AN EFFICIENT CLASS OF RATIO CUM PRODUCT ESTIMATORS WITH NON-RESPONSE AND MEASUREMENT ERROR USING ORRT MODELS: A SENSITIVE ESTIMATION PROCEDURE

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ABSTRACT

The ratio-cum-product estimator is utilized to estimate the population mean of sensitive study variable to tackle the problem of non-response and measurement error under simple random sampling by using ORRT models are explored in this study. The characteristics of the proposed class of estimator are studied up to the first order of approximation. The relative performance of the suggested estimator as compared with distinct classes of proposed estimator and Kumar et al. [12] estimator are performed. Besides that, the theoretical findings are shown through a simulation study based on an artificially generated population and a real population. From the simulation results and graphical representations, it is revealed that the proposed class of ratio-cum-product estimators had the lowest mean squared error than Kumar et al. [12] estimator in simple random sampling. The suggested class of estimators can be used to estimate the population mean of a sensitive variable in surveys from many sectors such as social, finance, business, health, and education. La clase sugerida de estimadores se puede utilizar para estimar la media poblacional de una variable sensible en encuestas de muchos sectores, como el social, el financiero, el empresarial, el sanitario y el educativo.

KEYWORDS: Sensitive variable, Non-response, Measurement Error, Simple Random Sampling, Optional Randomized Response Technique (ORRT).
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RESUMEN

En este estudio se utiliza el estimador de razón acumulada producto para estimar la media poblacional de una variable sensible para abordar el problema de la falta de respuesta y el error de medición en un muestreo aleatorio simple. Para ello se utiizan modelos de ORRT. Se estudian las características de la clase de estimador propuesta hasta el primer orden de aproximación. Se compara el rendimiento relativo del estimador sugerido con las distintas clases de estimador propuesto y el estimador Kumar et al. [12]. Además, los hallazgos teóricos se muestran a través de un estudio de simulación basado en una población generada artificialmente y una población real. A partir de los resultados de la simulación y las representaciones gráficas, se revela que la clase propuesta de razón acumulada producto tuvo el error cuadrático medio más bajo que el estimador de Kumar et al. [12] en muestreo aleatorio simple

PALABRAS CLAVE: Variable sensible, No respuesta, Error de medición, Muestreo simple aleatorio, Técnica opcional de respuesta aleatoria (ORRT)

1. INTRODUCTION

One of the most essential tasks is to maintain privacy while dealing with a sensitive problem in the sample survey. A sensitive variable is one that contains sensitive information about a person or company. It is impossible in sensitive surveys to collect direct information of a study variable i.e., a respondent may be embarrassed sharing the information that the interviewer asked for private or other reasons, such as inquiries about corruption, criminal activity, abortion, drug addiction, and so on. Warner [17] presented a randomised response technique (RRT) to eliminate the bias of evasive answer in such situations. In survey sampling, direct true responses on the study variable may be hard to gather, particularly if the variable is sensitive. Various survey statisticians, such as Eichhron and Hayre [5], Gupta et al. [6], Saha [14], and Diana and Perri [4] have resembled the population mean of sensitive variables when the study variable is sensitive in nature.

Both non-response and measurement error can be particularly problematic for sensitive variables, as individuals may be more reluctant to respond to questions about sensitive topics or may provide inaccurate information due to social desirability bias. Non-response occurs when survey participants do not respond to certain questions or refuse to participate in the survey. Measurement error, on the other hand, occurs when there are inaccuracies or inconsistencies in the measurement of the variables of interest. This can be due to various factors, such as respondent misunderstanding, data entry errors, or interviewer bias. Therefore, it is important to address these issues when estimating sensitive variables in survey research. Azeem and Hanif [2] proposed a technique to estimate the population mean in the presence of measurement error and non-response, Kumar et al. [9] developed the estimation of population mean in the simultaneously presence of non-response and measurement error, Singh and Sharma [16] proposed a method of estimation in the presence of non-response and measurement errors simultaneously. Furthermore, various authors including Kumar et al. [13], Singh and Vishwakarma [15], Khalil et al. [8], Zhang et al. [18], Audu et al. [1] and Choudhary et al. [3] studied the problem of mean estimation in the presence of non-response and measurement error simultaneously.

This paper aims to develop a ratio-cum-product estimator in the presence of non-response and measurement error at the same time under Simple random sampling by utilized two auxiliary variable(s) using ORRT models. Section 2 describes a sampling strategy and in section 3, an existing Kumar et al. [12] estimator is discussed. The Proposed estimator is described in section 4. In section 5, we have studied the efficiency comparisons of all considered estimator(s). To validate the theoretical findings an empirical study is performed in section 6. Finally, an ultimate conclusion is given in section 7.

