}_\tau): minf(x) \; s.t. \; x \in \mathcal{M}^{\mathcal{S}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \dots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \dots, m, \\ r_i(x)s_i(x) \leq \tau, & i = 1, \dots, m. \end{array} \right\} \tag{1.4}$$

$$(\mathcal{R}^{\mathcal{LF}}_\tau): minf(x) \; s.t \; x \in \mathcal{M}^{\mathcal{LF}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \dots, q, \\ r_i(x)s_i(x) - \tau^2 \leq 0, & i = 1, \dots, m, \\ (r_i(x) + \tau)(s_i(x) + \tau) - \tau^2 \geq 0, & i = 1, \dots, m. \end{array} \right\} \tag{1.5}$$

$$(\mathcal{R}^{\mathcal{K}}_\tau): minf(x) \; s.t. \; x \in \mathcal{M}^{\mathcal{K}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \dots, q, \\ r_i(x), s_i(x) \geq -\tau, & i = 1, \dots, m, \\ (r_i(x) - \tau)(s_i(x) - \tau) \leq 0, & i = 1, \dots, m. \end{array} \right\} \tag{1.6}$$

$$(\mathcal{R}_\tau^{\mathcal{SU}}) : minf(x)\ s.t.\ x \in \mathcal{M}_\tau^{\mathcal{K}} = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ r_i(x) + s_i(x) - & \\ \phi^{SU}(r_i(x) - s_i(x); \tau) \leq 0, & i = 1, \ldots, m. \end{array} \right\} \tag{1.7}$$

$$(\mathcal{R}_\tau^{\mathcal{SK}}) : minf(x) s.t. \quad x \in \mathcal{M}_\tau^{\mathcal{SK}} = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ \phi^{SK}(x, \tau) \leq 0, & i = 1, \ldots, m. \end{array} \right\} \tag{1.8}$$

where

$$\phi^{SU}(a, \tau) = \begin{cases} |a|, & \text{if } |a| \geq \tau, \\ \tau\theta(a/\tau), & \text{otherwise} \end{cases}$$

and

$$\phi^{SK}(x, \tau) = \begin{cases} (r_i(x) - \tau)(s_i(x) - \tau), & \text{if } r_i(x) + s_i(x) \geq 2\tau, \\ -\left[(r_i(x) - \tau)^2 + (s_i(x) - \tau)^2\right]/2, & \text{otherwise.} \end{cases}$$

Here θ is a C^2-function regularizing function, recall that a regularizing function fulfills $\theta(1) = \theta(-1) = 1,\ \theta'(1) = -\theta'(-1) = 1,\ \theta''(-1) = \theta(1) = 0$ and $\theta''(x) > 0,$ for all $x \in (-1, 1)$.

Some properties of problems (1.2), (1.4), (1.5), (1.6), (1.7), (1.8) have been obtained in papers as [5, 19, 10, 24, 13, 4], respectively. This paper aims to complete the comparative study of these approaches developed in [4]. In the first part, we completed and compared the hypothesis used to characterize the stability of the set of feasible solutions and critical points for the different approaches. In this part, we obtain the properties of the limit points of a sequence of stationary points of the parametric problems given above. Assuming that the constraints are given by smooth enough functions, we investigate the properties of the solutions in the generic case. This analysis leads to a good understanding of the performance of the parametric algorithms from a local viewpoint and the problems that a nonlocal, continuation-like algorithm may have.

For simplicity, we consider only inequality constraints, but under standard extensions of the linear independence constraint qualification (LICQ) and the Mangasarian Fromovitz constraint qualification (MFCQ), all results of this paper can be extended to problems with additional equality constraints.

The paper is organized as follows. In Section 2, we review some preliminary material on MPCC programs and parametric optimization problems needed in the subsequent sections. Section 3 studies the properties of the limit points of sequences of stationary points of the parametric problems for $\tau \to 0$. Some of the conditions given in the literature are strengthened. Then, we present the properties of the solutions of the parametric problems $\mathcal{P}_\tau$, $\mathcal{Q}_\tau$, $\mathcal{R}_\tau^{\mathcal{S}}$, $\mathcal{R}_\tau^{\mathcal{LF}}$ and $\mathcal{R}_\tau^{\mathcal{K}}$ for a generic MPCC. The last two approaches are not considered because C^3-differentiability is needed. In Section 5, some numerical examples are presented. The paper ends with some concluding remarks.

We end this section with some basic notation that will be used throughout the text. The i^{th}-canonical vector in $\mathbb{R}^n$ will be denoted by e_i. The open ball centered at $\bar{x} \in \mathbb{R}^n$ with radius $\epsilon > 0$ will be $B_\epsilon(\bar{x}) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| < \epsilon\}$, where $\|x\|$ is the Euclidean norm. On the other hand, I_R is the identity matrix of dimension r and $diag(v)$ is the matrix whose diagonal is the vector v.

2. PRELIMINARIES

In this part, we introduce the concepts and results used later on. We start with a subsection devoted to MPCC theory: the definition of the set of active indexes, constraints qualifications, the stationarity concepts and the necessary optimality conditions, are given there. Due to the parametric character of the studied approaches, the second part contains a brief introduction to the concepts of solutions for parametric optimization, the concept of generic problems and the properties of those solutions appearing in the generic case.

2.1. MPCC problems

For this class of optimization problems, first, we introduce the active index sets

$$I_g(x) = \{j \in \{1,\dots,q\} \mid g_j(x) = 0\},$$

$$I_r(x) = \left\{ i \in \{1,\dots,m\} \middle| \begin{matrix} r_i(x) & = & 0, \\ s_i(x) & > & 0 \end{matrix} \right\}, I_s(x) = \left\{ i \in \{1,\dots,m\} \middle| \begin{matrix} r_i(x) & > & 0, \\ s_i(x) & = & 0 \end{matrix} \right\},$$

$$I_{rs}(x) = \{i \in \{1,\dots,m\} \mid r_i(x) = 0, s_i(x) = 0\},$$

The constraint qualifications that will be used later on are presented in the following definition

Definition 2.1. *Let $\bar{x} \in \mathcal{M}$. We say that MPCC-LICQ holds at $\bar{x}$, if the system*

$$\{\nabla g_j(\bar{x}) | j \in I_g(\bar{x})\} \cup \{\nabla r_i(\bar{x}) | i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})\} \cup \{\nabla s_i(\bar{x}) | i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})\}$$

is linearly independent. MPCC-MFCQ is said to hold at $\bar{x}$ if the system

$$\{\nabla r_i(\bar{x}) | i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})\} \cup \{\nabla s_i(\bar{x}) | i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})\}$$

is linearly independent and there exists some $d \in \mathbb{R}^n$ such that

$$\nabla g_j(\bar{x})^T d < 0, \quad \forall\, j \in I_g(\bar{x}),$$

$$\nabla r_i(\bar{x})^T d = 0, \quad \forall\, i \in I_r(\bar{x}) \cup I_{rs}(\bar{x}), \qquad \nabla s_i(\bar{x})^T d = 0, \quad \forall\, i \in I_s(\bar{x}) \cup I_{rs}(\bar{x}).$$

For $\bar{x} \in \mathcal{M}$, we introduce the Lagrangian function (near $\bar{x}$),

$$L(x,\mu,\rho,\sigma) = f(x) + \sum_{j \in I_g(\bar{x})} \mu_j g_j(x) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i r_i(x) - \sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i s_i(x). \tag{2.1}$$

Assuming that the parametric problems are solvable and their solutions converge when $\tau \to 0^+$, one question we answer is which properties does the limit points have. Actually, they will correspond with different types of stationarity concepts for MPCC, which we will present now:

Definition 2.2. (Stationarity Concepts) *Let $\bar{x} \in \mathcal{M}$. Then, $\bar{x}$ is called weakly stationary (W-stationary) if there are multipliers $(\mu,\rho,\sigma) \in \mathbb{R}^{|I_g(\bar{x})|+|I_r(\bar{x})|+|I_s(\bar{x})|}$ with $\mu \geq 0$ such that*

$$0 = \nabla f(\bar{x}) + \sum_{j \in I_g(\bar{x})} \mu_j \nabla g_j(\bar{x}) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i \nabla r_i(\bar{x}) - \sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i \nabla s_i(\bar{x}).$$

A W-stationary point $\bar{x}$ with corresponding multipliers (μ,ρ,σ) is:

(a) Clarke stationary (C-stationary) if $\rho_i\sigma_i \geq 0$, for all $i \in I_{rs}(\bar{x})$.

(b) Mordukhovich stationary (M-stationary) if either $\rho_i\sigma_i = 0$ or $\rho_i, \sigma_i > 0$ holds, for all $i \in I_{rs}(\bar{x})$.

(c) Strongly stationary (S-stationary) if $\rho_i, \sigma_i \geq 0$ for all $i \in I_{rs}(\bar{x})$.

