ESTIMATION OF PARAMETERS OF TWO WEIBULL POPULATIONS UNDER THE JOINT RANKED SET SAMPLING

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ABSTRACT
In this paper, we have derived the maximum likelihood estimates(MLE) and Bayes estimates of parameters of the two Weibull populations under joint ranked set sampling(JRSS), joint modified ranked set sampling(JMRSS), and joint simple random sampling(JSRS). We compare the efficiency of these estimators under JRSS, JMRSS, and JSRS using mean square error(MSE) and bias. All these comparisons are done using the Monte Carlo simulation method. It is observed that Bayes estimates for scale parameters of the Weibull distribution for known shape parameters perform better than MLE under different sampling schemes. It is also observed that estimates under JRSS are more efficient than those under JMRSS and JSRS. Further, illustrates real-life examples to validate the performance of estimates under the mentioned sampling scheme.

KEYWORDS: Joint ranked set sampling, Weibull distribution, maximum likelihood estimates, Bayes estimates, inverted gamma prior.

MSC: 62D05,62F07,62F10,62F15,62P30

RESUMEN
En este artículo, se derivan estimaciones de máxima verosimilitud (MLE) y de Bayes de los parámetros de las dos poblaciones de Weibull bajo muestreo agregado de conjuntos clasificados (JRSS), muestreo agregado de conjuntos clasificados modificados (JMRSS) y muestreo aleatorio simple agregado (JSRS). Comparamos la eficiencia de estos estimadores bajo JRSS, JMRSS y JSRS utilizando el error cuadrático medio (MSE) y el sesgo. Todas estas comparaciones se realizan utilizando el método de simulación de Monte Carlo. Se observa que las estimaciones de Bayes para los parámetros de escala de la distribución de Weibull para los parámetros de forma conocidos funcionan mejor que MLE bajo diferentes esquemas de muestreo. También se observa que las estimaciones del JRSS son más eficientes que las del JMRSS y el JSRS. Además, ilustra ejemplos de la vida real para validar el rendimiento de las estimaciones bajo el esquema de muestreo mencionado

PALABRAS CLAVE: muestreo agregado de conjuntos clasificados, Distribución de Weibull, estimador máximo verosímil,estimación de Bayes, gamma invertida a priori

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1. INTRODUCTION

Ranked set sampling(RSS) was introduced by McIntyre [27] to estimate the forage yield of the crop. The RSS method is used when measurement units are costly and time-consuming. In the origin period of RSS, time and cost are reduced by using concomitant variables. This method is found to be a more efficient method than the classical one. Although the RSS method was developed as part of sampling theory this method can also used in various fields like life testing reliability, regression analysis, design of experiments, statistical quality control, statistical inference, and so on.RSS methods have been used in all these fields due to the performance of the RSS method is better as compared to existing classical methods.[17, 19, 30, 41],

In sampling theory, population mean is estimated using the RSS method and compared with the classical SRS method, and found that the RSS method performs better. Various researchers used the RSS method to estimate the mean when units are costly and time-consuming Martin et al. [26] used ranked set sampling for estimating shrub phytomas in Appalachian oak forests. Al-Saleh and Al-shaft [7] used the RSS technique to improve sampling precision of estimating the average milk yield, the study was conducted on a field of (402) sheep in Est. Jorden. Muttalk[29] established that the median Ranked set sampling estimator for population mean performs well as compared to RSS and regression estimator in terms of relative precision.RSS scheme is still popular among researchers and recently some authors discussed the various types of RSS methods suitable for the representation of population and minimization of cost and time. Taconeli [38] introduced dual-ranked set sampling. Yaruz [43] has done a comparative study of ranked set sampling and simple random sampling in agriculture studies a case study on Walnut trees. Jeelani et al. [24] improved the ratio estimator using ranked set sampling. Jeelani and et al. [23] discuss the role of rank set sampling in improving the estimates of the population mean under stratification. Al-Saleh and Al-Omari [6] used multi-stage ranked set sampling. Al-Saleh and Al-Kadri [5] used double-ranked set sampling and showed that ranking in the second stage is easier than in ranking the first stage.

Most of the real-life situations can be explained through the probability distribution and it is crucial to estimate the parameter involved in probability distribution. In classical theory, the SRS method is used at the sampling stage to estimate the parameter of the probability distribution using a suitable method. We use various RSS methods to estimate the parameters of a probability distribution. Gulzar et al. [20][15] estimate parameters of Erlang distribution at the sampling stage based on rank set sampling and some of its modifications. Hassan [21] estimates the parameter of exponentiated exponential distribution using rank set sampling. Al-Omari and et al. [4][4] estimate the parameter of two-parameter X-gamma distribution using a rank set sampling scheme. Al-Nasser et al. [11] introduced folded rank set sampling for asymmetric distribution. Chen and et al. [15] introduced the maximum likelihood of the parameter for a continuous one-parameter exponential family under the optimal ranked set sampling. Alotaibi and et al. [9] have done statistical inference for the Kavya-Manoharan Kumaraswamy model under ranked set sampling. Chen [16] discussed Pareto parameter estimation using moving extremes ranked set sampling and ranked set sampling. He et al. [22] estimate parameters of log-logistic distribution using moving extremes ranked set sampling design. Biradar et al. [13] estimate the parameters of the population mean using paired ranked set sampling. Zheng and Al-Saleh [44] show that when judgment error is present modified maximum likelihood estimator is more robust than the MLE using RSS.Xie et al. [42] estimate the correlation coefficient of morgenstern-type bivariate exponential distribution by ranked set sampling with concomitant variable. Mukhoti et al.[37][31] estimate the $p(X > Y)$ for exponential using order statistics with application to rank set sampling. Naeem Sadiq et al. [40] used the method of quartile for the determination of Weibull parameters for the assessment of wind potential. Kaday et al. [2] introduced a method to estimate Weibull parameters for wind energy applications. Al-Saleh and et al. [8] used steady-state ranked simulated sample(SRSIS) to improve the efficiency of an estimator for integrals. The ranked set sampling method was also used in the design of experiments. Ozturk et al. [31] embedded a Ranked set sampling into a cluster randomized design(CRD) for the selection of sampling units. Study shows that it reduced the expected mean square error. In comparative life testing experiments, the experimenter put two different types of products that are produced under the same facility for life tests simultaneously. The experiment is terminated under the type-II censoring scheme to reduce the cost and time. This scheme is known as the joint type-II censoring scheme. Rasoli and Balakrishnan [33] used a joint progressive type-II censoring scheme for two exponential populations and studied the exact likelihood inference. Ashour and Abo-Kasem [10] estimate the parameter of two Weibull populations under a joint type-II censoring scheme. Ding and Gui [18] studied the statistical inference for two gamma distributions under Joint type-II censoring. Ranked set sampling schemes are embedded in comparative life testing experiments under a joint type-II censoring scheme called a joint ranked set sampling scheme. Joint ranked set sampling(JRSS)and its modified version was introduced by Raykundaliya and Patel [35] to estimate the parameter of two exponential distributions. Patel et al. [39] developed joint ranked set sampling and joint modified ranked set sampling to estimate parameters and testing of two Rayleigh populations. Raykundaliya and Patel [34] estimate the parameter of the exponential population using joint percentile ranked set sampling. Estimating the parameters of the Weibull distribution is essential in accurately modeling real-world scenarios involving Weibull populations(or distribution). The Weibull distribution is widely used in,Reliability Engineering,Extreme Value Theory, survival analysis, Material Science and Engineering,Quality Control, and reliability engineering[25, 28, 14]. This distribution is used to forecast weather and wind speed determination. It is popularly used because of its relation with other distributions like exponential, uniform, and Rayleigh. The applicability of the Weibull distribution across various fields necessitates more efficient methods for parameter estimation and its can be done by using RSS method[3, 36, 1, 12].In scenarios where different products manufactured under the same conditions follow a Weibull distribution, efficiency can be improved by using Ranked Set Sampling (JRSS).

The remaining paper is arranged in the following manner. In section 2, maximum likelihood(ML) estimates of parameters of two Weibull populations are derived using the joint ranked set sampling(JRSS) method. Expression for, asymptotic variance of ML estimates of Weibull parameters are also derived. In subsection 2.4, Bayes estimates for scale parameters are also estimated when the shape parameter is known. In section 3, ML estimate, asymptotic variance for ML estimates, and Bayes estimates are found under joint modified ranked set sampling(JMRSS). Bayes estimates are derived when the shape parameter is known. In section 4, Ml and Bayes estimates for Weibull parameters are obtained using JSRS for comparison purposes. In section 5, a simulation study is conducted to see the performance of estimates under various mentioned schemes and with real examples.Some concluding remarks are given in section 6.

2. JOINT RANKED SET SAMPLING(JRSS)

In this section, we studied the estimation of parameter of the Weibull population under joint ranked set sampling.

2.1. pdf and cdf of Weibull populations

Suppose $X_1, X_2...X_{m_1}$ are i.i.d life time of product type-I from $Weibull(\alpha_1, \beta_1)$ population with pdf and cdf is given as follows,

$$f(x,\alpha_1,\beta_1) = \begin{cases} \frac{\alpha_1}{\beta_1{}^{\alpha_1}} x^{\alpha_1-1} e^{-(\frac{x}{\beta_1})^{\alpha_1}}, & \text{if } \alpha_1, \beta_1, x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$$F(x,\alpha_1,\beta_1) = 1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}} \tag{2.2}$$

Suppose $Y_1, Y_2...Y_{m_2}$ are i.i.d life time of product type-II from $Weibull(\alpha_2, \beta_2)$ population with pdf and cdf is given as follows,

$$g(y,\alpha_2,\beta_2) = \begin{cases} \frac{\alpha_2}{\beta_2{}^{\alpha_2}} y^{\alpha_2-1} e^{-(\frac{y}{\beta_2})^{\alpha_2}}, & \text{if } \alpha_2, \beta_2, y > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

$$G(y,\alpha_2,\beta_2) = 1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}} \tag{2.4}$$

2.2. Algorithm of JRSS

The algorithm to select JRSS from two populations under identical conditions is as follows.
Step 1: Select m_1 units from a product of type-I and m_2 units from a product of type-II randomly and combine these observations to create a joint sample of size $m = m_1 + m_2$.
Step 2: Arrange all these observations of the joint sample in increasing order.
Step 3: To have m joint sets of sizes m repeat Step 1. and Step 2. m times.
Step 4: To have a joint ranked set sample of size m, select i^{th} minimum from i^{th} set. where i=1,2,...,m.
Step 5: Repeat Step 1. to Step 4 to increase the size of the joint ranked set sample k times i.e. n=mk.
Represent w_{ij} as an i^{th} order element from i^{th} joint sample in j^{th} cycle.i=1,2,...,m; j=1,2,...,k.

2.3. Maximum Likelihood Estimation Under JRSS

The likelihood function according to the algorithm discussed in section 2. based on the observations $W_{ij}, i = 1, 2, 3, ..., m; j = 1, 2, 3, ..., k$ obtained in the joint rank set sample is given as follows

$$
\begin{aligned}
L &= (\alpha_1, \alpha_2, \beta_1, \beta_2, x, y) \\
&= \prod_{j=1}^{k}\prod_{i=1}^{m} \frac{m!}{(i-1)(m-1)!}[F_x(w_{ij})]^{a_{ij}}[1-F_x(w_{ij})]^{b_{ij}}[f_x(w_{ij})]^{z_{ij}} \times \\
&\qquad [G_y(w_{ij})]^{c_{ij}}[1-G_y(w_{ij})]^{d_{ij}}[g_y(w_{ij})]^{1-z_{ij}} \\
&= \prod_{j=1}^{k}\prod_{i=1}^{m} C.\left[1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right]^{a_{ij}}\left[e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right]^{b_{ij}}\left[\frac{\alpha_1}{{\beta_1}^{\alpha_1}}w_{ij}^{\alpha_1-1}e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right]^{z_{ij}}\left[1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right]^{c_{ij}} \times \\
&\qquad \left[e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right]^{d_{ij}}\left[\frac{\alpha_2}{{\beta_2}^{\alpha_2}}w_{ij}^{\alpha_2-1}e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right]^{1-z_{ij}}
\end{aligned} \tag{2.5}
$$

where,

$$
z_{ij} = \begin{cases} 1 & \text{if } w_{ij} \text{ is from first population(X)} \\ 0 & \text{if } w_{ij} \text{ is from second population(Y),} \end{cases} \qquad \text{i=1,2,...,m; j=1,2,...,k}
$$

$C = \frac{m!}{(i-1)(m-1)!}$

$w_{ij} = i^{th}$ minimum of i^{th} set of combined sample in j^{th} cycle.