2. SAMPLING STRATEGY

Imagine $U = U_1, U_2, ..., U_N$ be a finite population of size N units and from U, a sample of size n is taken by using simple random sampling without replacement (SRSWOR). Let Y be a sensitive study variable which cannot be observed directly and X_1 and X_2 be two non sensitive auxiliary variable(s) with mean and variance i.e. ($\bar{Y}$, $\bar{X}_1$ and $\bar{X}_2$) and variances (S_y^2, $S_{x_1}^2$ and $S_{x_2}^2$), respectively. Suppose S_1 and S_2 be two scrambling variables with means ($\bar{S}_1$, $\bar{S}_2$) and variances ($S_{S_1}^2$, $S_{S_2}^2$), respectively. Let W be the probability that respondent find the question sensitive. If the respondents consider the question sensitive then he/she is asked to report a scrambled response and else a correct response is recorded.

To collect sensitive information from the respondents, the researchers find difficulty due to the happening of non-response. If the variable of interest is sensitive in nature then to tackle the problem of non-response, Hansen and Hurwitz [7] technique has been modified by Zhang et al. [18], Kumar and Kour [11] and Choudhary et al. [3]. In this technique, the respondent gives direct answer in first phase then ORRT model is used to get answer from a sub-group of non-respondents in the second phase.

Therefore, ORRT model in the second phase is given as

$$Z = \begin{cases} Y & \text{with probability 1-W} \\ S_1 Y + S_2 & \text{with probability W,} \end{cases}$$

with mean $E(Z) = E(Y)$ and variance $Var(Z) = S_y^2 + S_{S_2}^2 W + S_{S_1}^2 (S_y^2 + \bar{Y}^2) W$. The RRT model is $Z = (S_1 Y + S_2)J + Y(1 - J)$, where $J \sim$ Bernoulli(π) with $E(J) = \pi, Var(J) = \pi(1 - \pi)$ and $E(J^2) = Var(J) + E^2(J) = \pi$. And the expectation and variance of randomized mechanism is $E_R(Z) = (\bar{S}_1 \pi + 1 - \pi)Y + \bar{S}_2 \pi$ and $V_R(Z) = (Y^2 S_{S_1}^2 + S_{S_2}^2)\pi$.

Let us take a transformation of the randomized response be $\hat{y}_i^*$ whose expectation under the randomization mechanism is the true response y_i and is given as

$$\hat{y}_i^* = \frac{z_i - \bar{S}_2}{\bar{S}_1 W + 1 - W}$$

with $E_R(\hat{y_i}^*) = y_i$ and $V_R(\hat{y_i}^*) = \frac{V_R(z_i)}{(\bar{S}\pi + 1 - \pi)^2} = \frac{(y_i^2 S_{S_1}^2 + S_{S_2}^2)\pi}{(\bar{S}_1 \pi + 1 - \pi)^2} = \gamma_i$

.

From previous discussions, we assume that out of 'n' sample units, only n_1 units provide response on first call and remaining $n_2 = (n - n_1)$ units do not respond. Then a sub-sample of $n_s (= n_2/k(k > 1))$ units are taken from non-responding units n_2, respectively. Then, a modified version of Hansen and Hurwitz estimator suggested by Zhang et al. [18] and Kumar and Kour [11] is given by

$$\hat{\bar{y}}^* = w_1\bar{y}_1^* + w_2\hat{\bar{y}}_2^*$$

with mean $E(\hat{\bar{y}}^*) = \bar{Y}$ and variance $Var(\hat{\bar{y}}^*) = \lambda S_y^2 + \lambda^* S_{y(2)}^2 + \frac{W_2 k}{n}\left[\frac{\{(S_{y(2)}^2 + \bar{y}_{(2)}^2)S_{S_1}^2 + S_{S_2}^2\}\pi}{(\bar{S}_1\pi + 1 - \pi)^2}\right]$.

Similarly, one can write the estimator for X_1 and X_2 as

$$\bar{x}_1^* = w_1\bar{x}_1^* + w_2\bar{x}_{12}^*$$

and

$$\bar{x}_2^* = w_1\bar{x}_2^* + w_2\bar{x}_{22}^*$$

with $E(\bar{x}_1^*) = \bar{X}_1$, $E(\bar{x}_2^*) = \bar{X}_2$ and $Var(\bar{x}_1^*) = \lambda S_{x_1}^2 + \lambda^* S_{x_{1(2)}}^2$, $Var(\bar{x}_2^*) = \lambda S_{x_2}^2 + \lambda^* S_{x_{2(2)}}^2$

In addition, let $U_i = y_i - Y_i$, $V_i = x_{1i} - X_{1i}$ and $W_i = x_{2i} - X_{2i}$ be the measurement error for the study variable (Y) and auxiliary variables (X_1, X_2) in the population. Let $P_i = z_i - Z_i$ represent the relative measurement error associated with the sensitive variables (Z) in the face-to-face interview phase. These measurement errors are considered to be random and uncorrelated, with mean zero and variances S_u^2, S_v^2, S_w^2, $S_{v(2)}^2$, $S_{w(2)}^2$ and S_p^2, respectively.