Clearly

$$\text{S-stationarity} \Rightarrow \text{M-stationarity} \Rightarrow \text{C-stationarity} \Rightarrow \text{W-stationarity}.$$

It is worth pointing out that the S-stationarity condition is equivalent to the standard KKT condition applied directly to problem (1.1). As in the case of no-linear problems, local minimizers where a certain CQ holds are stationary points. The following necessary condition illustrates this fact for MPCC:

Theorem 2.1. (First order necessary condition, cf. [7]) *Let $\bar{x}$ be a local minimizer in which MPCC-LICQ is satisfied. Then $\bar{x}$ is an S-stationary point.*

Now we give a short introduction to the theory of stationary points for parametric optimization problems.

2.2. Parametric optimization

A one-parametric optimization problem has the structure:

$$(P(\tau)): \quad \min\ f(x,\tau) \ \text{ s.t. } \ x \in M(\tau) = \left\{ x \in \mathbb{R}^n \ \middle|\ \begin{array}{ll} g_j(x,\tau) \leq 0, & j = 1,\dots,q, \\ h_i(x,\tau) = 0, & i = 1,\dots,l. \end{array} \right\} \tag{2.2}$$

and $\tau \in \mathbb{R}$ is a parameter. We assume that, at least, $f, g_1, \dots, g_q, h_1, \dots, h_l, \in C^3$. The set of active constraints reads:

$$I_g(x,\tau) = \{j \in \{1,\dots,q\} | g_j(x,\tau) = 0\}.$$

Similarly, classical concepts of optimization, such as the Lagrangian function (near a point $(\bar{x},\bar{\tau})$), the Lagrangian multipliers $\mu \in \mathbb{R}^l$, $\lambda \geq 0$, the LICQ or the MFCQ condition, the tangent space $T_{\bar{x},\bar{\tau}}$ and the Karush-Kuhn-Tucker condition, see [2], can naturally be extended to parametric programs. We also need the concept of generalized critical points (g.c.):

Definition 2.3. *The point $(\bar{x},\bar{\tau})$ is said to be a g.c. point for problem (2.2) if there exists a nonzero Lagrangean multiplier vector, $0 \neq (\bar{\lambda}_0, \bar{\lambda}, \bar{\mu})$ such that*

$$\nabla_x L(\bar{x},\bar{\tau},\bar{\lambda}_0,\bar{\lambda},\bar{\mu}) = \bar{\lambda}_0 \nabla_x f(\bar{x},\bar{\tau}) + \sum_{i=1}^{l} \bar{\lambda}_i \nabla_x h_i(\bar{x},\bar{\tau}) + \sum_{j \in I_g(\bar{x},\bar{\tau})} \bar{\mu}_j \nabla_x g_j(\bar{x},\bar{\tau}) = 0\ .$$

For parametric optimization problems, 5 classes of g.c. points were defined. We only list the main property of each type. Here *SC* (strict complementarity) means that the multipliers $\bar{\mu}_j, j \in I_g(\bar{x},\bar{\tau})$, associated with the active inequalities are non-zero and the second order condition (SOC) is said to hold if $\nabla_x^2 L(\bar{x},\bar{\tau},\bar{\lambda},\bar{\mu})|_{T_{\bar{x},\bar{\tau}}}$ is non-singular. For the detailed definitions, we refer to [11]:

Type 1 – LICQ, SC and SOC hold.

Type 2 – violation of SC.

Type 3 – violation of SOC.

Type 4 – violation of LICQ and $l + |I_g(\bar{x}, \bar{\tau})| < n + 1$.
Type 5 – violation of LICQ and $l + |I_g(\bar{x}, \bar{\tau})| = n + 1$.
We denote by Σ^i_{gc} the set of g.c. points of type i. Σ_{gc} is the set of g.c. points.
Points of Type 1 are also called non-degenerate critical points. At these points the Jacobian of the KKT system

$$\begin{aligned} \nabla_x L(\bar{x}, \bar{\tau}, 1, \bar{\lambda}, \bar{\mu}) &= 0, \\ \bar{\lambda}_j g_j(\bar{x}, \bar{\tau}) &= 0, \quad j \in I_g(\bar{x}, \bar{\tau}) \\ h_i(\bar{x}, \bar{\tau}) &= 0, \quad i = 1, \ldots, l. \end{aligned}$$

is non-singular. This is an important class since locally, the set of g.c. points is a curve of Σ^1_{gc} parametrized by τ, while around a point of type Σ^i_{gc}, $i = 2, 3, 4, 5$, although Σ_{gc} is characterized, it is given by bifurcations and turning points.
In the following, by $[C^3_S]_{n+1}$, we denote the set of functions that are three times continuously differentiable for $(x, \tau) \in \mathbb{R}^n \times \mathbb{R}$, endowed with the so-called strong topology (see, e.g., [11]). A problem (2.2) given by the data $(f, h, g) = (f, h_i, i = 1, \ldots, l, g_j, j = 1, \ldots, q) \in [C^3_S]^{1+l+q}_{n+1}$ is called JJT-regular in $[a, b]$ if $\Sigma_{g.c.} \cap (\mathbb{R}^n \times [a, b]) \subset \cup^5_{i=1} \Sigma^i_{gc}$.
Here is the famous general genericity result for problems (2.2). Due to the characteristics of the topology, a generic set of problems is a large set. Indeed, good properties such as the approximation of any parametric model through regular problems and the stability of the regularity, are fulfilled.

Theorem 2.2. (cf. [3]) *Fix the parametric problem $\mathcal{P}(\tau)$ given by $(f, h, g) \in [C^3_S]^{1+l+q}_{n+1}$. Consider now the parametric problem $P_\tau(A, b, c, d)$ defined by the perturbed functions $f(x, \tau) + x^T Ax + b^T x$, $h_i(x, \tau) + c^T_i x + d_i, i = 1, \ldots, l$, $g_j(x, \tau) + c^T_{j+l} x + d_{j+l}, j = 1, \ldots, q$, where A is a symmetric $n \times n$-matrix, and $(b, c, d) \in \mathbb{R}^{n+n(l+q)+l+q}$. Then, the set of perturbations (parameters) (A, b, c, d) such that the corresponding problem $P_\tau(A, b, c, d)$ is not JJT-regular on $[0, 1]$ has Lebesgue measure equal to zero. Moreover, the set of JJT-regular problems in $[0, 1]$ is open and dense with respect to the strong topology in $[C^3_S]^{1+l+q}_{n+1}$.*

Unfortunately for parametric problems with a different (special) structure, this genericity result cannot be used directly. So, for the special structured problems $\mathcal{P}_\tau$ and Q_τ, a special genericity analysis has to be done. This is performed in Section 4.
The density part of the preceding theorem is based on the following result from Differential Topology which will be used in Section 4:

Lemma 2.1. (Parameterized Sard Lemma, *cf.* **[11])** *Let us assume that $\phi \in [C^\kappa]^r_{n+p}$, $\kappa > max\,\{0, n - r\}$, $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$. We suppose that for all $(\bar{x}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^p$ such that $\phi(\bar{x}, \bar{z}) = 0$, we have $rank(\nabla_{x,z}\phi(\bar{x}, \bar{z})) = r$. Define $\phi_z : \mathbb{R}^n \to \mathbb{R}^r$, $\phi_z(x) = \phi(x, z)$. Then for almost all $z \in \mathbb{R}^p$, the matrix $\nabla_x \phi_z(x^*)$ has rank r for all x^* which are zeroes of the map $\phi_z(x)$.*

Here, almost all is understood in the sense of the Lebesgue measure.
For MPCC, a similar genericity analysis has been done, see [23]. As for non-linear programs, the following concept of regularity plays an important role in approximating general MPCC through regular ones and the stability. Moreover, in this paper, we discuss whether the results are fulfilled for regular MPCCs

because it is a way to evaluate whether the hypotheses are strong or not. The concept is given in the following definition.

Definition 2.4. *We say the problem $\mathcal{P}$ is regular in the MPCC sense, if MPCC-LICQ holds at all points of $\mathcal{M}$ and the generalized critical (g.c.) points $\bar{x}$ of $\mathcal{P}$ satisfy with (unique) multipliers $(\mu, \rho, \sigma, \rho)$ the condition MPCC-SC,*

$$(\textit{MPCC-SC}): \quad \rho_i, \sigma_i \neq 0,\ i \in I_{rs}(\bar{x}),\ \mu_j \neq 0,\ j \in I_g(\bar{x})$$

as well as MPCC-SOC (see (2.1)*):*

$$(\textit{MPCC-SOC}): d^T \nabla_x^2 L(\bar{x}, \bar{\mu}, \bar{\rho}, \bar{\sigma}) d \neq 0 \quad \forall d \in T_{\bar{x}}\mathcal{M} \setminus \{0\}\ ,$$

where $T_{\bar{x}}\mathcal{M} := \left\{ d \in \mathbb{R}^n \left| \begin{array}{lcll} \nabla_x g_j(\bar{x}) d & = & 0, & j \in I_g(\bar{x}), \\ \nabla_x r_i(\bar{x}) d & = & 0, & i \in I_r(\bar{x}) \cup Irs(\bar{x}), \\ \nabla_x s_i(\bar{x}) d & = & 0, & i \in I_s(\bar{x}) \cup Irs(\bar{x}). \end{array} \right. \right\}$