a_{ij}= Number of x's observation less than or equal to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

b_{ij}= Number of x's observation greater than or equal to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

c_{ij}= Number of y's observation less than or equal to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

d_{ij}= Number of y's observation greater than or equal to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

log likelihood of (2.5) is as follows

$$
\begin{aligned}
\log(L) &= \sum_{j=1}^{k}\sum_{i=1}^{m}\log(C) + \sum_{j=1}^{k}\sum_{i=1}^{m} a_{ij}\log\left[1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right] \\
&\quad - \sum_{j=1}^{k}\sum_{i=1}^{m} b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1} + \sum_{j=1}^{k}\sum_{j=1}^{m} z_{ij}\log\left[\frac{\alpha_1}{{\beta_1}^{\alpha_1}}w_{ij}^{\alpha_1-1}e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right] \\
&\quad + \sum_{j=1}^{k}\sum_{i=1}^{m} c_{ij}\log\left[1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right] \\
&\quad + \sum_{j=1}^{k}\sum_{j=1}^{m}(1-z_{ij})\log\left[\frac{\alpha_2}{{\beta_2}^{\alpha_2}}w_{ij}^{\alpha_2-1}e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right] \\
&\quad - \sum_{j=1}^{k}\sum_{i=1}^{m} d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}
\end{aligned} \tag{2.6}
$$

To get ML equations differentiate equation (2.6) with respect to α_1,β_1, α_2 and β_2

$$\frac{\partial logL}{\partial\alpha_1} = \sum_{j=1}^{k}\sum_{i=1}^{m}\frac{a_{ij}\log\left(\frac{w_{ij}}{\beta_1}\right)\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}}{1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}} + \sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}\left[\frac{1}{\alpha_1}-\log(\beta_1)-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log\left(\frac{w_{ij}}{\beta_1}\right)+\log(w_{ij})\right]-\sum_{j=1}^{k}\sum_{i=1}^{m}b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log\left(\frac{w_{ij}}{\beta_1}\right) \tag{2.7}$$

$$\frac{\partial logL}{\partial\beta_1} = -\frac{\alpha_1}{\beta_1}\sum_{j=1}^{k}\sum_{i=1}^{m}\frac{a_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}}{1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}}-\frac{\alpha_1}{\beta_1}\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}+\frac{\alpha_1}{\beta_1}\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1} + \frac{\alpha_1}{\beta_1}\sum_{j=1}^{k}\sum_{i=1}^{m}b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1} \tag{2.8}$$

$$\frac{\partial logL}{\partial\alpha_2} = \sum_{j=1}^{k}\sum_{i=1}^{m}\frac{c_{ij}\log\left(\frac{w_{ij}}{\beta_2}\right)\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}e^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}}{1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}} + \sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})\left[\frac{1}{\alpha_2}-\log(\beta_2)-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right)+\log(w_{ij})\right] - \sum_{j=1}^{k}\sum_{i=1}^{m}d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right) \tag{2.9}$$

$$\frac{\partial logL}{\partial\beta_2} = -\frac{\alpha_2}{\beta_2}\sum_{j=1}^{k}\sum_{i=1}^{m}\frac{c_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}}{1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}}-\frac{\alpha_2}{\beta_2}\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij}) + \frac{\alpha_2}{\beta_2}\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}+\frac{\alpha_2}{\beta_2}\sum_{j=1}^{k}\sum_{i=1}^{m}d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2} \tag{2.10}$$

By equating to zero and simplifying equation (2.7),(2.8),(2.9) and (2.10) we get the MLE estimate of parameters $\alpha_1, \beta_1, \alpha_2$ and β_2 respectively.

$$\hat{\alpha}_{1JRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}}{J_1+J_2-J_3} \tag{2.11}$$

$$\hat{\beta}_{1JRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}\beta_1(\frac{w_{ij}}{\beta_1})^{\alpha_1}(z_{ij}+b_{ij})-\sum_{j=1}^{k}\sum_{i=1}^{m}\frac{\beta_1 a_{ij}(\frac{w_{ij}}{\beta_1})^{\alpha_1}e^{-(\frac{w_{ij}}{\beta_1})^{\alpha_1}}}{1-e^{-(\frac{w_{ij}}{\beta_1})^{\alpha_1}}}}{\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}} \tag{2.12}$$

$$\hat{\alpha}_{2JRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})}{J_4+J_5-J_6} \tag{2.13}$$

$$\hat{\beta}_{2JRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}\beta_2(\frac{w_{ij}}{\beta_2})^{\alpha_2}(1-z_{ij}+d_{ij})-\sum_{j=1}^{k}\sum_{i=1}^{m}\frac{\beta_2 c_{ij}(\frac{w_{ij}}{\beta_2})^{\alpha_2}e^{-(\frac{w_{ij}}{\beta_2})^{\alpha_2}}}{1-e^{-(\frac{w_{ij}}{\beta_2})^{\alpha_2}}}}{\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})} \tag{2.14}$$

Where,$J_1, J_2, J_3, J_4, J_5, J_6$ are as follows

$$J_1 = \sum_{j=1}^{k}\sum_{i=1}^{m} b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log\left(\frac{w_{ij}}{\beta_1}\right)$$

$$J_2 = \sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}\left[\log(\beta_1)\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log\left(\frac{w_{ij}}{\beta_1}\right) - \log(w_{ij})\right]$$

$$J_3 = \sum_{j=1}^{k}\sum_{i=1}^{m} \frac{a_{ij}\log(\frac{w_{ij}}{\beta_1})\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1} e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}}{1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}}$$

$$J_4 = \sum_{j=1}^{k}\sum_{i=1}^{m} d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right)$$

$$J_5 = \sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})\left[\log(\beta_2) + \left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right)\log(w_{ij})\right]$$

$$J_6 = \sum_{j=1}^{k}\sum_{i=1}^{m} \frac{c_{ij}\log\left(\frac{w_{ij}}{\beta_2}\right)\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2} e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_1}}}{1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}}$$

Lets take second derivatives of log likelihood (2.6) with respect to α_1 β_1,α_2 β_2, to derive Fisher information matrix

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha_1^2} &= \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{J_7 - J_8}{\left[1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right]^2} - \sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left[\frac{1}{\alpha_1^2} + \left[\log\left(\frac{w_{ij}}{\beta_1}\right)\right]^2\left(\frac{w_{ij}}{\beta_1}\right)\right] \\ &\quad - \sum_{i=1}^{m}\sum_{j=1}^{k} b_{ij}\left[\log\left(\frac{w_{ij}}{\beta_1}\right)\right]^2\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\end{aligned} \tag{2.15}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\beta_1^2} &= \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{J_9 - J_{10}}{\left[1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right]^2} - \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{z_{ij}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\left[\alpha_1\log(\frac{w_{ij}}{\beta_1}) + 1\right] \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{k} b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\left[\alpha_1\log\left(\frac{w_{ij}}{\beta_1}\right) + 1\right]\end{aligned} \tag{2.16}$$

$$\frac{\partial \log(L)}{\partial \alpha_1 \beta_1} = \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij}\alpha_1 \ln\left(\frac{W_{ij}}{\beta_1}\right) \cdot \left(\frac{W_{ij}}{\beta_1}\right)^{2\alpha_1} \mathrm{e}^{-\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}}{\beta_1\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}\right)} \tag{2.17}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij}\alpha_1 \ln\left(\frac{W_{ij}}{\beta_1}\right) \cdot \left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1} \mathrm{e}^{-\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}}{\beta_1\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}\right)} \tag{2.18}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij} \cdot \left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1} \mathrm{e}^{-\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}}{\beta_1\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}\right)} + \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij}\alpha_1 \ln\left(\frac{W_{ij}}{\beta_1}\right) \cdot \left(\frac{W_{ij}}{\beta_1}\right)^{2\alpha_1} \mathrm{e}^{-2\cdot\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}}{\beta_1\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}\right)^2} \tag{2.19}$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left(\frac{\alpha_1 \ln\left(\frac{W_{ij}}{\beta_1}\right) \cdot \left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_{11}}}{\beta_1} - \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{\beta_1}\right) \tag{2.20}$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{b_{ij}\alpha_1 \ln\left(\frac{W_{ij}}{\beta_1}\right) \cdot \left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_{11}}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{b_{ij} \cdot \left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_{11}}}{\beta_1} \tag{2.21}$$

$$\frac{\partial^2 l}{\partial \alpha_2^2} = \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{J_{11}-J_{12}}{\left[1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right]^2} - \sum_{i=1}^{m}\sum_{j=1}^{k}(1-z_{ij})\left[\frac{1}{\alpha_2^2} + \left[\log\left(\frac{w_{ij}}{\beta_2}\right)\right]^2\left(\frac{w_{ij}}{\beta_2}\right)\right]$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{k} d_{ij}\left[\log\left(\frac{w_{ij}}{\beta_2}\right)\right]^2\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2} \tag{2.22}$$

$$\frac{\partial^2 l}{\partial \beta_2^2} = \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{J_{13}-J_{14}}{\left[1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right]^2} - \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{(1-z_{ij})}{\beta_2} + \sum_{i=1}^{m}\sum_{j=1}^{k}(1-Z_{ij})\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\left[\alpha_2 log\left(\frac{w_{ij}}{\beta_2}\right)+1\right]$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{k} d_{ij}\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}\left[\alpha_2 log\left(\frac{w_{ij}}{\beta_2}\right)+1\right] \tag{2.23}$$

$$\frac{\partial \log(L)}{\partial \alpha_2 \beta_2} = \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij}\alpha_2 \ln\left(\frac{W_{ij}}{\beta_2}\right)\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{2\alpha_2} \mathrm{e}^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}}{\beta_2\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}\right)}$$
$$- \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij}\alpha_2 \ln\left(\frac{W_{ij}}{\beta_2}\right)\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2} \mathrm{e}^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}}{\beta_2\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}\right)}$$
$$- \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij}\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2} \mathrm{e}^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}}{\beta_2\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}\right)} + \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{a_{ij}\alpha_2 \ln\left(\frac{W_{ij}}{\beta_2}\right)\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{2\alpha_2} \mathrm{e}^{-2\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}}{\beta_2\left(1-\mathrm{e}^{-\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}\right)^2}$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left(\frac{\alpha_2 \ln\left(\frac{W_{ij}}{\beta_2}\right)\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}{\beta_2} + \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_{2\,1}}}{\beta_2} - \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{\beta_2}\right)$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{b_{ij}\alpha_2 \ln\left(\frac{W_{ij}}{\beta_2}\right)\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{b_{ij}\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_{2\,1}}}{\beta_2}$$

$$J_7 = a_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1} e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}} \left[\log\left(\frac{w_{ij}}{\beta_1}\right)\right]^2 \left[1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right]\left[1-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\right]$$
$$J_8 = a_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{2\alpha_1} e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{2\alpha_1}} \left[\log\left(\frac{w_{ij}}{\beta_1}\right)\right]^2$$
$$J_9 = -a_{ij}\left[\frac{\alpha_1}{\beta_1^{(\alpha_1+1)}}\right]^2 w_{ij}^{2\alpha_1} e^{-2\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}$$
$$J_{10} = -\alpha_1\left[1-e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}\right]\left[\frac{a_{ij}w_{ij}^{\alpha_1}e^{-\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}}}{\beta_1^{\alpha_1+2}}\right]\left[\alpha_1\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}-(\alpha_1+1)\right]$$
$$J_{11} = c_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2} e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}} \left[\log\left(\frac{w_{ij}}{\beta_2}\right)\right]^2 \left[1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right]\left[1-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\right]$$
$$J_{12} = c_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{2\alpha_2} e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{2\alpha_2}} \left[\log\left(\frac{w_{ij}}{\beta_2}\right)\right]^2$$
$$J_{13} = -\left[\frac{\alpha_2}{\beta_2^{(\alpha_2+1)}}\right]^2 c_{ij} w_{ij}^{2\alpha_2} e^{-2\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}$$
$$J_{14} = -\alpha_2\left[1-e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}\right]\left[\frac{c_{ij}w_{ij}^{\alpha_2}e^{-\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}}}{\beta_2^{\alpha_2+2}}\right]\left[\alpha_2\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}-(\alpha_2+1)\right]$$

$$\frac{\partial^2 l}{\partial\alpha_1\partial\alpha_2} = 0; \frac{\partial^2 l}{\partial\beta_1\partial\alpha_2} = 0; \frac{\partial^2 l}{\partial\beta_1\partial\alpha_1} = 0 \tag{2.24}$$

Fisher information matrix(I) for the parameters of two Weibull populations under JRSS can be written as follow ,

$$I(\alpha_1,\beta_1,\alpha_2,\beta_2) = \begin{bmatrix} I_{\alpha_1\alpha_1} & I_{\alpha_1\beta_1} & I_{\alpha_1\alpha_2} & I_{\alpha_1\beta_2} \\ I_{\beta_1\alpha_1} & I_{\beta_1\beta_1} & I_{\beta_1\alpha_2} & I_{\beta_1\beta_2} \\ I_{\alpha_2\alpha_1} & I_{\alpha_2\beta_1} & I_{\alpha_2\alpha_2} & I_{\alpha_2\beta_2} \\ I_{\beta_2\alpha_1} & I_{\beta_2\beta_1} & I_{\beta_2\alpha_2} & I_{\beta_2\beta_2} \end{bmatrix} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha_1^2} & \frac{\partial^2 l}{\partial\alpha_1\partial\beta_1} & 0 & 0 \\ \frac{\partial^2 l}{\partial\beta_1\partial\alpha_1} & \frac{\partial^2 l}{\partial\beta_1^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 l}{\partial\alpha_2^2} & \frac{\partial^2 l}{\partial\alpha_2\partial\beta_2} \\ 0 & 0 & \frac{\partial^2 l}{\partial\beta_2\partial\alpha_2} & \frac{\partial^2 l}{\partial\beta_2^2} \end{bmatrix}$$