In the context of non-response and measurement error simultaneously, the variances of $\hat{\bar{y}}$, $\hat{\bar{x}}_1$ and $\hat{\bar{x}}_2$ are given by

$$Var(\hat{\bar{y}}^{**}) = \lambda(S_y^2 + S_u^2) + \lambda^*(S_{y(2)}^2 + S_p^2) + \kappa = A + \kappa$$

$$Var(\hat{\bar{x}}_1^{**}) = \lambda(S_{x_1}^2 + S_v^2) + \lambda^*(S_{x_{1(2)}}^2 + S_{v(2)}^2) = B$$

and

$$Var(\hat{\bar{x}}_2^{**}) = \lambda(S_{x_2}^2 + S_w^2) + \lambda^*(S_{x_{2(2)}}^2 + S_{w(2)}^2) = C$$

where $A = \left[\lambda(S_y^2 + S_u^2) + \lambda^*(S_{y(2)}^2 + S_p^2)\right]$, $B = \left[\lambda(S_{x_1}^2 + S_v^2) + \lambda^*(S_{x_{1(2)}}^2 + S_{v(2)}^2)\right]$,
$C = \left[\lambda(S_{x_2}^2 + S_w^2) + \lambda^*(S_{x_{2(2)}}^2 + S_{w(2)}^2)\right]$ and $\kappa = \frac{W_2 k}{n}\left[\frac{\{(S_{y(2)}^2 + \bar{y}_{(2)}^2)S_{S_1}^2 + S_{S_2}^2\}\pi}{(\bar{S}_1\pi + 1 - \pi)^2}\right]$.

3. KUMAR ET AL. [12] ESTIMATOR

Kumar et al. [12] suggest a ratio-cum-product type estimators when the auxiliary variable(s) $\bar{X}_1$ and $\bar{X}_2$ are known under measurement error and non-response using ORRT model is given by

$$\hat{T}_{ssz}^{**} = \left[\alpha^*\hat{\bar{y}}^{**} + \beta_{yx_1}^{**}(\bar{X}_1 - \bar{x}_1^{**}) + \beta_{yx_2}^{**}(\bar{X}_2 - \bar{x}_2^{**})\right]\left(\frac{\bar{X}_1}{\bar{x}_1^{**}} + \frac{\bar{x}_1^{**}}{\bar{X}_1}\right)\left(\frac{\bar{X}_2}{\bar{x}_2^{**}} + \frac{\bar{x}_2^{**}}{\bar{X}_2}\right), \quad (3.1)$$

where
$\beta_{yx_1}^{**} = s_{yx_1}^{**}/s_{x_1}^{**2}$ is the estimate of the population regression coefficient $\beta_{yx_1} = S_{yx_1}^{**}/S_{x_1}^{**2}$;
$\beta_{yx_2}^{**} = s_{yx_2}^{**}/s_{x_2}^{**2}$ is the estimate of the population regression coefficient $\beta_{yx_2} = S_{yx_2}^{**}/\hat{S}_{x_2}^{**2}$ and α^* be a finite quantity.

The bias and mean squared error of $\hat{T}^{**}_{ssz}$ is given by

$$Bias(\hat{T}^{**}_{ssz}) = \bar{Y}(4\alpha^* - 1) + 4\beta_{yx_1}\left[\left(\lambda\frac{\alpha^*_{03}}{\alpha^*_{02}} + \lambda^*\frac{\alpha^*_{03(2)}}{\alpha^*_{02(2)}}\right) + \left(\lambda\frac{\alpha^*_{12}}{\alpha^*_{11}} + \lambda^*\frac{\alpha^*_{12(2)}}{\alpha^*_{11(2)}}\right)\right] - 4\beta_{yx_2}$$
$$\left[\left(\lambda\frac{\mu^*_{03}}{\mu^*_{02}} + \lambda^*\frac{\mu^*_{03(2)}}{\mu^*_{02(2)}}\right) + \left(\lambda\frac{\mu^*_{12}}{\mu^*_{11}} + \lambda^*\frac{\mu^*_{12(2)}}{\mu^*_{11(2)}}\right)\right] + 2\alpha^*(R_1B + R_2C) \quad (3.2)$$

and

$$MSE(\hat{T}^{**}_{ssz}) = \bar{Y}^2(4\alpha^* - 1)^2 + 16\alpha^{*2}A + 16\beta^2_{yx_1}B + 16\beta^2_{yx_2}C - 32\alpha^*\beta_{yx_1}D +$$
$$32\beta_{yx_1}\beta_{yx_2}E - 32\alpha^*\beta_{yx_2}F + 16\alpha^{*2}\kappa \quad (3.3)$$

which is optimum when

$$\hat{\alpha}^*_{opt.} = \frac{\frac{1}{4}\bar{Y}^2 + \beta_{yx_1}D + \beta_{yx_2}E}{\bar{Y}^2 + A + \kappa}$$