The genericity and the perturbation results in this case are

Theorem 2.3. (cf. [23]) *Fix the parametric problem $\mathcal{P}(\tau)$ given by $(f, h, g) \in [C_S^3]_{n+1}^{1+l+q}$. Consider now the parametric problem $\mathcal{P}_\tau(A, b, c, d)$ defined by the perturbed functions $f(x, \tau) + x^T A x + b^T x$, $h_i(x, \tau) + c_i^T x + d_i,\ i = 1, \ldots, l,\ g_j(x, \tau) + c_{j+l}^T x + d_{j+l},\ j = 1, \ldots, q,\ r_i(x, \tau) + c_{j+l+q+i}^T x + d_{j+l+q+i},\ i = 1, \ldots, m,\ s_i(x, \tau) + c_{j+l+q+m+i}^T x + d_{j+l+q+m+i},\ i = 1, \ldots, m$, where A is a symmetric $n \times n$-matrix, and $(b, c, d) \in \mathbb{R}^{n+n(l+q+m+m)+l+q+m+m}$. Then, the set of perturbations (parameters) (A, b, c, d) such that the corresponding problem $\mathcal{P}_\tau(A, b, c, d)$ is not regular on $[0, 1]$ has Lebesgue measure equal to zero. Moreover, the set of regular problems in $[0, 1]$ is open and dense with respect to the strong topology in $[C_S^3]_{n+1}^{1+l+q}$.*

3. LIMIT POINTS OF SEQUENCES OF STATIONARY POINTS OF THE PARAMETRIC SCHEMES

In this section, we investigate under which conditions limits of sequences of stationary points of the parametric problems defined by the approaches, are stationary points of $\mathcal{P}$. In what follows, we make use of facts proven in [22] for $(\mathcal{R}_\tau^{\mathcal{ST}})$.

3.1. Case $\mathcal{P}_\tau$

Definition 3.1. *Let for a sequence $\tau^k \to 0$ be given a sequence $\{x^k\}$ of stationary points of the programs $\mathcal{P}_{\tau_k}$ with corresponding multipliers $(\mu^k, \rho^k, \sigma^k, \delta^k)$, i.e.,*

$$\begin{aligned} 0 \ = \ & \nabla f(x^k) + \sum_{j \in I_g(x^k)} \mu_j^k \nabla g_j(x^k) - \sum_{i: r_i(x^k)=0} \rho_i^k \nabla r_i(x^k) \\ & - \sum_{i: s(x^k)=0} \sigma_i^k \nabla s_i(x^k) + \delta^k \sum_{i=1}^m s_i(x^k) \nabla r_i(x^k) + \delta^k \sum_{i=1}^m r_i(x^k) \nabla s_i(x^k) \end{aligned} \tag{3.1}$$

where $(\mu^k, \rho^k, \sigma^k) \geq 0$ and δ^k is free. The sequence $\{x^k\}$ (with corresponding $(\mu^k, \rho^k, \sigma^k, \delta^k)$) is called proper if $\delta^k \leq 0$ for infinitely many values of k.

The following is the first main result of this part.

Theorem 3.1. *Let $\tau^k \to 0$ and let x^k be stationary points of $\mathcal{P}_{\tau_k}$ with $x^k \to \bar{x}$. Then:*

(a) If MPCC-MFCQ holds at $\bar{x}$ and the sequence $\{x^k\}$ is proper, then, $\bar{x}$ is a S-stationary point of $\mathcal{P}$.

(b) If MPCC-LICQ holds at $\bar{x}$, then, $\bar{x}$ is a C-stationary point of $\mathcal{P}$.

Proof: Recall that the stationary points x^k satisfy (3.1) with multipliers $(\mu^k, \rho^k, \sigma^k, \delta^k)$ such that $(\mu^k, \rho^k, \gamma^k) \geq 0$ and δ^k is free. By continuity, for τ^k small enough, the inclusions hold:

$$I_g(x^k) \subseteq I_g(\bar{x}), \qquad \{i : r_i(x^k) = 0\} \subseteq I_r(\bar{x}) \cup I_{rs}(\bar{x}), \qquad \{i : s_i(x^k) = 0\} \subseteq I_s(\bar{x}) \cup I_{rs}(\bar{x}).$$

So, defining

$$\mu_j^k = 0,\ j \in I_g(\bar{x}) \setminus I_g(x^k), \quad \rho_i^k = 0,\ i \in I_r(\bar{x}) \cup I_{rs}(\bar{x}) \setminus \{i : r_i(x^k) = 0\},$$
$$\sigma_i^k = 0,\ i \in I_s(\bar{x}) \cup I_{rs}(\bar{x}) \setminus \{i : s_i(x^k) = 0\},$$

we have

$$0 = \nabla f(x^k) + \sum_{j \in I_g(\bar{x})} \mu_j^k \nabla g_j(x^k) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i^k \nabla r_i(x^k) -$$
$$\sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i^k \nabla s_i(x^k) + \delta^k \sum_{i=1}^m s_i(x^k) \nabla r_i(x^k) + \delta^k \sum_{i=1}^m r_i(x^k) \nabla s_i(x^k).$$

After putting

$$\tilde{\rho}_i^k = \rho_i^k - \delta^k s_i(x^k), i \in I_r(\bar{x}) \cup I_{rs}(\bar{x}), \qquad \tilde{\sigma}_i^k = \sigma_i^k - \delta^k r_i(x^k), i \in I_s(\bar{x}) \cup I_{rs}(\bar{x}),$$

and using $\{1, \ldots, m\} = \{i : r_i(x^k) = 0\} \cup \{i : r_i(x^k) > 0\} = \{i : s_i(x^k) = 0\} \cup \{i : s_i(x^k) > 0\}$, we get

$$0 = \nabla f(x^k) + \sum_{j \in I_g(\bar{x})} \mu_j^k \nabla g_j(x^k) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \tilde{\rho}_i^k \nabla r_i(x^k) - \tag{3.2}$$
$$\sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \tilde{\gamma}_i^k \nabla s_i(x^k) + \sum_{i \in I_s(\bar{x})} \delta^k s_i(x^k) \nabla r_i(x^k) + \sum_{i \in I_r(\bar{x})} \delta^k r_i(x^k) \nabla s_i(x^k).$$

(a) Idea of the proof: Let us assume now that MPCC-MFCQ holds at $\bar{x}$ and that $\{x^k\}$ is proper. Then, without loss of generality, we can suppose that $\delta^k \leq 0$ for all $k \in \mathbb{N}$. Using (3.2) and continuity arguments we obtain that the multipliers are bounded. Taking a convergent sub-sequence it holds that $\bar{x}$ is a S-stationary point. For more details see Appendix A. □

(b) If the sequence $\{x^k\}$ is not proper, then, without loss of generality, we can assume $\delta^k > 0$ for all k. By the MPCC-LICQ, this implies that, x^k is a stationary point of the problem $(\mathcal{R}^{\mathcal{ST}}{}_{\tau^k})$ in (1.4). The result, then, follows from Theorem 8.2 in [22]. If the sequence $\{x^k\}$ is proper, recall that MPCC-LICQ implies the MPCC-MFCQ condition, then, we can use (a). So, $\bar{x}$ is S-stationary and, in particular, the C-stationarity holds.

Under additional conditions, in Theorem 4.1(a), we obtain a stronger statement.

Theorem 3.2. *Let $\tau^k \to 0$ and let x^k be stationary points of $\mathcal{P}_{\tau^k}$ with $x^k \to \bar{x}$ such that $\{x^k\}$ is not proper and MPCC-LICQ holds at $\bar{x}$. If in addition the points x^k satisfy the second-order necessary conditions for their respective problems, then, $\bar{x}$ is a M-stationary point of $\mathcal{P}$.*

Proof: The result is valid for the scheme $\mathcal{R}_\tau^{\mathcal{ST}}$ in (1.4), see [22, Theorem 8.4]. Let x^k be stationary for $\mathcal{P}_{\tau^k}$ with multipliers $(\mu^k, \tilde{\rho}^k, \tilde{\gamma}^k, \delta^k)$. Since the sequence $\{x^k\}$ is not proper, we may assume, without loss of generality, that $\delta^k > 0 \; \forall \; k$, and, hence, the second order necessary conditions in x^k for problem $\mathcal{P}_{\tau^k}$ are the same as the second order necessary conditions for x^k with respect to problem $\mathcal{R}_\tau^{\mathcal{ST}}$. The result now follows from [22, Theorem 8.4]. □

3.2. Case $\mathcal{Q}_\tau$.

Now we prove a similar result for this approach.