2.4. Bayes Estimation Using JRSS

Bayesian theory is widely used in estimation theory when information about parameters of interest is available. If this information can be described by distribution is known as prior distribution.We know that, if X follows Weibull(α,β) and if substitute $\theta=\beta^\alpha$ then $U=X^\alpha$ is exponentially distributed with mean θ.We used this approach and estimate the θ first and then used $\beta=\theta^{\frac{1}{\alpha}}$.Here we consider shape parameter α is known to obtain an estimate of β.let's substitute $\theta_1=\beta_1^{\alpha_1}$ and $\theta_2=\beta_2^{\alpha_2}$ and using transformation $U=X^{\alpha_1}$ and $V=Y^{\alpha_1}$ in equations (2.1) and (2.3) we have U and V are exponential with mean θ_1 and θ_2 respectively.To obtain Bayes estimate we Considered inverted gamma prior and square error loss function(SELF) for θ_1 and θ_2. Under square error loss function(SELF), Bayes estimate is the mean of posterior with an inverted gamma distribution as the prior distribution.
Therefore,

$$f(U)=\frac{1}{\theta_1}e^{-(\frac{u}{\theta_1})} \tag{2.25}$$

$$f(V)=\frac{1}{\theta_2}e^{-(\frac{v}{\theta_2})} \tag{2.26}$$

The pdf inverted gamma prior $(igamma(\gamma_i,\eta_i))$ for parameters θ_1 and θ_2 for known shape parameters α_1 and α_2 is given as,

$$\pi(\beta_i)=\frac{\eta_i^{\gamma_i}}{\Gamma\gamma_i}\beta_i^{-(\gamma_i+1)}e^{-(\frac{\eta_i}{\beta_i})};\beta_i>0,\eta_i>0,\gamma_i>0 \tag{2.27}$$

;i=1,2

The likelihood function for two exponential populations under JRSS is given as

$$L(\alpha_1,\alpha_2,\beta_1,\beta_2,w)=\prod_{j=1}^{k}\prod_{i=1}^{m}\frac{m!}{(i-1)(m-1)!}\left[1-e^{-\left(\frac{w_{ij}}{\theta_1}\right)}\right]^{a_{ij}}\left[e^{-\left(\frac{w_{ij}}{\theta_1}\right)}\right]^{b_{ij}}\left[\frac{1}{\theta_1}e^{-\left(\frac{w_{ij}}{\theta_1}\right)}\right]^{z_{ij}}\times \\ \left[1-e^{-\left(\frac{w_{ij}}{\theta_2}\right)}\right]^{c_{ij}}\left[\frac{1}{\theta_2}e^{-(\frac{v}{\theta_2})}\right]^{1-z_{ij}}\left[e^{-\left(\frac{w_{ij}}{\theta_2}\right)}\right]^{d_{ij}} \tag{2.28}$$

Joint posterior of θ_1 and θ_2 given $\underline{W}$ is derived as follows

$$h(\theta_1,\theta_2|\underline{w})\propto\prod_{j=1}^{k}\prod_{i=1}^{m}\left[1-e^{-\left(\frac{w_{ij}}{\theta_1}\right)}\right]^{a_{ij}}\theta_1^{-\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}}e^{-\frac{\sum_{j=1}^{k}\sum_{i=1}^{m}w_{ij}(b_{ij}+z_{ij})}{\theta_1}}\theta_1^{-(\gamma_1+1)}e^{-\frac{\eta_1}{\theta_1}}\times \\ \prod_{j=1}^{k}\prod_{i=1}^{m}\left[1-e^{-\left(\frac{w_{ij}}{\theta_2}\right)}\right]^{c_{ij}}\theta_2^{-\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})}e^{-\frac{\sum_{j=1}^{k}\sum_{i=1}^{m}w_{ij}(d_{ij}+1-z_{ij})}{\theta_2}}\theta_2^{-(\gamma_1+1)}e^{-\frac{\eta_2}{\theta_2}} \tag{2.29}$$

$$\propto \theta_1^{-\sum_{j=1}^{k}\sum_{i=1}^{m}(z_{ij}+\gamma_1+1)} e^{\frac{(A_1+\eta_1)}{\theta_1}} \theta_2^{-\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij}+\gamma_2+1)} e^{\frac{(A_2+\eta_2)}{\theta_2}} \times$$
$$\prod_{j=1}^{k}\prod_{i=1}^{m}\left[1-e^{-\left(\frac{w_{ij}}{\theta_1}\right)}\right]^{a_{ij}} \prod_{j=1}^{k}\prod_{i=1}^{m}\left[1-e^{-\left(\frac{w_{ij}}{\theta_2}\right)}\right]^{c_{ij}} \tag{2.30}$$
$$= h_1(\theta_1|\underline{w})h_2(\theta_1|\underline{w})$$

Where,
$A_1 = \sum_{j=1}^{k}\sum_{i=1}^{m} w_{ij}(b_{ij}+z_{ij})$ and $A_2 = \sum_{j=1}^{k}\sum_{i=1}^{m} w_{ij}(d_{ij}+1-z_{ij})$

$$h_1(\theta_1|\underline{w}) \propto \theta_1^{-\sum_{j=1}^{k}\sum_{i=1}^{m}(z_{ij}+\gamma_1+1)} e^{\frac{(A_1+\eta_1)}{\theta_1}} \prod_{j=1}^{k}\prod_{i=1}^{m}\left[1-e^{-(\frac{w_{ij}}{\theta_1})}\right]^{a_{ij}} \tag{2.31}$$

$$h_2(\theta_2|\underline{w}) \propto \theta_2^{-\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij}+\gamma_2+1)} e^{\frac{(A_2+\eta_2)}{\theta_2}} \prod_{j=1}^{k}\prod_{i=1}^{m}\left[1-e^{-\left(\frac{w_{ij}}{\theta_2}\right)}\right]^{c_{ij}} \tag{2.32}$$

$h_1(\theta_1|\underline{w})$ and $h_2(\theta_1|\underline{w})$ are marginal posterior pdf of parameters θ_1 and θ_2 respectively.this can be written as,

$$h_1(\theta_1|\underline{w}) = igamma(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}+\gamma_1, A_1+\eta_1)h_{13}(\theta_1) \tag{2.33}$$

$$h_2(\theta_2|\underline{w}) = igamma(\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})+\gamma_1, A_1+\eta_1)h_{23}(\theta_2) \tag{2.34}$$

Where,

$$h_{13}(\theta_1) = \prod_{j=1}^{k}\prod_{i=1}^{m}[1-e^{-(\frac{w_{ij}}{\theta_1})}]^{a_{ij}} \tag{2.35}$$

and

$$h_{23}(\theta_1) = \prod_{j=1}^{k}\prod_{i=1}^{m}[1-e^{-(\frac{w_{ij}}{\theta_2})}]^{c_{ij}} \tag{2.36}$$

The Bayes estimates of parameters θ_1 and θ_2 under the SELF are given respectively

$$\tilde{\theta}_{1JRSS} = \int_0^{\infty} \theta_1 h_1(\theta_1|\underline{w})d\theta_1, \tag{2.37}$$

$$\tilde{\theta}_{2JRSS} = \int_0^{\infty} \theta_2 h_2(\theta_2|\underline{w})d\theta_2, \tag{2.38}$$

$$\tilde{\beta}_{1JRSS} = (\tilde{\theta}_{1JRSS})^{\frac{1}{\alpha_1}} \tag{2.39}$$

$$\tilde{\beta}_{2JRSS} = (\tilde{\theta}_{2JRSS})^{\frac{1}{\alpha_2}} \tag{2.40}$$

α_1 and α_2 are specified values
Here, we observed that $h_1(\theta_1|\underline{w})$ and $h_2(\theta_2|\underline{w})$ are complex function, it is difficult to obtain closed form Bayes estimates of θ_1 and θ_2 .
To approximate complex function we use importance sampling method The algorithm is given below:

Step 1: Specify the values of m and k
Step 2: Generate two independent samples of sizes m_1 and m_2 respectively from exponential distribution with parameter $\theta_1(=\beta_1^{\alpha_1})$ and $\theta_2(=\beta_2^{\alpha_2})$
Step 3: Combined m_1 and m_2 observation drawn from exponential distributions with parameters θ_1 and θ_2 respectively.
Step 4: Arranged observations in ascending order and choose i^{th} lowest observation for i^{th}JRSS observation
Step 5: Repeat Step 1 and Step 2, m times to get JRSS of size m
Step 6: Identify the ranked set sample $w_{11}, w_{21}, ..., w_{m1}$
Step 7: Repeat Steps 1 to Step 6 for k times to get JRSS sample of $w_{1j}, w_{2j}, ..., w_{mj}$; j=1,2,3,...,k for k cycles.
Step 8: Generate N values, lets say$\theta_{11}, \theta_{12}, ..., \theta_{1N}$ of θ_1 from inverted gamma distribution $igamma(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}+\gamma_1, A_1+\eta_1)$
Step 9: Generate N values, lets say$\theta_{21}, \theta_{22}, ..., \theta_{2N}$ of θ_2 from inverted gamma distribution $igamma(\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})+\gamma_2, A_2+\eta_2)$
Step 10: For N values of θ_1 and θ_2,calculate N values of of functions $h_{13}(\theta_1)$ and $h_{23}(\theta_2)$ respectively using (2.35) and (2.36)
Step 11: For SELF,the Bayes estimate for parameter θ_1 and θ_2 are obtain as follows

$$\tilde{\theta}_{1JRSS} = \frac{\sum_{i=1}^{N}\theta_{1i}h_{13}(\theta_{1i})}{h_{13}(\theta_{1i})}$$

$$\tilde{\theta}_{2JRSS} = \frac{\sum_{i=1}^{N}\theta_{2i}h_{23}(\theta_{2i})}{h_{23}(\theta_{2i})}$$

Step 12: Based on step 11 Bayes estimate for scale parameters β_1 and β_2 of Weibull distribution are obtain as

$$\tilde{\beta}_{1JRSS} = (\hat{\theta_1})^{\frac{1}{\alpha_1}} \tag{2.41}$$

$$\tilde{\beta}_{2JRSS} = (\hat{\theta_2})^{\frac{1}{\alpha_2}} \tag{2.42}$$

3. JOINT MODIFIED RANKED SET SAMPLING

In this section, we studied the estimation of parameters of the Weibull population under joint modified ranked set sampling.
The algorithm to select JMRS sample from two populations under identical conditions is as follows.
Step 1. Select m_1 units from a product of type-I and m_2 units from a product of type-II randomly and combine these observations to create a joint sample of size $m = m_1 + m_2$.
Step 2. Arrange all these observations of the joint sample in increasing order.
Step 3. To have m joint sets of sizes m, repeat the Step 1. and Step 2. m times.
Step 4. To have a JMRS sample of size m, select the first minimum from all m sets.
Step 5. Repeat Step 1. to Step 4 to increase the size of the JMRS sample k times i.e. n=mk.