Theorem 3.3. *Let $\tau^k \to 0$ and let x^k be stationary points of $\mathcal{Q}_{\tau^k}$ with $x^k \to \bar{x}$ and such that MPCC-MFCQ holds at $\bar{x}$. Then, $\bar{x}$ is a C-stationary point of $\mathcal{P}$.*

Proof: Since x^k is stationary for Q_{τ^k}, there exist multipliers (λ^k, δ^k) such that

$$0 = \nabla f(x^k) + \sum_{j \in I_g(x^k)} \mu_j^k \nabla g_j(x^k) + \sum_{i=1}^{m} \delta_i^k s_i(x^k) \nabla r_i(x^k) + \sum_{i=1}^{m} \delta_i^k r_i(x^k) \nabla s_i(x^k),$$

where $\mu_j^k \geq 0 \quad \forall \; j \in I_g(x^k)$ and δ^k free. Since for τ^k small enough the inclusion $I_g(x^k) \subseteq I_g(\bar{x})$ holds, defining

$$\rho_i^k := -\delta_i^k s_i(x^k), i \in I_r(\bar{x}) \cup I_{rs}(\bar{x}), \quad \sigma_i^k := -\delta_i^k r_i(x^k), i \in I_s(\bar{x}) \cup I_{rs}(\bar{x}) \;,$$

we find

$$\begin{aligned} 0 = \nabla f(x^k) + \sum_{j \in I_g(\bar{x})} \mu_j^k \nabla g_j(x^k) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i^k \nabla r_i(x^k) - \\ \sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i^k \nabla s_i(x^k) + \sum_{i \in I_s(\bar{x})} \delta_i^k s_i(x^k) \nabla r_i(x^k) + \sum_{i \in I_r(\bar{x})} \delta_i^k r_i(x^k) \nabla s_i(x^k). \end{aligned} \tag{3.3}$$

We introduce the abbreviation $\hat{\delta}^k := \delta^k_{I_s(\bar{x}) \cup I_r(\bar{x})}$. Due to the MPCC-MFCQ, the sequence $(\mu^k, \rho^k, \sigma^k, \hat{\delta}^k)$ is bounded. Then, taking the limit $k \to \infty$ in (3.3), we obtain

$$0 = \nabla f(\bar{x}) + \sum_{j \in I_g(\bar{x})} \mu_j \nabla g_j(\bar{x}) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i \nabla r_i(\bar{x}) - \sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i \nabla s_i(\bar{x})$$

where $\mu \geq 0, \; \rho, \sigma, \hat{\delta}$ free.
In order to prove C-stationarity, note that $\forall i \in I_{rs}(\bar{x})$ the relation

$$\rho_i^k \sigma_i^k = (\delta_i^k)^2 r_i(x^k) s_i(x^k) = (\delta_i^k)^2 \tau^k \geq 0$$

is valid, which for $k \to \infty$ yields $\rho_i, \sigma_i \geq 0$, as claimed. □

The properties of the limits of the sequences generated by the regularization approaches have been studied, under MPCC-LICQ in [22], [16], [12], [26]. As MPCC-LICQ is generic, the results under this hypothesis hold for a large class of problems. The genericity of the proper condition is a matter of future research.

4. GENERIC PROPERTIES OF THE SCHEMES

In this part, we present the genericity results. We want to remark that our analysis is different from the study done in [25]. In that paper, the authors provide properties under which small enough variations on the functions defining the MPCC guarantee non-degeneracy. However, the parametric problems we are dealing with are not MPCC, but nonlinear programs with a particular structure.

For the genericity analysis, we are going to develop, for fixed n, q, m, we regard an MPCC problem as $(f, g, r, s) = (f, g_1, \ldots, g_q, r_1, \ldots, r_m, s_1, \ldots, s_m) \in [C^2_S]^{1+q+2m}_n$, (see Section 2.2). Then, we consider the schemes $\tau \in (0, 1]$. A genericity analysis explores which type of singular behavior can be excluded for a generic (open and dense) subset of $[C^2_S]$ or $[C^3_S]$. We introduce the sets

$$\begin{aligned} \mathcal{M}^Y_0 &:= \{(x,\tau) \mid x \in \mathcal{M}^Y_\tau, \tau \in (0,1],\ \text{LICQ fails at } (x,\tau)\}, \\ \mathcal{M}^Y_1 &:= \left\{ (x,\tau) \in \mathcal{M}^Y_0 \mid \begin{array}{l} \text{the coefficients associated to inequalities at the} \\ \text{0-combination are non zero} \end{array} \right\}. \end{aligned} \tag{4.1}$$

Here Y denotes the different approaches, $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{LF}, \mathcal{K}$.

We start with the study of the feasible sets $\mathcal{M}^{\mathcal{P}}{}_\tau$ of $\mathcal{P}_\tau$. Recall that, by definition, the LICQ condition fails for (g, r, s) at a point (x, τ), $x \in \mathcal{M}^{\mathcal{P}}{}_\tau$, $\tau \in (0, 1]$, if there exists a solution $(\lambda, \gamma, \nu) \neq 0$ of

$$\sum_{j \in I_g} \lambda_j \nabla g_j(x) + \sum_{i \in I_r} \gamma_i \nabla r_i(x) + \sum_{i \in I_s} \nu_i \nabla s_i(x) + \delta \sum_{i=1}^m [s_i(x)\nabla r_i(x) + r_i(x) \nabla s_i(x)] = 0. \tag{4.2}$$

In this formula, for the index sets of active inequalities, we use the abbreviation $I_g = I_g(x), \quad I_r = I_r(x), \quad I_s = I_s(x)$.

In the next theorem, for fixed (g, h, r, s), we consider linearly perturbed functions

$$\begin{gathered} \tilde{g}_j(x) = g_j(x) + b^T_{g_j} x + c_{g_j},\ j = 1, \ldots, q, \\ \tilde{r}_i(x) = r_i(x) + b^T_{r_i} x + c_{r_i},\ i = 1, \ldots, m,\ \ \tilde{s}_i(x) = s_i(x) + b^T_{s_i} x + c_{s_i},\ i = 1, \ldots, m, \end{gathered} \tag{4.3}$$

with perturbation parameters $B_g = [b_{g_1}, \ldots, b_{g_q}]$, $B_r = [b_{r_1}, \ldots, b_{r_m}]$, $B_s = [b_{s_1}, \ldots, b_{s_m}]$, $c_g = (c_{g_1}, \ldots, c_{g_q})$, $c_r = (c_{r_1}, \ldots, c_{r_m})$, $c_s = (c_{s_1}, \ldots, c_{s_m})$.

In this case,

$$\mathcal{M}^P_1 = \{(x,\tau) \in \mathcal{M}^P_0 \mid \text{ LICQ fails with } \lambda_j \neq 0,\ j \in I_g,\ \gamma_i \neq 0, i \in I_r,\ \nu_i \neq 0, i \in I_s\}.$$

Theorem 4.1. *Let $(g, r, s) \in [C^2]^{q+2m}_n$ be given. Then, for almost all linear perturbation in* (4.3)*, the corresponding set $\mathcal{M}^0$ is a discrete set with $\mathcal{M}^P_0 = \mathcal{M}^P_1$.*

Proof: The feasible set of the perturbed problem (see (4.3)) is given by

$$\mathcal{M}^{\mathcal{P}}{}_\tau = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \tilde{g}_j(x) \leq 0,\ j = 1, \ldots, q, \\ \tilde{r}_i(x) \geq 0,\ \tilde{s}_i(x) \geq 0,\ i = 1, \ldots, m, \\ \tilde{r}(x)^T \tilde{s}(x) = \tau, \end{array} \right\} \tag{4.4}$$

First, we consider the set given only by the inequality constraints. A point (x, τ), in which LICQ fails for these inequalities, solves the system

$$\begin{pmatrix} \sum_{j\in I_g} \lambda_j[\nabla g_j(x) + b_{g_j}] + \sum_{i\in I_r} \gamma_i[\nabla[r_i(x) + b_{r_i}] + \sum_{i\in I_s} \nu_i[\nabla s_i(x) + b_{r_i}] \\ g_j(x) + b_{g_j}^T x + c_{g_j},\ j \in I_g, \\ r_i(x) + b_{r_i}^T x + c_{r_i},\ i \in I_r, \\ s_i(x) + b_{s_i}^T x + c_{s_i},\ i \in I_s, \end{pmatrix} = 0, \tag{4.5}$$

for some $(\lambda, \gamma, \nu) \neq 0$. Without loss of generality, we assume that $\lambda_1 = 1$ holds. The Jacobian of this system with respect to all variables and perturbation parameters has the form

∂_x	$\partial_{\lambda,\gamma,\nu}$	$\partial_{b_{g_1}}$	$\partial_{c_{I_g}}$	$\partial_{c_{r_{I_r}}, c_{s_{I_s}}}$
$\otimes$	$\otimes$	I_n	0	$\otimes$
$\otimes$	0	0	I	0
$\otimes$	0	0	0	$I\|0$
$\otimes$	0	0	0	$0\|I$

This Jacobian has full row rank $v = n + |I_g| + |I_r| + |I_s|$. By Lemma 2.1., for almost all parameter (b_{g_1}, c_g, c_r, c_s), the Jacobian only with respect to the $v-1$ variables $(x_1, \dots x_n, \lambda_j,\ j \in I_g \setminus \{1\}, \gamma_i,\ i \in I_r, \nu_i,\ i \in I_s)$ (recall, $\lambda_1 = 1$ is not a variable) has full row rank v at all solutions of the system (4.5). But since the number of variables is strictly smaller than v, for almost all perturbations the system has no solution, i.e., LICQ must hold.

Now, suppose that LICQ fails at the set (4.4), i.e., there exists a solution $(x, \tau, \lambda, \gamma, \nu, \delta)$, $(\lambda, \gamma, \nu, \delta) \neq 0$ of

$$\begin{array}{c} \sum_{j\in I_g} \lambda_j[\nabla g_j(x) + b_{g_j}] + \sum_{i\in I_r} \gamma_i[\nabla r_i(x) + b_{r_i}] + \sum_{i\in I_s} \nu_i[\nabla s_i(x) + b_{s_i}] + \\ \delta \sum_{i=1}^m [(s_i(x) + b_{s_i}^T x + c_{s_i})(\nabla r_i(x) + b_{r_i}) + (r_i(x) + b_{r_i}^T x + c_{r_i})(\nabla s_i(x) + b_{s_i})] = 0, \\ g_j(x) + b_{g_j}^T x + c_{g_j} = 0,\ i \in I_g, \\ r_i(x) + b_{r_i}^T x + c_{r_i} = 0,\ i \in I_r, \\ s_i(x) + b_{s_i}^T x + c_{s_i} = 0,\ i \in I_s, \\ (r(x) + B_r^T x + c_r)^T (s(x) + B_s^T x + c_s) = \tau. \end{array} \tag{4.6}$$

Since, by the preceding arguments, we can assume that LICQ holds for the system without the last equation, we must have $\delta \neq 0$. Without loss of generality, we take $\delta = 1$. On the other hand, as $(r(x) + B_r^T x + c_r)^T (s(x) + B_s^T x + c_s) = \tau$ there exists i^* such that $r_{i^*}(x) \cdot s_{i^*}(x) > 0$. Without loss of generality, we take $i^* = 1$. The Jacobian of the system (4.6) reads:

∂_x	$\partial_{\lambda,\gamma,\nu}$	$\partial_{b_{r_1}}$	$\partial_{c_{r_1}}$	$\partial_{c_{I_g}}$	$\partial_{c_{r_{I_r}}, c_{s_{I_s}}}$
$\otimes$	$\otimes$	$s_1(x) I_n + \nabla s(x)[x_1, \dots, x_n]$	$\nabla s_1(x)$	0	$\otimes$
$\otimes$	0	0	0	I	0
$\otimes$	0	0	0	0	$I\|0$
$\otimes$	0	0	0	0	$0\|I$
$\otimes$	0	$s_1(x)[x_1, \dots, x_n]$	$s_1(x)$	0	$\otimes$

Since $s_1(x) > 0$, the matrix $\begin{pmatrix} s_1(x) I_n + \nabla s_1(x)[x_1, \dots, x_n] & \nabla s_1(x) \\ s_1(x)[x_1, \dots, x_n] & s_1(x) \end{pmatrix}$ is non singular. So, the Jacobian of the system with respect to all variables and perturbation parameters has a full-row rank $v = n + |I_g| +$

$l + |I_r| + |I_s| + 1$ and, by Lemma 2.1., for almost all parameters, also the Jacobian with respect to the $v = n + 1 + |I_g| + l + |I_r| + |I_s|$ variables $(x, \tau, \lambda, \mu, \gamma, \nu)$ ($\delta = 1$ is not a variable) has full row rank v at all solutions of (4.6). Consequently, for almost all perturbation parameters, the set $\mathcal{M}^0$ where LICQ fails is a discrete set. To prove that for almost all parameters also $\lambda_i \neq 0,\ i \in I_g,\ \gamma_i \neq 0,\ i \in I_r$, and $\nu_i \neq 0,\ i \in I_s$, holds for the points (x, τ) in $\mathcal{M}^0$, we apply the same analysis to the system,

$$\begin{array}{c} \sum_{j\in I_g} \lambda_j[\nabla g_j(x) + b_{g_j}] + \sum_{i\in I_r} \gamma_i[\nabla r_i(x) + b_{r_i}] + \sum_{i\in I_s} \nu_i[\nabla s_i(x) + b_{s_i}]+ \\ \delta \sum_{i=1}^{m}[(s_i(x) + b_{s_i}^T x + c_{s_i})(\nabla r_i(x) + b_{r_i}) + (r_i(x) + b_{r_i}^T x + c_{r_i})(\nabla s_i(x) + b_{s_i})] = 0, \\ g_j(x) + b_{g_j}^T x + c_{g_j} = 0,\ j \in I_g, \\ r_i(x) + b_{r_i}^T x + c_{r_i} = 0,\ i \in I_r,\ \ s_i(x) + b_{s_i}^T x + c_{s_i} = 0,\ i \in I_s, \\ (r(x) + B_r^T x + c_r)^T (s(x) + B_s^T x + c_s) = \tau, \\ \lambda_j = 0, j \in I_g^0 \subset I_g, \quad \gamma_i = 0, i \in I_r^0 \subset I_r, \quad \nu_i = 0, i \in I_s^0 \subset I_s. \end{array} \tag{4.7}$$

By the Sard Lemma, it follows that, for almost all parameters, the relation $I_g^0 = I_r^0 = I_s^0 = \varnothing$ must hold at all solutions of (4.6), i.e., for almost all parameters, we have $\mathcal{M}_0^P = \mathcal{M}_1^P$. □

The preceding theorem is the basis result for the density part of the main genericity theorem for $\mathcal{P}_\tau$. Recall that a parametric program $\mathcal{P}_\tau$ ($\mathcal{Q}_\tau, \mathcal{R}^S, \mathcal{R}^{LF}$ or $\mathcal{R}^K$) is said to be JJT-regular on $(0, 1]$ if all g.c. points (x, τ) of $\mathcal{P}_\tau$, $\tau \in (0, 1]$ are of Type 1 - Type 5 (cf., Section 2.2).

Theorem 4.2. (cf. [21]) *Let $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ be fixed. Then for almost all linear perturbation of (g, r, s) and quadratic perturbation of f the corresponding program $\mathcal{P}_\tau$ is JJT-regular on $(0, 1]$.*
In particular the set of functions $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ such that the corresponding programs $\mathcal{P}_\tau$ are JJT-regular on $(0, 1]$ is generic (open, dense) in $[C^3]_n^{1+q+2m}$.

Proof: The density part follows from Theorem 4.1. as in [21, Theorem 6.21] (see also the proof of [3, Proposition 4.6.1] for $\mathcal{Q}_\tau$). The openness part can be done with the techniques and arguments as used in the proof of [3, Proposition 4.6.1,4.6.2]. □

The genericity analysis for $\mathcal{Q}_\tau$ has been presented in [3, Section 4.6]. So, we only state the results, see [3, Proposition 4.6.1]) and [3, Proposition 4.6.2], respectively.

Theorem 4.3. *Let $(g, r, s) \in [C^2]_n^{q+2m}$ be fixed. Then, for almost all linear perturbation of (g, r, s), the set ${\mathcal{M}^{\mathcal{Q}}}_0$ is a discrete set with ${\mathcal{M}^{\mathcal{Q}}}_0 = {\mathcal{M}^{\mathcal{Q}}}_1$.*

Theorem 4.4. *Let $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ be fixed. Then, for almost all linear perturbation of (g, r, s) and quadratic perturbation of f, the corresponding program $\mathcal{Q}_\tau$ is JJT-regular on $(0, 1]$.*
In particular, the set of functions $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ such that the corresponding programs $\mathcal{Q}_\tau$ are JJT-regular on $(0, 1]$, is generic (open, dense) in $[C^3]_n^{1+q+2m}$.

The parametric problems defined by the approaches $\mathcal{R}_\tau^K$, ${\mathcal{R}^{\mathcal{SU}}}_\tau$ and ${\mathcal{R}^{\mathcal{SK}}}_\tau$ satisfies that there exists a curve of feasible points such that LICQ fails. It is well known that JJT-regularity implies that LICQ is violated at most at a set of isolated points. This fact does not hold in the approaches $\mathcal{R}_\tau^K$, ${\mathcal{R}^{\mathcal{SU}}}_\tau$ and ${\mathcal{R}^{\mathcal{SK}}}_\tau$ because there exists a sequence of feasible points in which the LICQ fails. So, we will only analyze the regularizations given by $\mathcal{R}_\tau^S$ and ${\mathcal{R}^{\mathcal{LF}}}_\tau$. The following results show that JJT-regularity is also generic in these cases.

Theorem 4.5. *Let $(g, r, s) \in [C^2]_n^{q+2m}$ be fixed. Then, for almost all linear perturbation of (g, r, s), the set $\mathcal{M}^{\mathcal{S}}{}_0$ is a discrete set with $\mathcal{M}^{\mathcal{S}}{}_0 = \mathcal{M}^{\mathcal{S}}{}_1$.*
Moreover if $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ be fixed. Then, for almost all linear perturbation of (g, r, s) and quadratic perturbation of f, the corresponding program $\mathcal{R}_\tau^S$ is JJT-regular on $(0, 1]$.
In particular, the set of functions $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ such that the corresponding programs $\mathcal{R}_\tau^S$ are JJT-regular on $(0, 1]$, is generic (open, dense) in $[C^3]_n^{1+q+2m}$.