3.1.Maximum Likelihood Estimation Under JMRSS

The likelihood function according to the algorithm discussed in a previous section based on the observations $w_{ij}, i = 1,2,3,...,m; j = 1,2,3,...,k$ can be obtained by putting $a_{ij} = c_{ij} = 0$ for all $i = 1,2,3,...,m; j = 1,2,...,k$ in equation(4.2)

$$\begin{aligned} \log(L) = \sum_{j=1}^{k}\sum_{i=1}^{m}\log(C) + \sum_{j=1}^{k}\sum_{j=1}^{m}(1-z_{ij})\log\left[\frac{\alpha_2}{{\beta_2}^{\alpha_2}}w_{ij}^{\alpha_2-1}e^{-(\frac{w_{ij}}{\beta_2})^{\alpha_2}}\right] \\ + \sum_{j=1}^{k}\sum_{j=1}^{m}z_{ij}\log\left[\frac{\alpha_1}{{\beta_1}^{\alpha_1}}w_{ij}^{\alpha_1-1}e^{-(\frac{w_{ij}}{\beta_1})^{\alpha_1}}\right] - \sum_{j=1}^{k}\sum_{i=1}^{m}b_{ij}(\frac{w_{ij}}{\beta_1})^{\alpha_1} \\ - \sum_{j=1}^{k}\sum_{i=1}^{m}d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2} \end{aligned} \tag{3.1}$$

$$\frac{\partial logL}{\partial\alpha_1} = \sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}\left[\frac{1}{\alpha_1} - \log(\beta_1) - \left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log(\frac{w_{ij}}{\beta_1}) + \log(w_{ij})\right] - \sum_{j=1}^{k}\sum_{i=1}^{m}b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log(\frac{w_{ij}}{\beta_1}) \tag{3.2}$$

$$\begin{aligned} \frac{\partial logL}{\partial\beta_1} = -\frac{\alpha_1}{\beta_1}\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij} + \frac{\alpha_1}{\beta_1}\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1} \\ + \frac{\alpha_1}{\beta_1}\sum_{j=1}^{k}\sum_{i=1}^{m}b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1} \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\partial logL}{\partial\alpha_2} = \sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})\left[\frac{1}{\alpha_2} - \log(\beta_2) - \left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right) + \log(w_{ij})\right] \\ - \sum_{j=1}^{k}\sum_{i=1}^{m}d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right) \end{aligned} \tag{3.4}$$

$$\frac{\partial logL}{\partial\beta_2} = -\frac{\alpha_2}{\beta_2}\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij}) + \frac{\alpha_2}{\beta_2}\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2} + \frac{\alpha_2}{\beta_2}\sum_{j=1}^{k}\sum_{i=1}^{m}d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2} \tag{3.5}$$

By equating to zero and simplifying the equation we get the MLE estimate of parameters $\alpha_1, \beta_1, \alpha_2$ andβ_2

$$\hat{\alpha}_{1JMRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}}{J_{15}+J_{16}} \tag{3.6}$$

$$\hat{\beta}_{1JMRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}\beta_1\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}(z_{ij}+b_{ij})}{\sum_{j=1}^{k}\sum_{i=1}^{m}z_{ij}} \tag{3.7}$$

$$\hat{\alpha}_{2JMRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})}{J_{17}+J_{18}} \tag{3.8}$$

$$\hat{\beta}_{2JMRSS} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}\beta_2\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}(1-z_{ij}+d_{ij})}{\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})} \tag{3.9}$$

Where,J_{15}, J_{16}, J_{17} andJ_{18} are as follows

$$J_{15} = \sum_{j=1}^{k}\sum_{i=1}^{m} b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log\left(\frac{w_{ij}}{\beta_1}\right)$$

$$J_{16} = \sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}\left[\log(\beta_1)\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\log\left(\frac{w_{ij}}{\beta_1}\right) - \log(w_{ij})\right]$$

$$J_{17} = \sum_{j=1}^{k}\sum_{i=1}^{m} d_{ij}\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right)$$

$$J_{18} = \sum_{j=1}^{k}\sum_{i=1}^{m}(1 - z_{ij})\left[\log(\beta_2) + \left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\log\left(\frac{w_{ij}}{\beta_2}\right)\log(w_{ij})\right]$$

Lets take second derivatives of log likelihood (2.6) with respect to α_1 β_1,α_2 β_2, to derive Fisher information matrix

$$\begin{aligned}\frac{\partial^2 l}{\partial \alpha_1^2} &= \sum_{i=1}^{m}\sum_{j=1}^{k} - \sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left[\frac{1}{\alpha_1^2} + \left[\log\left(\frac{w_{ij}}{\beta_1}\right)\right]^2\left(\frac{w_{ij}}{\beta_1}\right)\right] \\ &- \sum_{i=1}^{m}\sum_{j=1}^{k} b_{ij}\left[\log\left(\frac{w_{ij}}{\beta_1}\right)\right]^2\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \beta_1^2} &= \sum_{i=1}^{m}\sum_{j=1}^{k} - \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{z_{ij}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\left[\alpha_1 \log(\frac{w_{ij}}{\beta_1}) + 1\right] \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{k} b_{ij}\left(\frac{w_{ij}}{\beta_1}\right)^{\alpha_1}\left[\alpha_1 \log\left(\frac{w_{ij}}{\beta_1}\right) + 1\right]\end{aligned} \tag{3.11}$$

$$\frac{\partial \log(L)}{\partial \alpha_1 \beta_1} = \sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left(\frac{\alpha_1 \ln\left(\frac{W_{ij}}{\beta_1}\right)\cdot\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_1}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_{1_1}}}{\beta_1} - \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{\beta_1}\right) \tag{3.12}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{b_{ij}\alpha_1 \ln\left(\frac{W_{ij}}{\beta_1}\right)\cdot\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_{1_1}}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{b_{ij}\cdot\left(\frac{W_{ij}}{\beta_1}\right)^{\alpha_{1_1}}}{\beta_1} \tag{3.13}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \alpha_2^2} &= \sum_{i=1}^{m}\sum_{j=1}^{k} - \sum_{i=1}^{m}\sum_{j=1}^{k}(1 - z_{ij})\left[\frac{1}{\alpha_2^2} + \left[\log\left(\frac{w_{ij}}{\beta_2}\right)\right]^2\left(\frac{w_{ij}}{\beta_2}\right)\right] \\ &- \sum_{i=1}^{m}\sum_{j=1}^{k} d_{ij}\left[\log\left(\frac{w_{ij}}{\beta_2}\right)\right]^2\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\end{aligned} \tag{3.14}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \beta_2^2} &= \sum_{i=1}^{m}\sum_{j=1}^{k} - \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{(1 - z_{ij})}{\beta_2} + \sum_{i=1}^{m}\sum_{j=1}^{k}(1 - Z_{ij})\left(\frac{w_{ij}}{\beta_2}\right)^{\alpha_2}\left[\alpha_2 log\left(\frac{w_{ij}}{\beta_2}\right) + 1\right] \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{k} d_{ij}\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}\left[\alpha_2 log\left(\frac{w_{ij}}{\beta_2}\right) + 1\right]\end{aligned} \tag{3.15}$$

$$\begin{aligned}\frac{\partial \log(L)}{\partial \alpha_2 \beta_2} &= \sum_{i=1}^{m}\sum_{j=1}^{k} z_{ij}\left(\frac{\alpha_2 \ln\left(\frac{W_{ij}}{\beta_2}\right)\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}{\beta_2} + \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_{2_1}}}{\beta_2} - \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{\beta_2}\right) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{b_{ij}\alpha_2 \ln\left(\frac{W_{ij}}{\beta_2}\right)\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_2}}{\beta_1} + \sum_{i=1}^{m}\sum_{j=1}^{k}\frac{b_{ij}\cdot\left(\frac{W_{ij}}{\beta_2}\right)^{\alpha_{2_1}}}{\beta_2}\end{aligned}$$

$$\frac{\partial^2 l}{\partial \alpha_1 \partial \alpha_2} = 0; \frac{\partial^2 l}{\partial \beta_1 \partial \alpha_2} = 0; \frac{\partial^2 l}{\partial \beta_1 \partial \alpha_1} = 0 \tag{3.16}$$

Fisher information matrix(I) for the parameters of two Weibull populations under JMRSS can be written as follow ,

$$I(\alpha_1,\beta_1,\alpha_2,\beta_2)=\begin{bmatrix} I_{\alpha_1\alpha_1} & I_{\alpha_1\beta_1} & I_{\alpha_1\alpha_2} & I_{\alpha_1\beta_2} \\ I_{\beta_1\alpha_1} & I_{\beta_1\beta_1} & I_{\beta_1\alpha_2} & I_{\beta_1\beta_2} \\ I_{\alpha_2\alpha_1} & I_{\alpha_2\beta_1} & I_{\alpha_2\alpha_2} & I_{\alpha_2\beta_2} \\ I_{\beta_2\alpha_1} & I_{\beta_2\beta_1} & I_{\beta_2\alpha_2} & I_{\beta_2\beta_2} \end{bmatrix} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha_1^2} & \frac{\partial^2 l}{\partial\alpha_1\partial\beta_1} & 0 & 0 \\ \frac{\partial^2 l}{\partial\beta_1\partial\alpha_1} & \frac{\partial^2 l}{\partial\beta_1^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 l}{\partial\alpha_2^2} & \frac{\partial^2 l}{\partial\alpha_2\partial\beta_2} \\ 0 & 0 & \frac{\partial^2 l}{\partial\beta_2\partial\alpha_2} & \frac{\partial^2 l}{\partial\beta_2^2} \end{bmatrix}$$

3.2.Bayes Estimation Under JMRSS

To obtain Bayes estimates under JMRSS scheme substitute $a_{ij} = c_{ij} = 0$ in equation (2.33) and (2.34).Then marginal posterior distributions of θ_1 and θ_2 will be given as follows.

$$h_{1M}(\theta_1|w) = igamma(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij} + \gamma_1, A_1 + \eta_1) \tag{3.17}$$

$$h_{2M}(\theta_2|w) = igamma(\sum_{j=1}^{k}\sum_{i=1}^{m}(1 - z_{ij}) + \gamma_2, A_2 + \eta_2) \tag{3.18}$$

Bayes estimates of θ_1 and θ_2 are mean of marginal posterior under SELF.Hence Bayes estimates $\tilde{\theta}_{1JMRSS}$ and $\tilde{\theta}_{2JMRSS}$ are given below

$$\begin{aligned}\tilde{\theta}_{1JMRSS} &= E\left(h_{1M}(\theta_1)\right) \\ &= \frac{A_1 + \eta_1}{\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij} + \gamma_1 - 1}\end{aligned} \tag{3.19}$$

$$\begin{aligned}\tilde{\theta}_{2JMRSS} &= E\left(h_{2M}(\theta_2)\right) \\ &= \frac{A_2 + \eta_2}{\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij} + \gamma_2 - 1}\end{aligned} \tag{3.20}$$

Hence, Bayes estimates of parameters β_1 and β_2 for known α_1 and α_2 of the Weibull distributions can be written as

$$\tilde{\beta}_{1JMRSS} = \left(\tilde{\theta}_{1JMRSS}\right)^{\frac{1}{\alpha_1}} \tag{3.21}$$

$$\tilde{\beta}_{2JMRSS} = \left(\tilde{\theta}_{2JMRSS}\right)^{\frac{1}{\alpha_2}} \tag{3.22}$$

4. JOINT SIMPLE RANDOM SAMPLING

According to available literature joining two samples taken from two independent populations and estimating parameters of two different populations have more efficiency than estimating parameters independently based on individual samples. Hence for comparison purposes, We derive MLE and Bayesd estimates of the parameters for joint simple random sampling.

4.1. Maximum likelihood estimation under JSRS

We draw simple random sample of size $n_1 = m_1k$ from first population of $Weibull(\alpha_1,\beta_1)$ and SRS of size $n_2 = m_1k$ from second population $Weibull(\alpha_2,\beta_2)$for k cycle.Hence joint SRS of size $n = n_1 + n_2$

is observed.

The likelihood function under JSRS is as follows

$$L = \prod_{i=1}^{n_1} f(x_i; \alpha_1, \beta_1) \prod_{i=1}^{n_2} f(y_i; \alpha_2, \beta_2)$$

used equation (2.1) and equation(2.3) in above so we get following equation

$$L = \left(\frac{\alpha_1}{\beta_1^{\alpha_1}}\right)^{n_1} \left(\frac{\alpha_2}{\beta_2^{\alpha_2}}\right)^{n_2} e^{-\sum_{i=1}^{n_1}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1} + \sum_{i=1}^{n_1}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}} \prod_{i=1}^{n_1} x_i^{\alpha_1} \prod_{i=1}^{n_2} y_i^{\alpha_2} \tag{4.1}$$

$$\begin{aligned} l = \log(L) = n_1 \log(\alpha_1) - n_1 \log(\beta_1) + n_2 \log(\alpha_2) - n_2\alpha_2 \log(\beta_2) - \sum_{i=1}^{n_1} (\frac{x_i}{\beta_1})^{\alpha_1} - \sum_{i=1}^{n_2} (\frac{y_i}{\beta_2})^{\alpha_2} \\ + (\alpha_1 - 1) \sum_{i=1}^{n_1} \log(x_i) + (\alpha_2 - 1) \sum_{i=1}^{n_2} \log(y_i) \end{aligned} \tag{4.2}$$

Differentiate equation(4.2) with respect to $\alpha_1, \beta_1, \alpha_2$ and β_2 we got the following equations.

$$\frac{\partial l}{\partial \alpha_1} = \frac{n_1}{\alpha_1} - n_1 \log(\beta_1) + \sum_{i=1}^{n_1} \log(x_i) - \sum_{i=1}^{n_1} \left(\frac{x_i}{\beta_1}\right)^{\alpha_1} \log\left(\frac{x_i}{\beta_1}\right) \tag{4.3}$$

$$\frac{\partial l}{\partial \beta_1} = \frac{\alpha_1}{\beta_1} \sum_{i=1}^{n_1} \left(\frac{x_i}{\beta_1}\right)^{\alpha_1} - \frac{n_1\alpha_1}{\beta_1} \tag{4.4}$$

$$\frac{\partial l}{\partial \alpha_2} = \frac{n_2}{\alpha_2} - n_2 \log(\beta_2) + \sum_{i=1}^{n_2} \log(y_i) - \sum_{i=1}^{n_2} \left(\frac{y_i}{\beta_2}\right)^{\alpha_2} \log\left(\frac{y_i}{\beta_2}\right) \tag{4.5}$$

$$\frac{\partial l}{\partial \beta_2} = \frac{\alpha_2}{\beta_2} \sum_{i=1}^{n_2} \left(\frac{y_i}{\beta_2}\right)^{\alpha_2} - \frac{n_2\alpha_2}{\beta_2} \tag{4.6}$$