Proof: Define $I_r = \{i : r_i(x) = 0\}$, $I_s = \{i : s_i(x) = 0\}$ and $I_{rs} = \{i : r_i(x)s_i(x) = \tau\}$. Since $I_{rs} \cap [I_r \cup I_s] = \emptyset$, it is enough to prove the result for problems $\min f(x)$ s.t. $x \in \mathcal{M}_\tau^I$ where $\mathcal{M}_\tau^I$ is defined by the inequality constraints $g_i(x) \leq 0,\ i = 1, \ldots, q\ r_i(x) \geq 0, i \in I_r, s_i(x) \geq 0, i \in I_s$, and $r_i(x)s_i(x) = \tau, i \in I_{rs}$.
It is a problem with the same structure of $\mathcal{Q}_\tau$, so the result follows as in Theorems 4.3. and 4.4. □
Finally, we present the genericity result for the parametric problem $\mathcal{R}_\tau^{LF}$.

Theorem 4.6. *Let $(g, r, s) \in [C^2]_n^{q+2m}$ be fixed. Then, for almost all linear perturbation of (g, r, s), the set $\mathcal{M}^{\mathcal{LF}}{}_0$ is a discrete set with $\mathcal{M}^{\mathcal{LF}}{}_0 = \mathcal{M}^{\mathcal{LF}}{}_1$.*
Moreover if $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ be fixed. Then, for almost all linear perturbation of (g, r, s) and quadratic perturbation of f, the corresponding program $\mathcal{R}_\tau^{LF}$ is JJT-regular on $(0, 1]$.
The set of functions $(f, g, r, s) \in [C^3]_n^{1+q+2m}$ such that the corresponding programs $\mathcal{R}_\tau^{LF}$ are JJT-regular on $(0, 1]$, is generic (open, dense) in $[C^3]_n^{1+q+2m}$.

Proof: The only difference with respect to the previous analysis is the structure of the set of feasible solutions. So, after showing the results concerning the violation of the LICQ condition, the proof follows the same lines as the proof of the Theorems 4.4. and 4.2.. So, we will only include this part. First note that the points where LICQ fails are solutions of the system

$$\begin{array}{rcl} \sum_{i \in I_g} \mu_j \nabla g_j(x) + \sum_{i \in I_{rs}^1} \alpha_i [s_i(x)\nabla r_i(x) + r_i(x)\nabla s_i(x)] & & \\ + \sum_{i \in I_{rs}^2} \beta_i [(s_i(x) + \tau)\nabla r_i(x) + (r_i(x) + \tau)\nabla s_i(x)] & = & 0 \\ g_i(x) & = & 0, i \in I_g, \\ r_i(x)s_i(x) - \tau^2 & = & 0, i \in I_{rs}^1, \\ (r_i(x) + \tau)(s_i(x) + \tau) - \tau^2 & = & 0, i \in I_{rs}^2, \end{array}$$

where $(\mu, \alpha, \beta) \neq 0$. For the equality $\mathcal{M}^{\mathcal{LF}}{}_0 = \mathcal{M}^{\mathcal{LF}}{}_1$, we add the condition that some components of μ, α, β are zero. It is clear that $I_{rs}^1 \cap I_{rs}^2 = \emptyset$. W.l.o.g. we assume that $I_{rs}^1 = \{1, \ldots, p_1\}$ and $I_{rs}^2 = \{p_1 + 1, \ldots, p_1 + p_2\}$.
Now, we consider the previous system for the perturbed functions $g_i(x) + c_{g,i}x + d_{g,i}$, $r_i(x) + c_{r,i}x + d_{r,i}$, $s_i(x) + c_{s,i}x + d_{s,i}$. The derivatives with respect to the parameters and μ, α, β have always full row rank. Indeed if $\mu_1 \neq 0$ we assume $\mu_1 = 1$ and we get the matrix

$$\begin{array}{ccccc} \partial_x & \partial_{\mu_{2,\ldots|I_g|},\alpha,\beta} & \partial_{c_g} & \partial_{d_g} & \partial_{d_r} \\ \otimes & \otimes & I|\otimes & 0 & \otimes \\ \otimes & 0 & \otimes & I & 0 \\ \otimes & 0 & 0 & 0 & \begin{array}{ccc} diag(s_1, \ldots s_{p_1}) & 0 & 0 \\ 0 & diag(s_{p_1+1} + \tau, \ldots s_{p_1+p_2}\tau) & 0 \end{array} \\ 0 & I & 0 & 0 & 0 \end{array}$$

If $\mu = 0$ and $\alpha_1 \neq 0$ we obtain

$$\begin{array}{cccccc}
\partial_x & \partial_{\alpha_1,\ldots\alpha_{p_1},\beta} & \partial_{d_g} & \partial_{c_{r,1}} & \partial_{d_{r_{1,\ldots,p_1}}} & \partial_{d_{r_{p_1+1,\ldots,p_1+p_2}}} \\
\otimes & \otimes & 0 & s_1 I + \nabla s_1 x|\otimes & \nabla s_1|\otimes & \otimes \\
\otimes & 0 & I & 0 & 0 & 0 \\
\otimes & 0 & 0 & \frac{s_1 x|0}{0} & diag \begin{pmatrix} s_1 \\ \vdots \\ s_{p_1} \end{pmatrix} & 0 \\
0 & \otimes & 0 & 0 & 0 & diag \begin{pmatrix} s_{p_1+1} + \tau \\ \vdots \\ s_{p_1+p_2+\tau} \end{pmatrix} |0 \\
0 & I & 0 & 0 & 0 & 0
\end{array}$$

Finally if $\mu, \alpha_1 = 0$ and $\alpha_2 \neq 0$ assuming, w.l.o.g. that $\alpha_{2,1} = 1$, we get

$$\begin{array}{cccccc}
\partial_x & \partial_{\alpha_1,\ldots\alpha_{p_1},\beta} & \partial_{d_g} & \partial_{c_r} & \partial_{d_{r_{1,\ldots,p_1}}} & \partial_{d_{r_{p_1+1,\ldots,p_1+p_2}}} \\
\otimes & \otimes & 0 & (s_{p_1+1} + \tau) I + \nabla s_{p_1+1} x|\otimes & 0 & \nabla s_1|\otimes \\
\otimes & 0 & I & 0 & 0 & 0 \\
\otimes & 0 & 0 & 0 & diag \begin{pmatrix} s_1 \\ \vdots \\ s_{p_1} \end{pmatrix} & 0 \\
\otimes & 0 & 0 & \frac{(s_{p_1+1}+\tau) x|0}{0} & 0 & diag \begin{pmatrix} s_{p_1+1} \\ \vdots \\ s_{p_1+p_2} \end{pmatrix} \\
0 & I & 0 & 0 & 0 & 0
\end{array}$$

As $r_{1,\ldots,p_1}, s_{1,\ldots,p_1}, r_{p_1+1} + \tau, \ldots, r_{p_1+p_2} + \tau, s_{p_1+1} + \tau, \ldots, s_{p_1+p_2} + \tau \neq 0$, in the three cases the matrices have full row rank. Again by Lemma 2.1. the dimensions of the spaces and the full-row rank condition imply that all the components of μ, α, β are non-zero. So, only the first case is possible. In particular, ${\mathcal{M}^{\mathcal{LF}}}_0 = {\mathcal{M}^{\mathcal{LF}}}_1$ and it is a zero-dimensional manifold. From this, the result follows. □

5. NUMERICAL EXAMPLES

In this part, we illustrate the numerical behavior of the smoothing approaches $\mathcal{P}_\tau$ and $\mathcal{Q}_\tau$, defined in (1.3) and (1.2) respectively. We compare the results with the following regularization schemes studied in [22], [16] and [12], respectively.
The parametric problems (denoted by E_τ) are solved by the following basic algorithm, implemented in MATLAB 7.10.0.

Algorithm

Initialization Take $x^0 \in \mathbb{R}^n$, $\tau_0 > 0$, $\sigma \in (0,1)$, $\tau_{min} > 0$.
$k = 0$.
while $(\tau_k \geq \tau_{min})$ **do**
  Take x^k as starting point and compute x^{k+1} as approximate solution of E_{τ_k}.
  $\tau_{k+1} \leftarrow \sigma\tau_k$ y $k \leftarrow k+1$.
end while

The point x_0 is computed by solving the relaxed problem

$$(\mathcal{P}): \quad minf(x) \quad s.t. \quad x \in \mathcal{M} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_j(x) \leq 0, & j = 1,\dots,q, \\ r_i(x), s_i(x) \geq 0.0001, & i = 1,\dots,m. \end{array} \right\} \tag{5.1}$$

The parameter τ is taken in such a way that the feasibility of the corresponding parametric problem is guaranteed. In the case of $\mathcal{P}_\tau$, we simply need to take $\tau_0 = r_i(x_0)s_i(x_0)$. For the regularizations $\mathcal{R}^S_\tau$, $\mathcal{R}^{SU}_\tau$ $\mathcal{R}^{SK}_\tau$, τ_0 is $\max_i r_i(x_0)s_i(x_0)$ and $\tau_0 = \max_i \sqrt{r_i(x_0)s_i(x_0)}$ if $\mathcal{R}^{LF}_\tau$ is considered. In the other two cases $\tau_0 = 1$ and the constraints are parametrized as follows: for $\mathcal{Q}_\tau$, we take $r_i(x)s_i(x) = \tau r_i(x_0)s_i(x_0)$ and and in the case of $\mathcal{R}^K_\tau$, $(r_i(x) - t * r_i(x_0))(s_i(x) - t * y_i)$ are considered. The final value of the parameter is $\tau_{min} = 10^{-20}$ and the geometric decreasing rate is $\sigma = .001$. The intermediate non-linear problems were solved using the SQP method provided by the MatLab solver *fmincon*. As output, for the parametric problem E_τ, we will compute x^{opt}_E, the value of the objective function $f(x^{opt}_E)$ and the maximum violation of the constraints of the MPCC model,

$$maxViol(x^{opt}_E) := max\{max\{0, g(x^{opt}_E)\}, |h(x^{opt}_E)|, |min\{r(x^{opt}_E), s(x^{opt}_E)\}|\}.$$

The CPU consumption is also taken into account. So, we can measure the quality of the solutions computed by the approaches concerning feasibility and the difference between $f(x^{opt}_E)$ and the best-known value of the objective function. For the experiments, we used some examples of MacMPEC library [15]. They were chosen in such a way that different types of objective functions and constraints (described by linear, nonlinear, quadratic, convex or non-linear of the functions) were considered. The results are shown in Table 1 Concerning the value of the objective function, $\mathcal{R}^{\mathcal{SU}}{}_\tau$ computed the best value in three cases, while the smoothing approaches $\mathcal{Q}_\tau$ and $\mathcal{P}_\tau$ did it two times. It is important to remark that there were cases in which the evaluation of the objective function at the computed solutions was far from the best-known value. Only $\mathcal{R}^{\mathcal{SK}}{}_\tau$ computed good values at all the instances. $\mathcal{R}^{\mathcal{S}}{}_\tau$ and $\mathcal{R}^{\mathcal{LF}}{}_\tau$ failed in Dempe, but computed points close to feasibility. For the $\mathcal{Q}_\tau$ approach this was observed in the two cases. Also in two instances $f(x^{opt}_E) - f^{opt}$ was larger for $\mathcal{R}^K_\tau$, but only in one case the computed point was at least close to the feasibility. In the case of $\mathcal{P}_\tau$ points with inadequate values of $f(x^{opt}_E) - f^{opt}$ were obtained in three cases and, in one case, $maxViol(x^{opt}_E)$ was also large.
We can observe that $\mathcal{Q}_\tau$ is the fastest approach in 7 of the 8 cases. In 2 cases the best value of the objective function was obtained by $\mathcal{Q}_\tau$ and, in other 2 instances, was the second better approach concerning this

criterion. Concerning the fulfillment of the constraints, this smoothing approach computed the solution with the smallest value of $maxViol(x_E^{opt})$ in 2 cases and the second best value in other three cases. So, it is a good option for the fast computation of a point close to feasibility. The main disadvantage is that $f(x_E^{opt}) - f^{opt}$ may be large in unsuccessful cases. In the case of $\mathcal{P}_\tau$ feasibility may be compromised and it is very slow. From this point of view, regularization approaches are more stable. A better equilibrium between feasibility and optimality is reached. So, $\mathcal{R}^{\mathcal{SK}}{}_\tau$ will be recommended as a solution approach because it attains these desired properties, relative small values of $f(x_E^{opt}) - f^{opt}$ and $maxViol(x_E^{opt})$. Moreover, although $\mathcal{R}^{\mathcal{SK}}{}_\tau$ is not the quickest approach, their CPU-time values are not very large.

6. CONCLUSIONS

Smoothing and regularization schemes have been widely used for solving MPCC. These approaches lead to parametric optimization problems E_τ depending on a parameter $\tau \geq 0$ such that, for $\tau = 0$, program E_0 coincides with the original MPCC problem $\mathcal{P}$. In this paper, we completed the study of the properties of the seven parametric schemes. We studied the properties of the limit points of a sequence of stationary points of the schemes for $\tau \to 0$. In the case of C-stationarity, the approach $\mathcal{P}_\tau$ uses the strongest assumption, while in the other cases, only the fulfillment of the MPCC-MFCQ is need, see [24]. The only advantage of using the scheme $\mathcal{P}_\tau$ is that if the sequence generated by the approach is proper, the S-stationarity is guaranteed under MPCC-MFCQ. This means that only the sign of one constraint $\sum_{i=1}^{m} r_i(x)s_i(x) = \tau$ has to be considered. A similar result can be sketched for $\mathcal{Q}_\tau$, but then the sign of the multipliers of the constraints $r_i(x)s_i(x) = \tau$, $i \in I_{rs}(\bar{x})$ have to be taken into account. Since $I_{rs}(\bar{x})$ is not known in advance, the type of stationary point can be predicted under very strong conditions.
We, also proved that the smoothing schemes and the regularization approaches $\mathcal{R}_\tau^S$ and $\mathcal{R}_\tau^{LF}$, generically, define parametric JJT-regular problems. In this generic case the curves $x(\tau)$ of local minimizers of $\mathcal{P}_\tau, \mathcal{Q}_\tau, \mathcal{R}_\tau^S$ and $\mathcal{R}_\tau^{LF}$, for $\tau \geq 0$, can only possess specific (simple) bifurcations or turning points (corresponding to g.c. points of type 2 - type 5). Moreover, these curves can be traced numerically for $\tau \to 0$ by (standard) path-following methods. Due to the violation of the LICQ, JJT is not generic for the other approaches. We want to point out that this analysis provides global properties of the parametric problems, while classical results such as those obtained in Section 3. and [4] have a local character. The generic analysis we have conducted relates an MPCC to the seven parametric approaches we have considered here. So, the convergence holds for a large (generic) class of problems. However, there are important particular cases with a certain structure as those resulting after substituting the lower level problem in bilevel optimization models by the Karush-Kuhn-Tucker necessary optimality condition. As proven in [1], the generic properties of these MPCC models are different. So, this analysis Due to its importance, it is interesting to conduct this study for this particular class of MPCC. A similar study of those MPCCs resulting after discretizing piece-wise smooth optimal control models, see [17], is also important. On the other hand, the performance of algorithms as those proposed in [9, 8], can be analyzed under generic properties for general or particular suitable cases
From a numerical viewpoint, more experiments need to be considered. In many cases, we observed that with (relative) good CPU times, $\mathcal{R}^{\mathcal{SK}}{}_\tau$ computes better solutions. Indeed, the evaluations of the objective function are closer to the best-known value and feasibility. In the unsuccessful cases, other approaches

Scholtes1	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau^{SU}$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	1.0e-05	1.0e-05	2.0e-04	4.0e-05	2.0e-04	3.4e-07	2.0e-04
	$maxViol(x_E^{opt})$	2.0e-18	2.0e-18	2.6e-13	7.8e-13	2.6e-13	1.6e-12	2.6e-13
	CPUtime	3.8	5747	6	5749	2.1	5751	23.2
Bard1	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau^{SU}$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	1.0e-05	2.0e-04	2.0e-04	1.0e-05	2.0e-04	2.0e-06	2.0e-04
	$maxViol(x_E^{opt})$	1.5e-15	3.7e-13	2.0e-11	1.0e-14	2.0e-11	3.0e-08	2.0e-11
	CPUtime	8.8	5921	11.3	5922.2	16.4	5924	17.8
Bilevel2	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau SU$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	1 e+03	1.1e-07	2.0e-06	2.0e-06	6.6e+3	-9.6e-6	4.1e-07
	$maxViol(x_E^{opt})$	3.7e-08	5.6e-9	3.4e-10	1.1e-11	2.8e-06	9.3e-07	1.3e-13
	CPUtime	9.8	850.9	19.3	856.3	21.4	857.4	34.7
Dempe	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau^{SU}$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	3.0	3.0	3.0	3.0	1.0e-04	2.7e-03	6.4e-02
	$maxViol(x_E^{opt})$	1.1e-08	9.8e-09	1.0e-07	1.0e-07	5.3e-02	2e-04	1.1e-04
	CPUtime	6.1	549.8	15.8	565.9	19.6	574.7	27.3
Desilva	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau^{SU}$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	4.3e-03	1.3e-07	-7.1e-1	-7.1e-1	1.9e-04	-7.1e-1	-7.1e-1
	$maxViol(x_E^{opt})$	6.4e-14	6.4e-17	1.4e-01	1.4e-01	3.5e-11	1.4e-01	1.4e-01
	CPUtime	8.8	920	9.1	920.2	16.1	920.3	16.4
df1	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau^{SU}$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	3.1e-12	1.1	2.5e-01	2.5e-01	6.4e-03	2.5e-01	2.5e-01
	$maxViol(x_E^{opt})$	5.6e-7	7.9e-02	3.5e-01	3.5e-01	8.1e-12	3.5e-01	3.5e-01
	CPUtime	13.2	960.9	13.4	961.0	28.8	961.2	28.9
Outrata31	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau^{SU}$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	2.5e-01	7.6	2.5e-01	2.5e-01	1.1e+1	5.1e-01	2.5e-01
	$maxViol(x_E^{opt})$	2.1e-9	2.7 e+0	4.8e-11	4.2e-13	8.8 e+0	2.8e-8	9.7e-11
	CPUtime	3.4	390.6	31.5	398.9	51.8	400.7	69.7
qpec1	Approach	$\mathcal{Q}_\tau$	$\mathcal{P}_\tau$	$\mathcal{R}_\tau^{S}$	$\mathcal{R}_\tau^{LF}$	$\mathcal{R}_\tau^{K}$	$\mathcal{R}_\tau^{SU}$	$\mathcal{R}_\tau^{SK}$
	$f(x_E^{opt}) - f^{opt}$	4.0e-02	9.8e-09	1.2e-5	1.3e-05	5.6e-6	3.0	4.9e-06
	$maxViol(x_E^{opt})$	1.0e-06	1.2e-13	4.4e-14	1.1e-13	4.4e-14	1.1e-10	1.0e-14
	CPUtime	3.3	537.2	11.5	548.5	22.3	550.2	29.7