By differentiating equations (4.3), (4.4), (4.5), and (4.6) with respect to α_1, β_1, α_2, and β_2, we obtain the following equations:

$$\frac{\partial^2 l}{\partial \alpha_1^2} = -\frac{n_1}{\alpha_1^2} - \sum_{i=1}^{n_1} \left[\log\left(\frac{x_i}{\beta_1}\right)\right]^2 \left(\frac{x_i}{\beta_1}\right)^{\alpha_1} \tag{4.7}$$

$$\frac{\partial^2 l}{\partial \beta_1^2} = \frac{n_1\alpha_1}{\beta_1^2} - \frac{\alpha_1(\alpha_1+1)}{\beta_1^2} \sum_{i=1}^{n_1} \left(\frac{x_i}{\beta_1}\right)^{\alpha_1} \tag{4.8}$$

$$\frac{\partial^2 l}{\partial \alpha_2^2} = -\frac{n_2}{\alpha_2^2} - \sum_{i=1}^{n_2} \left[\log\left(\frac{y_i}{\beta_2}\right)\right]^2 \left(\frac{y_i}{\beta_2}\right)^{\alpha_2} \tag{4.9}$$

$$\frac{\partial^2 l}{\partial \beta_2^2} = \frac{n_2\alpha_2}{\beta_2^2} - \frac{\alpha_2(\alpha_2+1)}{\beta_2^2} \sum_{i=1}^{n_2} \left(\frac{y_i}{\beta_2}\right)^{\alpha_2} \tag{4.10}$$

$$\frac{\partial^2 l}{\partial \alpha_1 \partial \beta_1} = -\frac{n_1}{\beta_1} - \sum_{i=1}^{n_1} \frac{\alpha_1 \ln\left(\frac{x}{\beta_1}\right) \cdot \left(\frac{x}{\beta_1}\right)^{\alpha_1}}{\beta_1} - \frac{\left(\frac{x}{\beta_1}\right)^{\alpha_1}}{\beta_1} \tag{4.11}$$

$$\frac{\partial^2 l}{\partial \alpha_2 \partial \beta_2} = -\frac{n_2}{\beta_1} - \sum_{i=1}^{n_2} \frac{\alpha_1 \ln\left(\frac{y}{\beta_1}\right) \cdot \left(\frac{y}{\beta_1}\right)^{\alpha_1}}{\beta_1} - \frac{\left(\frac{y}{\beta_1}\right)^{\alpha_1}}{\beta_1} \tag{4.12}$$

$$\frac{\partial^2 l}{\partial\alpha_1\partial\alpha_2}=0;\frac{\partial^2 l}{\partial\beta_1\partial\alpha_2}=0;\frac{\partial^2 l}{\partial\beta_1\partial\alpha_1}=0 \tag{4.13}$$

Thus, the Fisher information matrix (I) for the Weibull parameters α_1, β_1, α_2, and β_2 under JSRS can be expressed as follows

$$I(\alpha_1,\beta_1,\alpha_2,\beta_2)=\begin{bmatrix} I_{\alpha_1\alpha_1} & I_{\alpha_1\beta_1} & I_{\alpha_1\alpha_2} & I_{\alpha_1\beta_2}\\ I_{\beta_1\alpha_1} & I_{\beta_1\beta_1} & I_{\beta_1\alpha_2} & I_{\beta_1\beta_2}\\ I_{\alpha_2\alpha_1} & I_{\alpha_2\beta_1} & I_{\alpha_2\alpha_2} & I_{\alpha_2\beta_2}\\ I_{\beta_2\alpha_1} & I_{\beta_2\beta_1} & I_{\beta_2\alpha_2} & I_{\beta_2\beta_2}\end{bmatrix}=-E\begin{bmatrix}\frac{\partial^2 l}{\partial\alpha_1^2} & \frac{\partial^2 l}{\partial\alpha_1\partial\beta_1} & 0 & 0\\ \frac{\partial^2 l}{\partial\beta_1\partial\alpha_1} & \frac{\partial^2 l}{\partial\beta_1^2} & 0 & 0\\ 0 & 0 & \frac{\partial^2 l}{\partial\alpha_2^2} & \frac{\partial^2 l}{\partial\alpha_2\partial\beta_2}\\ 0 & 0 & \frac{\partial^2 l}{\partial\beta_2\partial\alpha_2} & \frac{\partial^2 l}{\partial\beta_2^2}\end{bmatrix}$$

4.2. Byes estimation under JSRS

For the Bayes estimation, lets take same transformation $U=X^{\alpha_1}$ and $V=Y^{\alpha_2}$ and put $\theta_1=\beta_1^{\alpha_1}$ and $\theta_2=\beta_2^{\alpha_2}$ for known shape parameter α_1 and α_2.Consider the same inverted gamma prior for the exponential parameters θ_1 and θ_2.Then joint posterior distribution for parameter θ_1 and θ_2 can be written as follows.

$$\begin{aligned} h_1(\theta_1,\theta_2|x,y)\propto L\pi_1(\theta_1)\pi_2(\theta_2) &\propto \frac{1}{\theta_1^{n_1}}e^{-\sum_{i=1}^{n_1}\frac{x_i}{\theta_1}}\frac{\eta_1^{\gamma_1}}{\Gamma\gamma_1}\theta_1^{-(\gamma_1+1)}e^{-\frac{\eta_1}{\theta_1}}\\ &\times\frac{1}{\theta_2^{n_2}}e^{-\sum_{i=1}^{n_2}\frac{y_i}{\theta_2}}\frac{\eta_2^{\gamma_2}}{\Gamma\gamma_2}\theta_2^{-(\gamma_2+1)}e^{-\frac{\eta_2}{\theta_2}}\end{aligned}$$

$$h_1(\theta_1,\theta_2|x,y)=\frac{\eta_1^{\gamma_1}}{\Gamma\gamma_1}\theta_1^{-(n_1+\gamma_1+1)}e^{-\sum_{i=1}^{n_1}\frac{x_i+\eta_1}{\theta_1}}\frac{\eta_2^{\gamma_2}}{\Gamma\gamma_2}\theta_2^{-(n_2+\gamma_2+1)}e^{-\sum_{i=1}^{n_2}\frac{y_i+\eta_2}{\theta_2}}$$

$$=h_1(\theta_1|x)h_2(\theta_2|y)$$

This is product of two marginal posterior distribution of θ_1 and θ_2
Where,

$$h_1(\theta_1|x)=igamma(n_1+\gamma_1,\sum_{i=1}^{n_1}x_i+\eta_1) \tag{4.14}$$

$$h_2(\theta_2|y)=igamma(n_2+\gamma_2,\sum_{i=1}^{n_2}y_i+\eta_2) \tag{4.15}$$

Hence ,Under SELF Bayes estimates for parameters θ_1 and θ_2 are respectively given below.

$$\tilde{\theta}_{1JSRS}=\frac{\sum_{i=1}^{\eta_1}x_i}{\eta_1+\gamma_1-1} \tag{4.16}$$

$$\tilde{\theta}_{2JSRS}=\frac{\sum_{i=1}^{\eta_2}y_i}{\eta_2+\gamma_2-1} \tag{4.17}$$

Hence,Bayes estimate for parameter β_1 and β_2 under SELF are given as,

$$\tilde{\beta}_{1JSRS}=(\tilde{\theta}_{1JSRS})^{\frac{1}{\alpha_1}} \tag{4.18}$$

$$\tilde{\beta}_{1JSRS}=(\tilde{\theta}_{2JSRS})^{\frac{1}{\alpha_2}} \tag{4.19}$$

5. NUMERICAL STUDY

In this section, we examine the performance of both ML and Bayes estimators under the JRSS and JMRSS schemes using simulation methods. Additionally, a real-world example is provided to validate the findings.

5.1. SIMULATION STUDY

We simulate results for two Weibull population with parameters $\alpha_1 = 0.8$, $\beta_1 = 0.5$ and $\alpha_1 = 1.2, \beta_1 = 1.5$.Two independent samples of sizes $m_1 = 3, 4, 5$ and $m_2 = 2, 3, 4$ are taken from two Weibull populations for k=4,6,8 cycles. The sample is selected using various sampling methods such as JRSS, JMRSS, and JSRS, following the algorithm outlined in the previous section. We compute the MLE, MSE, and bias for the estimator by repeating the procedure 100 times. We also compute Bayes estimates for scale parameters with fixed shape parameters for the same sampling schemes by repeating the procedure 100 times under SELF. For Bayesian estimation inverted gamma prior is considered. We show the results for different hyperparameters and random samples of size $m_1 = 3, 4$ and $m_2 = 2, 3$ for k=4,5 cycle.
We used R studio to perform all these simulations. Further, we summarise the results in tables 1 and 2 for the MLE method based on different values of m_1,m_2, and k. We present the results of Bayesian estimates in tables from 3 to 8 for $m_1 = 3, 4$ $m_2 = 2, 3$ k=4,5 and different values of hyperparameter

Table 1: Simulation Result of MLE for Weibull parameters $\alpha_1 = 0.8, \beta_1 = 0.5$ and $\alpha_1 = 1.2, \beta_1 = 1.5$

		JRSS				JSRS				JMRSS			
(m_1, m_2)	k	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]
		0.85902	0.48561	1.30095	1.51027	0.93021	0.51341	1.4725	1.48895	0.98939	1.07586	1.85503	2.10161
	4	0.02372	0.0148	0.09791	0.0947	0.06616	0.02661	0.32966	0.19061	0.07289	0.42632	1.02489	0.55021
		0.05902	-0.01439	0.10095	0.01027	0.13021	0.01341	0.2725	-0.01105	0.18939	0.57586	0.65503	0.60161
		0.84981	0.51616	1.28769	1.52578	0.85714	0.50112	1.36666	1.50301	0.93313	1.10633	1.63829	2.09729
(3,2)	6	0.01691	0.00835	0.07648	0.06907	0.04002	0.03318	0.19257	0.1699	0.04603	0.4358	0.34423	0.52726
		0.04981	0.01616	0.08769	0.02578	0.05714	0.00112	0.16666	0.00301	0.13313	0.60633	0.43829	0.59729
		0.83726	0.50518	1.24764	1.53587	0.85347	0.51967	1.31706	1.51708	0.95209	1.07199	1.67087	2.13211
	8	0.01316	0.00783	0.04014	0.04086	0.02419	0.0189	0.10194	0.14129	0.0422	0.37276	0.31381	0.507
		0.03726	0.00517	0.04764	0.03587	0.05347	0.01967	0.11706	0.01708	0.15209	0.57199	0.47087	0.63211
		0.86023	0.51327	1.31999	1.49754	0.91701	0.54217	1.50587	1.55597	0.97975	1.19651	1.63412	2.33149
	4	0.03522	0.01251	0.07439	0.05216	0.05937	0.04432	0.28556	0.15619	0.07162	0.64372	0.32847	0.83904
		0.06023	0.01327	0.11999	-0.00246	0.11701	0.04217	0.30587	0.05597	0.17975	0.69651	0.43412	0.83149
		0.81475	**0.49251**	**1.25983**	**1.49695**	0.8754	0.50247	1.25979	1.51392	0.96609	1.25964	1.64523	2.33275
(3,3)	6	**0.01067**	**0.00863**	**0.02801**	**0.03186**	0.03724	0.02449	0.04716	0.10876	0.05732	0.71021	0.29821	0.81911
		0.01475	-0.00749	0.05983	-0.00305	0.0754	0.00247	0.05979	0.01392	0.16609	0.75964	0.44523	0.83275
		0.82372	0.51582	1.23372	1.48078	0.86211	0.50087	1.27947	1.54382	0.94987	1.29094	1.55687	2.29837
	8	0.01206	0.00631	0.01771	0.02568	0.02532	0.01647	0.05011	0.0788	0.04422	0.71702	0.16713	0.71584
		0.02372	0.01582	0.03372	-0.01922	0.06211	0.00087	0.07947	0.04382	0.14987	0.79094	0.35687	0.79837
		0.83545	0.49828	1.24733	1.50297	0.94637	0.52604	1.28241	1.52396	0.9924	1.51114	1.63303	2.47491
	4	0.02316	0.01	0.03802	0.03696	0.11179	0.04112	0.08134	0.09614	0.08983	1.50715	0.27579	1.08771
		0.03545	-0.00172	0.04733	0.00297	0.14637	0.02604	0.08241	0.02396	0.1924	1.01114	0.43303	0.97491
		0.80776	0.49742	1.23765	1.50664	0.87841	0.53081	1.23652	1.52651	0.92649	1.44695	1.64865	2.45463
(3,4)	6	0.01081	0.00533	0.01451	0.03068	0.04451	0.0284	0.04559	0.06885	0.04314	1.15825	0.25838	1.0087
		0.00776	-0.00258	0.03765	0.00664	0.07841	0.03081	0.03652	0.02651	0.12649	0.94695	0.44865	0.95463
		0.81107	0.50592	1.23244	1.4755	0.86443	0.51902	1.27927	1.49242	0.93073	1.23826	1.66415	2.28566
	8	0.00878	0.00545	0.01533	0.01771	0.02263	0.02474	0.04193	0.04969	0.03492	1.10194	0.21215	1.03516
		0.01107	0.00592	0.03244	-0.0245	0.06443	0.01902	0.07927	-0.00758	0.12973	0.9811	0.40716	0.99139
		0.81877	0.50946	1.28042	1.54059	0.89192	0.51287	1.46781	1.49056	0.97855	1.20569	1.81009	2.16164
	4	0.01647	0.00958	0.07702	0.09091	0.05803	0.0284	0.37947	0.18507	0.05763	0.57981	0.69367	0.61676
		0.01877	0.00946	0.08042	0.04059	0.09192	0.01287	0.26781	-0.00944	0.17855	0.70569	0.61009	0.66164
		0.82401	0.498	1.26233	1.50314	0.87889	0.49776	1.38655	1.47675	0.95462	1.21838	1.79764	2.23506
(4,2)	6	0.00922	0.00574	0.05676	0.06158	0.0297	0.02215	0.18429	0.13961	0.04323	0.5813	0.59447	0.7447
		0.02401	-0.002	0.06233	0.00314	0.07889	-0.00224	0.18655	-0.02325	0.15462	0.71838	0.59764	0.73506
		0.81798	0.50927	1.23056	1.52267	0.82219	0.51941	1.34119	1.52916	0.9306	1.23826	1.66415	2.28566
	8	0.0067	0.00478	0.0358	0.04479	0.01169	0.0172	0.10346	0.11193	0.02612	0.59812	0.31635	0.77298
		0.01798	0.00927	0.03056	0.02267	0.02219	0.01941	0.14119	0.02916	0.1306	0.73826	0.46415	0.78566