Table 1: Numerical results

compute points that are not close to the optimum. So, $\mathcal{R}^{\mathcal{SK}}{}_{\tau}$ is more stable. We want to point out that $\mathcal{Q}_{\tau}$ is fast and calculates solutions that are very close to feasibility. It can be used for computing solutions with relatively small evaluations of the objective function in a short time.

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A PROOF OF THEOREM 3.1. (A)

We analyse two cases: $I_{rs}(\bar{x}) \neq \{1,\dots,m\}$, or $I_{rs}(\bar{x}) = \{1,\dots,m\}$,.

Case 1, $I_{rs}(\bar{x}) \neq \{1,\dots,m\}$: We will show that the sequence $(\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k)$ in (3.2) is bounded. To the contrary, assume that the sequence is not bounded. Then, we consider the normalized sequence

$$\frac{(\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k)}{\|\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k\|}$$

and assume (for a sub-sequence) that

$$\lim_{k\to\infty} \frac{(\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k)}{\|\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k\|} = (\mu, \tilde{\rho}, \tilde{\sigma}, \delta).$$

Now multiply (3.2) by $\frac{1}{\|\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k\|}$ and take the limit $k \to \infty$. In view of $s_i(\bar{x}) = 0,\ i \in I_s(\bar{x})$ and $r_i(\bar{x}) = 0,\ i \in I_r(\bar{x})$, we obtain

$$0 = \sum_{j\in I_g(\bar{x})} \mu_j \nabla g_j(\bar{x}) - \sum_{i\in I_r(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\rho}_i \nabla r_i(\bar{x}) - \sum_{i\in I_s(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\sigma}_i \nabla s_i(\bar{x}), \tag{A1}$$

where $\mu \geq 0,\ \tilde{\rho}, \tilde{\sigma}, \delta$ free. Since MPCC-MFCQ holds at $\bar{x}$, by Lemma 2.1, we obtain

$$(\mu, \tilde{\rho}, \tilde{\sigma}) = 0.$$

Clearly $\|\mu, \tilde{\rho}, \tilde{\sigma}, \delta\| = 1$, from which we find $|\delta| = 1$ and $\delta = -1$ due to $\delta^k \leq 0\ \forall\ k$. As $I_{rs}(\bar{x}) \neq \{1,\dots,m\}$, it can be assumed, without loss of generality, that $I_r(\bar{x}) \neq \emptyset$. Take $i_0 \in I_r(\bar{x})$. Then, for k large enough, we have $s_{i_0}(x^k) \geq \frac{s_{i_0}(\bar{x})}{2} > 0$. Recalling

$$\lim_{k\to\infty} \frac{\tilde{\rho}^k_{i_0}}{\|(\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k)\|} = \tilde{\rho}_{i_0}, \qquad \lim_{k\to\infty} \frac{\delta^k}{\|(\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k)\|} = \delta = -1,$$

for $\rho^k_i = \tilde{\rho}^k_{i_0} - \delta^k s_{i_0}(x^k) \geq 0$ we conclude the existence of the limit

$$\lim_{k\to\infty} \frac{\rho^k_i}{\|(\mu^k, \rho^k, \sigma^k, \delta^k)\|} = \sigma_{i_0} \geq 0.$$

Hence, $\tilde{\rho}_{i_0} = \rho_{i_0} + s_{i_0}(\bar{x}) > 0$, in contradiction to $\tilde{\rho}_{i_0} = 0$.

Consequently, the sequence $(\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k)$ must be bounded and, without loss of generality, we have

$$\lim_{k\to\infty} (\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k, \delta^k) = (\mu, \tilde{\rho}, \tilde{\sigma}, \delta).$$

Taking limits as $k \to \infty$ in (3.2), we obtain

$$0 = \nabla f(\bar{x}) + \sum_{j\in I_g(\bar{x})} \lambda_j \nabla g_j(\bar{x}) - \sum_{i\in I_r(\bar{x})} \tilde{\gamma}_i \nabla r_i(\bar{x}) - \sum_{i\in I_s(\bar{x})} \tilde{\nu}_i \nabla s_i(\bar{x}). \tag{A2}$$

where $\lambda \geq 0,\ \tilde{\gamma}, \tilde{\nu}$ free. Furthermore, $\forall i \in I_{rs}(\bar{x})$ due to $\gamma^k_i, \nu^k_i \geq 0,\ s_i(x^k), r_i(x^k) \geq 0$ and $\delta^k \leq 0$ it holds $\tilde{\gamma}^k_i = \gamma^k_i - \delta^k s_i(x^k) \geq 0$, $\tilde{\nu}^k_i = \nu^k_i - \delta^k r_i(x^k) \geq 0$. Letting $k \to \infty$ yields

$$\tilde{\gamma}_i, \tilde{\nu}_i \geq 0. \tag{A3}$$

and $\bar{x}$ is S-stationary for $\mathcal{P}$.

<u>Case 2, $I_{rs}(\bar{x}) = \{1,\ldots,m\}$:</u> Then from (3.2) we obtain

$$0 = \nabla f(x^k) + \sum_{j\in I_g(\bar{x})} \mu_j^k \nabla g_j(x^k) - \sum_{i\in I_r(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\rho}_i^k \nabla r_i(x^k) - \sum_{i\in I_s(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\sigma}_i^k \nabla s_i(x^k). \tag{A4}$$

Assume again that the sequence $(\mu^k, \tilde{\rho^k}, \tilde{\sigma^k})$ is not bounded. Then, considering the normalized sequence, a similar analysis as in the previous case leads to a relation as in (A1),

$$0 = \sum_{j\in I_g(\bar{x})} \mu_j \nabla g_j(\bar{x}) - \sum_{i\in I_r(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\rho}_i \nabla r_i(\bar{x}) - \sum_{i\in I_s(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\sigma}_i \nabla s_i(\bar{x}),$$

where $\mu \geq 0$, $\tilde{\rho}, \tilde{\sigma}$ free. Again by MPCC-MFCQ at $\bar{x}$, from Lemma 2.1, we deduce $(\mu, \tilde{\rho}, \tilde{\sigma}) = 0$, which contradicts the fact that $\|\mu, \tilde{\rho}, \tilde{\sigma}\| = 1$.
Hence, the sequence $(\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k)$ must be bounded and a convergent sub-sequence can be chosen, i.e., without loss of generality, we may assume

$$\lim_{k\to\infty} (\mu^k, \tilde{\rho}^k, \tilde{\sigma}^k) = (\mu, \tilde{\rho}, \tilde{\sigma}).$$

Then, taking the limit $k \to \infty$ in (A4), we obtain

$$0 = \nabla f(\bar{x}) + \sum_{j\in I_g(\bar{x})} \mu_j \nabla g_j(\bar{x}) - \sum_{i\in I_r(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\rho}_i \nabla r_i(\bar{x}) - \sum_{i\in I_s(\bar{x})\cup I_{rs}(\bar{x})} \tilde{\sigma}_i \nabla s_i(\bar{x}),$$

where $\mu \geq 0$, ρ, σ free. As before, using $\rho_i^k, \sigma_i^k \geq 0$, $r_i(x^k), s_i(x^k) \geq 0$, $\delta^k \leq 0$, we obtain $\tilde{\rho}_i^k = \rho_i^k - \delta^k s_i(x^k) \geq 0$, $\tilde{\sigma}_i^k = \sigma_i^k - \delta^k r_i(x^k) \geq 0$ and letting $k \to \infty$ finally $\tilde{\rho}_i, \tilde{\sigma}_i \geq 0$. So, $\bar{x}$ is S-stationary, as stated. $\square$