Table 2: Simulation Result of MLE for Weibull parameters $\alpha_1 = 0.8, \beta_1 = 0.5$ and $\alpha_1 = 1.2, \beta_1 = 1.5$

		JRSS				JSRS				JMRSS			
(m_1, m_2)	k	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]
(4,3)		0.82666	0.51098	1.27786	1.50345	0.85424	0.49826	1.3271	1.50002	0.96394	1.39265	1.77076	2.36951
	4	0.0171	0.00816	0.05438	0.0568	0.02925	0.02894	0.13095	0.12487	0.05162	0.91493	0.50634	0.92056
		0.02666	0.01098	0.07786	0.00345	0.05424	-0.00174	0.1271	0.00002	0.16394	0.89265	0.57076	0.86951
		0.81305	0.50062	1.25041	1.48049	0.85734	0.52018	1.32333	1.47938	0.93247	1.39703	1.66812	2.43709
	6	0.00833	0.00438	0.02492	0.03588	0.02425	0.01484	0.0721	0.09984	0.03117	0.90066	0.30224	1.01547
		0.01305	0.00062	0.05041	-0.01951	0.05734	0.02018	0.12333	-0.02062	0.13247	0.89703	0.46812	0.93709
		0.82188	0.51457	1.23489	1.46842	0.84302	0.49841	1.3101	1.47199	0.9484	1.39097	1.63656	2.43191
	8	0.00828	0.00422	0.02358	0.02727	0.01695	0.01564	0.05955	0.06989	0.03169	0.84735	0.2471	0.97101
		0.02188	0.01457	0.03489	-0.03158	0.04302	-0.00159	0.1101	-0.02801	0.1484	0.89097	0.43656	0.93191
(4,4)		0.82659	0.51186	1.26033	1.48447	0.87233	0.5083	1.29874	1.5386	0.94111	1.62899	1.75336	2.50249
	4	0.01555	0.00431	0.02953	0.0343	0.03534	0.02352	0.1186	0.11816	0.04604	1.67314	0.41152	1.13182
		0.02659	0.01186	0.06033	-0.01553	0.07233	0.0083	0.09874	0.0386	0.14111	1.12899	0.55336	1.00249
		0.82419	0.51576	1.24494	1.50084	0.86452	0.49937	1.30482	1.50001	0.94249	1.55683	1.69142	2.57854
	6	0.01199	0.00545	0.01826	0.01677	0.03203	0.01438	0.07341	0.07787	0.03564	1.23972	0.29972	1.26577
		0.02419	0.01576	0.04494	0.00084	0.06452	-0.00063	0.10482	0.00001	0.14249	1.05683	0.49142	1.07854
		0.83639	0.50658	1.21934	1.50371	0.83328	0.52053	1.27495	1.51081	0.94586	1.53673	1.64882	2.53189
	8	0.00769	0.00292	0.01238	0.02046	0.0172	0.01253	0.0404	0.05522	0.03834	1.16689	0.25001	1.13773
		0.03639	0.00658	0.01934	0.00371	0.03328	0.02053	0.07495	0.01081	0.14586	1.03673	0.44882	1.03189
(5,2)		0.82399	0.50526	1.31103	1.51848	0.85995	0.516	1.5521	1.56929	0.97377	1.35098	1.89262	2.2898
	4	0.01292	0.00787	0.06071	0.08385	0.0331	0.02077	0.51613	0.19643	0.05401	0.88119	0.82274	0.88653
		0.02399	0.00526	0.11103	0.01848	0.05995	0.016	0.3521	0.06929	0.17377	0.85098	0.69262	0.7898
		0.81904	0.4985	1.26573	1.48841	0.85983	0.52905	1.35031	1.49307	0.97162	1.34447	1.77482	2.32785
	6	0.00714	0.00288	0.05813	0.05655	0.02659	0.01363	0.12334	0.11359	0.0407	0.76699	0.52441	0.9282
		0.01904	-0.0015	0.06573	-0.01159	0.05983	0.02905	0.15031	-0.00693	0.17162	0.84447	0.57482	0.82785
		0.81779	0.5	1.26661	1.49543	0.82957	0.51598	1.28141	1.50526	0.98045	1.31971	1.74872	2.31401
	8	0.00718	0.00316	0.03361	0.03548	0.01081	0.01405	0.07752	0.10897	0.04189	0.70788	0.41788	0.79733
		0.01779	0	0.06661	-0.00457	0.02957	0.01598	0.08141	0.00526	0.18045	0.81971	0.54872	0.81401
(5,3)		0.8295	0.50737	1.23838	1.52142	0.86329	0.51809	1.3933	1.52366	0.97136	1.48737	1.8133	2.4939
	4	0.01111	0.00571	0.03851	0.04141	0.03001	0.02293	0.19578	0.11794	0.05482	1.10425	0.55278	1.1964
		0.0295	0.00737	0.03838	0.02142	0.06329	0.01809	0.1933	0.02366	0.17136	0.98737	0.6133	0.9939
		0.81414	0.5122	1.25268	1.48915	0.83796	0.51134	1.35646	1.56744	0.98119	1.45383	1.75463	2.46422
	6	0.00651	0.00315	0.0332	0.03447	0.01428	0.01802	0.09166	0.10278	0.04645	0.96666	0.41067	1.08018
		0.01414	0.0122	0.05268	-0.01085	0.03796	0.01134	0.15646	0.06744	0.18119	0.95383	0.55463	0.96422
		0.80042	0.48949	1.22131	1.50957	0.84233	0.51281	1.31746	1.57179	0.96662	1.49093	1.66487	2.45228
	8	0.00465	0.0029	0.01337	0.03226	0.0127	0.01094	0.06831	0.05932	0.03755	1.03241	0.27014	1.0198
		0.00042	-0.01051	0.02131	0.00957	0.04233	0.01281	0.11746	0.07179	0.16662	0.99093	0.46487	0.95228
(5,3)		0.82714	0.49716	1.24771	1.46292	0.85348	0.50057	1.32489	1.5213	0.99994	1.59425	1.78838	2.63354
	4	0.01097	0.00458	0.028	0.03066	0.02998	0.01919	0.08216	0.11594	0.06196	1.35373	0.45528	1.50261
		0.02714	-0.00284	0.04771	-0.03708	0.05348	0.00057	0.12489	0.0213	0.19994	1.09425	0.58838	1.13354
		0.81927	0.49754	1.20233	1.51037	0.82201	0.50909	1.30053	1.52163	0.9598	1.60721	1.70784	2.64259
	6	0.00652	0.00295	0.01406	0.02485	0.01373	0.01443	0.04977	0.06595	0.03686	1.31001	0.32604	1.40862
		0.01927	-0.00246	0.00233	0.01037	0.02201	0.00909	0.10053	0.02163	0.1598	1.10721	0.50784	1.14259
		0.8066	**0.50572**	**1.2153**	**1.51083**	0.81754	0.49811	1.27523	1.50703	0.97022	1.5497	1.71402	2.64632
	8	0.00463	0.00223	0.01281	0.01626	0.01043	0.00835	0.04196	0.04678	0.03784	1.16417	0.29614	1.4069
		0.0066	0.00572	0.0153	0.01083	0.01754	-0.00189	0.07523	0.00703	0.17022	1.0497	0.51402	1.14632

Table 3: Bayesian estimation for parameters $\beta_1 = 0.5$ and $\beta_1 = 1.5$

(3,2), k=4				JRSS		SRS		JMRSS	
γ_1	η_1	γ_1	η_2	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias
90	40	60	100	0.40026 0.01076 -0.09974	1.54722 0.00748 0.04722	0.48148 0.01389 -0.01852	1.60074 0.13812 0.10074	0.44764 0.00349 -0.05236	1.66668 0.03186 0.16668
90	40	60	120	0.40313 0.01013 -0.09687	1.73684 0.06017 0.23684	0.49455 0.02056 -0.00545	1.54873 0.11961 0.04873	0.4453 0.00398 -0.0547	1.89514 0.16087 0.39514
90	40	80	100	0.39961 0.01081 -0.10039	1.27687 0.05262 -0.22313	0.49743 0.0211 -0.00257	1.55979 0.10093 0.05979	0.44478 0.00394 -0.05522	1.33697 0.0282 -0.16303
90	60	60	100	0.58568 0.00824 0.08568	1.54958 0.00625 0.04958	0.47291 0.01642 -0.02709	1.48762 0.06833 -0.01238	0.66632 0.02868 0.16632	1.67638 0.03525 0.17638
90	60	60	120	0.57668 0.00702 0.07668	1.73898 0.06132 0.23898	0.46992 0.0155 -0.03008	1.5505 0.07936 0.0505	0.66302 0.02754 0.16302	1.89546 0.16056 0.39546
90	60	80	100	0.57736 0.00671 0.07736	1.27386 0.05371 -0.22614	0.47767 0.01944 -0.02233	1.5069 0.121 0.0069	0.66693 0.02898 0.16693	1.33381 0.02951 -0.16619
90	60	80	120	0.58239 0.00774 0.08239	1.43461 0.00781 -0.06539	0.48684 0.02197 -0.01316	1.53398 0.12262 0.03398	0.66554 0.02842 0.16554	1.52848 0.00362 0.02848
110	40	60	100	0.32931 0.02963 -0.17069	1.5476 0.00627 0.0476	0.47523 0.01548 -0.02477	1.53972 0.11608 0.03972	0.35378 0.02186 -0.14622	1.65649 0.02809 0.15649
110	40	60	120	0.32871 0.02988 -0.17129	1.74022 0.0636 0.24022	0.45668 0.01714 -0.04332	1.58229 0.09367 0.08229	0.35489 0.02151 -0.14511	1.91048 0.17242 0.41048
110	40	80	100	0.32737 0.03048 -0.17263	1.26462 0.05796 -0.23538	0.473 0.01765 -0.027	1.53735 0.09349 0.03735	0.35661 0.02106 -0.14339	1.3551 0.02363 -0.1449
110	40	80	120	0.32892 0.02976 -0.17108	1.43351 0.0081 -0.06649	0.49197 0.02067 -0.00803	1.51861 0.09227 0.01861	0.35611 0.02126 -0.14389	1.51929 0.00209 0.01929
110	60	60	100	0.48014 0.00081 -0.01986	1.5445 0.00699 0.0445	0.50681 0.01825 0.00681	1.52819 0.08127 0.02819	0.52852 0.00133 0.02852	1.67011 0.03295 0.17011
				0.48092	1.74411	0.52053	1.50806	0.5299	1.8958

Table 4: Bayesian estimation for parameters $\beta_1 = 0.5$ and $\beta_1 = 1.5$

(3,2) k=5				JRSS		JSRS		JMRSS	
γ_1	η_1	γ_1	η_2	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias
90	40	60	100	0.40573 0.00977 -0.09427	1.53866 0.00804 0.03866	0.43374 0.01655 -0.06626	1.26141 0.1565 -0.23859	0.46304 0.00244 -0.03696	1.69118 0.04092 0.19118
90	40	60	120	0.40904 0.00921 -0.09096	1.73718 0.06324 0.23718	0.47016 0.01914 -0.02984	1.24588 0.14368 -0.25412	0.46236 0.00203 -0.03764	1.89355 0.15913 0.39355
90	40	80	100	0.39768 0.01118 -0.10232	1.27739 0.05316 -0.22261	0.46571 0.01477 -0.03429	1.21319 0.17952 -0.28681	0.46577 0.00223 -0.03423	1.3618 0.02128 -0.1382
90	60	60	100	0.5734 0.00655 0.0734	1.52251 0.00498 0.02251	0.45649 0.01902 -0.04351	1.31529 0.1643 -0.18471	0.67947 0.03371 0.17947	1.68671 0.04077 0.18671
90	60	60	120	0.57674 0.00713 0.07674	1.73807 0.06223 0.23807	0.45704 0.01632 -0.04296	1.29291 0.15351 -0.20709	0.6761 0.03188 0.1761	1.91118 0.17276 0.41118
90	60	80	100	0.58139 0.008 0.08139	1.27121 0.05559 -0.22879	0.44634 0.01945 -0.05366	1.26783 0.15564 -0.23217	0.67615 0.03223 0.17615	1.36712 0.02025 -0.13288
90	60	80	120	0.58509 0.00887 0.08509	1.43504 0.00822 -0.06496	0.44187 0.01622 -0.05813	1.21284 0.16924 -0.28716	0.67825 0.03327 0.17825	1.54228 0.00416 0.04228
110	40	60	100	0.34057 0.02601 -0.15943	1.53959 0.0068 0.03959	0.45569 0.01753 -0.04431	1.20893 0.1761 -0.29107	0.36996 0.01752 -0.13004	1.70265 0.04589 0.20265
110	40	60	120	0.33511 0.02787 -0.16489	1.72117 0.05489 0.22117	0.44046 0.02151 -0.05954	1.3007 0.15462 -0.1993	0.36993 0.0174 -0.13007	1.91188 0.17418 0.41188
110	40	80	100	0.3394 0.02647 -0.1606	1.275 0.05339 -0.225	0.47354 0.01644 -0.02646	1.27916 0.15106 -0.22084	0.37517 0.01615 -0.12483	1.36871 0.01994 -0.13129
110	40	80	120	0.33682 0.02714 -0.16318	1.4386 0.00677 -0.0614	0.47102 0.01832 -0.02898	1.23864 0.18007 -0.26136	0.36516 0.01867 -0.13484	1.54803 0.00478 0.04803
110	60	60	100	0.48408 0.00085 -0.01592	1.53887 0.00725 0.03887	0.46924 0.01801 -0.03076	1.27586 0.14725 -0.22414	0.54861 0.0032 0.04861	1.70036 0.04491 0.20036
				0.48378 [illegible]	1.7411 [illegible]	0.45541 [illegible]	1.26627 [illegible]	0.54712 [illegible]	1.91682 [illegible]

Table 5: Bayesian estimation for parameters $\beta_1 = 0.5$ and $\beta_1 = 1.5$

(3,3), k=4				JRSS		JSRS		JMRSS	
γ_1	η_1	γ_1	η_2	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias
90	40	60	100	0.46235 0.00237 -0.03765	1.76372 0.07375 0.26372	0.55108 0.02373 0.05108	1.30866 0.12897 -0.19134	0.4614 0.00235 -0.0386	1.75311 0.06809 0.25311
90	40	60	120	0.4572 0.00274 -0.0428	1.97257 0.22883 0.47257	0.55038 0.02078 0.05038	1.33206 0.12396 -0.16794	0.46178 0.00231 -0.03822	1.96892 0.22623 0.46892
90	40	80	100	0.46132 0.0023 -0.03868	1.42202 0.00984 -0.07798	0.53248 0.01409 0.03248	1.32819 0.11639 -0.17181	0.4637 0.00236 -0.0363	1.43899 0.00696 -0.06101
90	60	60	100	0.68047 0.03361 0.18047	1.76656 0.07758 0.26656	0.56291 0.02421 0.06291	1.31426 0.11997 -0.18574	0.68435 0.03509 0.18435	1.78389 0.08626 0.28389
90	60	60	120	0.67319 0.03113 0.17319	1.98094 0.23629 0.48094	0.55731 0.02007 0.05731	1.31825 0.11202 -0.18175	0.68446 0.03505 0.18446	1.99047 0.24777 0.49047
90	60	80	100	0.68047 0.03355 0.18047	1.42792 0.00759 -0.07208	0.55798 0.02367 0.05798	1.32542 0.14302 -0.17458	0.68016 0.03373 0.18016	1.43762 0.00717 -0.06238
90	60	80	120	0.68546 0.03522 0.18546	1.60652 0.01439 0.10652	0.55565 0.02034 0.05565	1.34066 0.15171 -0.15934	0.68121 0.03403 0.18121	1.5889 0.01029 0.0889
110	40	60	100	0.36624 0.01855 -0.13376	1.7742 0.07992 0.2742	0.56078 0.01939 0.06078	1.30665 0.11568 -0.19335	0.37136 0.01713 -0.12864	1.77617 0.08156 0.27617
110	40	60	120	0.37014 0.01737 -0.12986	1.96266 0.21937 0.46266	0.5544 0.0156 0.0544	1.35335 0.12118 -0.14665	0.37281 0.01688 -0.12719	1.9757 0.23295 0.4757
110	40	80	100	0.36696 0.01855 -0.13304	1.43082 0.00773 -0.06918	0.55273 0.01486 0.05273	1.35744 0.08784 -0.14256	0.36615 0.01857 -0.13385	1.43447 0.00828 -0.06553
110	40	80	120	0.36464 0.01876 -0.13536	1.60386 0.01415 0.10386	0.54134 0.02 0.04134	1.3277 0.11656 -0.1723	0.37092 0.01739 -0.12908	1.61048 0.01502 0.11048
110	60	60	100	0.54195 0.00248 0.04195	1.75586 0.07118 0.25586	0.56818 0.02213 0.06818	1.35214 0.13451 -0.14786	0.54315 0.0025 0.04315	1.76094 0.07328 0.26094
				0.54588	1.97919	0.56725	1.36999	0.54806	1.97984

Table 6: Bayesian estimation for parameters $\beta_1 = 0.5$ and $\beta_1 = 1.5$

(3,3), k=5				JRSS		JSRS		JMRSS	
γ_1	η_1	γ_1	η_2	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias
90	40	60	100	0.48656	1.81823	0.47674	1.55523	0.47734	1.82032
				0.00118	0.10912	0.01665	0.06631	0.00167	0.11079
				-0.01344	0.31823	-0.02326	0.05523	-0.02266	0.32032
90	40	60	120	0.48324	2.01412	0.45827	1.60705	0.48345	2.00891
				0.00166	0.27083	0.01543	0.06657	0.00143	0.26587
				-0.01676	0.51412	-0.04173	0.10705	-0.01655	0.50891
90	40	80	100	0.47499	1.48374	0.47045	1.63131	0.48072	1.48896
				0.00165	0.00441	0.01611	0.09097	0.00157	0.00433
				-0.02501	-0.01626	-0.02955	0.13131	-0.01928	-0.01104
90	60	60	100	0.69208	1.81845	0.48341	1.65086	0.70309	1.80852
				0.03856	0.10659	0.0217	0.07651	0.04254	0.10119
				0.19208	0.31845	-0.01659	0.15086	0.20309	0.30852
90	60	60	120	0.69137	1.99485	0.4672	1.63797	0.69791	2.01115
				0.03819	0.25033	0.01326	0.0734	0.04019	0.26769
				0.19137	0.49485	-0.0328	0.13797	0.19791	0.51115
90	60	80	100	0.69947	1.48772	0.47141	1.54336	0.69552	1.47031
				0.04149	0.00449	0.01478	0.06568	0.03983	0.00452
				0.19947	-0.01228	-0.02859	0.04336	0.19552	-0.02969
90	60	80	120	0.69405	1.65828	0.47482	1.58424	0.69008	1.64775
				0.03885	0.03	0.01798	0.06769	0.03765	0.0265
				0.19405	0.15828	-0.02518	0.08424	0.19008	0.14775
110	40	60	100	0.38638	1.81661	0.46765	1.60096	0.3843	1.80227
				0.01359	0.10677	0.01489	0.06731	0.01387	0.0963
				-0.11362	0.31661	-0.03235	0.10096	-0.1157	0.30227
110	40	60	120	0.38575	2.01149	0.48294	1.58622	0.38862	2.01639
				0.01365	0.26705	0.01686	0.06853	0.01302	0.27411
				-0.11425	0.51149	-0.01706	0.08622	-0.11138	0.51639
110	40	80	100	0.38472	1.47948	0.45132	1.55314	0.38808	1.48478
				0.01413	0.00466	0.01682	0.0612	0.01336	0.0036
				-0.11528	-0.02052	-0.04868	0.05314	-0.11192	-0.01522
110	40	80	120	0.38692	1.64902	0.48825	1.60614	0.38384	1.63987
				0.01371	0.02677	0.01729	0.07642	0.01417	0.02392
				-0.11308	0.14902	-0.01175	0.10614	-0.11616	0.13987
110	60	60	100	0.55753	1.8168	0.48976	1.64935	0.55514	1.80131
				0.00406	0.10807	0.01177	0.08658	0.0041	0.09665
				0.05753	0.3168	-0.01024	0.14935	0.05514	0.30131
				0.55651	2.00929	0.47924	1.5484	0.55969	1.99701

Table 7: Bayesian estimation for parameters $\beta_1 = 0.5$ and $\beta_1 = 1.5$

(4,3), k=4				JRSS		JSRS		JMRSS	
γ_1	η_1	γ_1	η_2	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias
90	40	60	100	0.50746	1.80538	0.52796	1.49113	0.50717	1.80981
				0.00113	0.09902	0.02029	0.09685	0.00133	0.10022
				0.00746	0.30538	0.02796	-0.00887	0.00717	0.30981
90	40	60	120	0.51157	2.00882	0.51023	1.36882	0.51652	2.00718
				0.00122	0.26322	0.01764	0.10663	0.00199	0.26341
				0.01157	0.50882	0.01023	-0.13118	0.01652	0.50718
90	40	80	100	0.50975	1.46743	0.51106	1.38842	0.50872	1.47078
				0.00123	0.00458	0.01151	0.10808	0.00097	0.00352
				0.00975	-0.03257	0.01106	-0.11158	0.00872	-0.02922
90	60	60	100	0.7348	1.81612	0.52421	1.44678	0.73321	1.80077
				0.05673	0.10588	0.01616	0.10459	0.0565	0.09604
				0.2348	0.31612	0.02421	-0.05322	0.23321	0.30077
90	60	60	120	0.72458	2.00441	0.51419	1.4026	0.72424	2.01012
				0.05158	0.25954	0.01683	0.09667	0.0517	0.26718
				0.22458	0.50441	0.01419	-0.0974	0.22424	0.51012
90	60	80	100	0.72588	1.47157	0.51637	1.46088	0.72079	1.4726
				0.05296	0.0042	0.01321	0.11041	0.05055	0.00354
				0.22588	-0.02843	0.01637	-0.03912	0.22079	-0.0274
90	60	80	120	0.72856	1.64252	0.51377	1.45854	0.72188	1.64041
				0.05368	0.02351	0.016	0.07216	0.05086	0.02349
				0.22856	0.14252	0.01377	-0.04146	0.22188	0.14041
110	40	60	100	0.40943	1.80281	0.51836	1.44924	0.41312	1.81322
				0.00905	0.09611	0.01824	0.07587	0.00849	0.10363
				-0.09057	0.30281	0.01836	-0.05076	-0.08688	0.31322
110	40	60	120	0.40612	2.00751	0.50803	1.43721	0.41317	2.02219
				0.00976	0.26144	0.01428	0.08158	0.00852	0.27954
				-0.09388	0.50751	0.00803	-0.06279	-0.08683	0.52219
110	40	80	100	0.41834	1.47467	0.51535	1.43695	0.40847	1.48116
				0.00766	0.00385	0.01491	0.08296	0.00913	0.00414
				-0.08166	-0.02533	0.01535	-0.06305	-0.09153	-0.01884
110	40	80	120	0.41756	1.65084	0.50825	1.39364	0.40972	1.64259
				0.00767	0.02695	0.01136	0.08671	0.00892	0.02371
				-0.08244	0.15084	0.00825	-0.10636	-0.09028	0.14259
110	60	60	100	0.58057	1.79643	0.51906	1.48981	0.59011	1.80761
				0.00739	0.09475	0.01405	0.07338	0.00897	0.10031
				0.08057	0.29643	0.01906	-0.01019	0.09011	0.30761
				0.58081	2.02546	0.50525	1.4256	0.58532	2.02381

Table 8: Bayesian estimation for parameters $\beta_1 = 0.5$ and $\beta_1 = 1.5$

(4,3), k=5				JRSS		JSRS		JMRSS	
γ_1	η_1	γ_1	η_2	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias	$\tilde{\beta}_1$ MSE Bias	$\tilde{\beta}_2$ MSE Bias
				0.54215	1.84987	0.49707	1.72328	0.5457	1.85313
90	40	60	100	0.00298	0.1287	0.0144	0.10893	0.00392	0.13251
				0.04215	0.34987	-0.00293	0.22328	0.0457	0.35313
				0.54028	2.06051	0.48134	1.72394	0.53737	2.04697
90	40	60	120	0.00331	0.32129	0.01326	0.11769	0.0028	0.30636
				0.04028	0.56051	-0.01866	0.22394	0.03737	0.54697
				0.53935	1.51237	0.50079	1.72945	0.544	1.51627
90	40	80	100	0.00294	0.00454	0.01458	0.11144	0.00355	0.00502
				0.03935	0.01237	0.00079	0.22945	0.044	0.01627
				0.75436	1.84786	0.49304	1.68315	0.75931	1.85612
90	60	60	100	0.06687	0.12768	0.01161	0.0942	0.06924	0.13333
				0.25436	0.34786	-0.00696	0.18315	0.25931	0.35612
				0.75317	2.0652	0.49524	1.70331	0.75266	2.06083
90	60	60	120	0.06595	0.3262	0.01267	0.09243	0.0658	0.3219
				0.25317	0.5652	-0.00476	0.20331	0.25266	0.56083
				0.75322	1.52399	0.49322	1.72299	0.75518	1.51302
90	60	80	100	0.06633	0.00589	0.01317	0.12067	0.06698	0.00483
				0.25322	0.02399	-0.00678	0.22299	0.25518	0.01302
				0.75814	1.67034	0.50048	1.70685	0.74839	1.68261
90	60	80	120	0.0687	0.0327	0.01255	0.1102	0.06343	0.03697
				0.25814	0.17034	0.00048	0.20685	0.24839	0.18261
				0.44033	1.85536	0.47405	1.74752	0.4377	1.86684
110	40	60	100	0.00449	0.13475	0.0124	0.12568	0.00493	0.14163
				-0.05967	0.35536	-0.02595	0.24752	-0.0623	0.36684
				0.43287	2.05746	0.49164	1.72804	0.43852	2.05351
110	40	60	120	0.00564	0.31798	0.01208	0.09557	0.00482	0.31194
				-0.06713	0.55746	-0.00836	0.22804	-0.06148	0.55351
				0.44036	1.52091	0.47752	1.77134	0.44417	1.53141
110	40	80	100	0.00463	0.00513	0.0114	0.12837	0.00419	0.00509
				-0.05964	0.02091	-0.02248	0.27134	-0.05583	0.03141
				0.44251	1.68242	0.49018	1.75652	0.43984	1.67353
110	40	80	120	0.00412	0.03708	0.01431	0.1301	0.00462	0.03522
				-0.05749	0.18242	-0.00982	0.25652	-0.06016	0.17353
				0.61362	1.86518	0.50013	1.7173	0.60972	1.85955
110	60	60	100	0.01415	0.13999	0.01263	0.10619	0.0131	0.13537
				0.11362	0.36518	0.00013	0.2173	0.10972	0.35955
				0.60794	2.03973	0.48689	1.72271	0.60763	2.05441
110	60	60	120	0.0132	0.29867	0.00936	0.10392	0.01299	0.31577
				0.10794	0.53973	-0.01311	0.22271	0.10763	0.55441

We can easily observe that from Table 1 and Table 2, all the MLE estimators have sufficient performance under different sampling schemes. JRSS outperforms both JSRS and JMRSS in terms of mean squared error (MSE). From Table 1 and Table 2, it is also observed that as the sample size from each population increases MSE decreases, and for an increase in the cycles MSE decreases. From this table we see that, MSE for shape parameter α_1 under JRSS is 0.01067 for $m_1 = 3, m_2 = 3$, and K=6.To achieve the approximate MSE=0.01043, we need (5,3) and k=8 cycles under simple random sampling. We need 36 units under JRSS and 64 units under SRS to achieve the same efficiency. Hence we can say that by using JRSS one can reduce the sample size, time, and cost when measurement units are time-consuming and costly. All Bayes estimators perform better compared to MLE estimates in the case of JRSS, JMRSS, and JSRS. In Bayesian analysis table-3 to table-8, it is seen that the estimators perform excellently in the case of JRSS compared to JMRSS and JSRS. It is also observed that there is no remarkable effect of sample sizes m_1,m_2 and cycle k on the efficiency of an estimator$\tilde{\beta}_1$ and $\tilde{\beta}_2$.In Bayesian analysis it most important to choice of hyper parameter(i.e (γ_1,η_1),(γ_2,η_2)) of the prior distribution.It is observed that for the most accurate combination of hyperparameters, MSE becomes minimum. It is generally observed here that, an increase in the value of the shape parameter of inverted gamma prior MSE decreases. For an increase in the scale parameter of inverted gamma prior MSE gets increases.

5.2. REAL EXAMPLE

To validate the results shown in the simulation study we perform a real example presented by Proschan [32]. He took the time between failures hours of the air conditioning system of a Boeing 720 jet airplane "7913" and "7914".This data was later used by Ashour and Abo-Kasem [10] for two Weibull populations under the joint Typed two censored schemes.We also find that Weibull with parameters $\alpha_1 = 1.024920, \beta_1 = 64.792336 \alpha_2 = 1.123145, \beta_2 = 79.923959$ distribution fit good for this 7913 and 7914 jet respectively.

Table 9: Failure times of air-conditioning system for 7913 and 7914 air planes

Planes	Failure times
7914	3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210
7913	1, 4, 11, 16, 18, 18, 18, 24, 31, 39,46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216

From airplanes 7913 and 7914, we generate an independent sample of size $m_1 = 3$ and $m_2 = 4$ respectively.this procedure is repeated k=6 times (cycle). A JRSS and JMRSS are obtained according to the algorithm given in the respective section, This process is repeated 100 times, and average values of estimator, and mean square errors(MSE) are obtained. Results are obtained for the Maximum

Likelihood(ML) estimator and Bayes estimator and are shown in the table

Table 10: Results of MLE for $(m_1, m_2) = (3, 4)$ and $k = 6$

		JRSS				JSRS				JMRSS			
(m_1, m_2)	k	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]	$\hat{\alpha}_1$ (MSE) [Bias]	$\hat{\beta}_1$ (MSE) [Bias]	$\hat{\alpha}_2$ (MSE) [Bias]	$\hat{\beta}_2$ (MSE) [Bias]
(3,4)	6	1.06021 0.0204 0.03529	65.79376 69.44707 1.00142	1.14622 0.01579 0.02308	80.26162 86.34364 0.33766	1.11965 0.05665 0.09473	66.85556 272.4736 2.06322	1.20298 0.05147 0.07984	79.14989 191.0388 -0.77407	1.34936 0.16696 0.32444	141.4796 8591.213 76.68728	1.50652 0.20702 0.38338	153.0364 5863.328 73.11243

6.CONCLUSIONS

In this study, we used JRSS and JMRSS methods to estimate the parameter of the Weibull distribution. To estimate parameters, we considered two approaches ML and the Bayesian under SELF. In the ML method, we examined that a JRSS method performs superior as compared to JMRSS and JSRS in terms of MSE. JMRSS does not perform well in ML estimates as compared to JSRS but it is easy to implement and reduce time also by selecting a minimum from all sets. Using JRSS reduced the required sample size for ML estimates compared to JSRS. This reduction in sample size occurs because, to achieve approximately the same MSE, JSRS necessitates a larger number of sample units compared to JRSS. In the Bayesian approach, we estimate the scale parameter when the shape parameter is known. We compared the performance of these estimators in JRSS, JMRSS, and JSRS and noted that JRSS outperformed compared to other methods. We observed that Bayes estimates of parameters under SELF are more effective as compared to ML estimates. From simulation results, we noticed that the increase in the shape parameters γ_1 and γ_2 MSE becomes less but the increase in the scale hyperparameter MSE increases. As discussed, we consider Bayesian estimation only for scale parameters under SELF. We can further extend this study in this direction by Bayesian estimates for the shape parameter of the Weibull distribution. In addition to this, we can also estimate parameters for different loss functions like linear exponential loss function, etc. We can also change different prior for Weibull parameters like improper prior and both are unknown.

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Table 11: Results for Bayes estimation

(3,4), k=6				JRSS		JSRS		JMRSS	
γ_1	η_1	γ_1	η_2	$\hat{\beta}_1$ MSE Bias	$\hat{\beta}_2$ MSE Bias	$\hat{\beta}_1$ MSE Bias	$\hat{\beta}_2$ MSE Bias	$\hat{\beta}_1$ MSE Bias	$\hat{\beta}_1$ MSE Bias
				70.68756	85.51423	63.36218	79.34515	70.51528	85.48636
300	21831	1179	169708	36.67053	31.4569	240.7654	213.875	34.53224	31.11637
				5.89522	5.59027	-1.43016	-0.57881	5.72294	5.5624
				70.86896	85.47435	63.74006	79.4609	70.90484	85.43313
300	21831	1179	169733	38.93927	31.01473	202.4651	167.47	39.18069	30.51037
				6.07662	5.55039	-1.05228	-0.46306	6.1125	5.50917
				70.93162	84.19511	64.84869	80.54702	70.6349	84.1959
300	21831	1200	169708	39.43895	18.41044	195.2062	157.6168	36.16436	18.44957
				6.13928	4.27115	0.05635	0.62306	5.84256	4.27194
				70.44005	84.15251	65.85835	79.00961	70.69752	84.16896
300	21831	1200	169733	33.55545	18.02584	159.0614	171.5738	36.55093	18.18636
				5.64771	4.22855	1.06601	-0.91435	5.90518	4.245
				70.78135	85.51599	65.99727	81.78007	70.59652	85.48562
300	21856	1179	169708	38.33758	31.47664	210.1684	272.9061	35.28619	31.12864
				5.98901	5.59203	1.20493	1.85611	5.80418	5.56166
				70.77155	85.52935	66.7948	79.60626	70.89035	85.49814
300	21856	1179	169733	37.36304	31.58835	239.3858	193.3016	39.30155	31.27734
				5.97921	5.60539	2.00246	-0.3177	6.09801	5.57418
				71.0204	84.18447	68.06237	78.96341	70.68194	84.22094
300	21856	1200	169708	40.74966	18.38496	209.4251	198.2681	36.10928	18.67159
				6.22806	4.26051	3.27003	-0.96055	5.8896	4.29698
				70.87412	84.20287	65.54603	80.29783	71.07455	84.19956
300	21856	1200	169733	38.57354	18.4905	145.026	178.0461	41.69545	18.50108
				6.08178	4.27891	0.75369	0.37387	6.28221	4.2756
				65.55563	85.52003	67.11095	79.52959	65.66661	85.47468
325	21831	1179	169708	2.06649	31.47427	228.089	212.6481	2.25233	31.00658
				0.76329	5.59607	2.31861	-0.39437	0.87427	5.55072
				65.7291	85.50469	67.06044	82.29273	65.8908	85.49034
325	21831	1179	169733	2.34671	31.31513	174.6731	232.2927	2.84146	31.19218
				0.93676	5.58073	2.2681	2.36877	1.09846	5.56638
				65.79386	84.17357	65.41662	81.48454	65.72449	84.10515
325	21831	1200	169708	2.90013	18.22993	288.5648	219.0387	2.52024	17.64414
				1.00152	4.24961	0.62428	1.56058	0.93215	4.18119
				65.70305	84.10018	66.0915	82.46561	65.6949	84.12914
325	21831	1200	169733	2.31772	17.68475	163.9408	208.3276	2.52999	17.88789
				0.91071	4.17622	1.29916	2.54165	0.90256	4.20518
				65.81019	85.54727	66.80027	80.80138	65.68838	85.45846
325	21856	1179	169708	2.97279	31.85194	219.9924	186.7449	2.18682	30.82359
				1.01785	5.62331	2.00793	0.87742	0.89604	5.5345
				65.87277	85.52045	66.21085	81.01049	65.74287	85.46834
325	21856	1179	169733	2.38287	31.57559	204.6322	188.4586	2.34135	30.89308
				1.08043	5.59649	1.41851	1.08653	0.95053	5.54438
				65.82411	84.17047	65.94103	79.94244	65.97645	84.15329
325	21856	1200	169708	2.23626	18.15984	194.2124	226.6491	3.07399	18.05889
				1.03177	4.24651	1.14869	0.01848	1.18411	4.22933
				65.86145	84.25311	64.18336	79.11986	65.88323	84.19928
325	21856	1200	169733	2.70625	18.90294	218.2361	195.9955	2.88518	18.45983
				1.06911	4.32915	-0.60898	-0.8041	1.09089	4.27532

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