Putting the value of $\hat{\alpha}^{**}_{opt.}$ in (3), we get the minimum MSE of the proposed estimator as

$$min.MSE(\hat{T}^{**}_{ssz}) = \bar{Y}^2(4\hat{\alpha}^*_{opt.} - 1)^2 + 16\hat{\alpha}^{*2}_{opt.}A + 16\beta^2_{yx_1}B + 16\beta^2_{yx_2}C - 32\hat{\alpha}^*_{opt.}\beta_{yx_1}D$$
$$+32\beta_{yx_1}\beta_{yx_2}E - 32\hat{\alpha}^*_{opt.}\beta_{yx_2}F + 16\hat{\alpha}^{*2}_{opt.}\kappa \quad (3.4)$$

where $D = \left(\lambda\rho_{yx_1}S_yS_{x_1} + \lambda^*\rho_{yx_{1(2)}}S_{y(2)}S_{x_{1(2)}}\right)$, $E = \left[\lambda\rho_{x_1x_2}S_{x_1}S_{x_2} + \lambda^*\rho_{x_1x_{2(2)}}S_{x_{1(2)}}S_{x_{2(2)}}\right]$
and $F = \left(\lambda\rho_{yx_2}S_yS_{x_2} + \lambda^*\rho_{yx_{2(2)}}S_{y(2)}S_{x_{2(2)}}\right)$.

In the next section, we develop a ratio-product type exponential estimator and studied its properties. The proposed class of estimator will be useful for the situation of exponential type-data and for mean estimation of sensitive variable.

4. PROPOSED CLASS OF ESTIMATORS

When the regression line of y on x passes through the neighbourhood of the origin, in that case the ratio (product) estimator is more efficient and the efficiencies of these estimators are almost equal. Under certain efficiency conditions, exponential-type estimators are known to outperform the related existing estimators in terms of lesser mean square errors. Kumar et al. [12] proposed a ratio-product type exponential estimator for estimating the finite population mean of sensitive variable using ORRT models. Following Zhang et al. [18], Kumar and Kour [10] and Kumar et al. [12] we propose a class of ratio-product type estimators for estimating population mean $\bar{Y}$ of the sensitive study variable 'y' in presence of non-response and measurement error, as

$$\hat{T}^{**} = w_0\hat{\bar{y}}^{**}\left(\frac{\bar{X}_1}{\bar{x}^{**}_1}\right)^{\alpha}\left(\frac{\bar{x}^{**}_2}{\bar{X}^{**}_2}\right)^{\eta}\exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}^{**}_1)}{\bar{X}_1 + \bar{x}^{**}_1}\right\}\exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}^{**}_2)}{\bar{X}_2 + \bar{x}^{**}_2}\right\} +$$
$$w_1(\bar{x}^{**}_1 - \bar{X}_1) + w_2(\bar{x}^{**}_2 - \bar{X}_2) \quad (4.1)$$

where $(\alpha, \eta, \alpha_1, \alpha_2)$ are suitably chosen scalars generating different form of the estimators for suitable values of $(\alpha, \eta, \alpha_1, \alpha_2)$; and (w_0, w_1, w_2) are suitably chosen constants to be determined such that MSE of $\hat{T}^{**}$ is minimum.

To obtain the bias and MSE of $\hat{T}^{**}$ we write

$\hat{\bar{y}}^{**} = \bar{Y}(1+\hat{e}_0^{**})$, $\bar{x}_1^{**} = \bar{X}_1(1+e_1^{**})$, $\bar{x}_2^{**} = \bar{X}_2(1+e_2^{**})$ such that

$E(\hat{e}_0^{**}) = E(e_1^{*}) = E(e_2^{**}) = 0$ and

$E(\hat{e}_0^{**2}) = C_0^2 = \frac{1}{\bar{Y}^2}(A+\kappa)$; $E(e_1^{**2}) = C_1^2 = \frac{1}{\bar{X}_1^2}B$; $E(e_2^{**2}) = C_2^2 = \frac{1}{\bar{X}_2^2}C$;

$E(\hat{e}_0^{**}e_1^{**}) = C_{01} = \frac{1}{\bar{Y}\bar{X}_1}D$; $E(\hat{e}_0^{**}e_2^{**}) = C_{02} = \frac{1}{\bar{Y}\bar{X}_2}F$; $E(e_1^{**}e_2^{**}) = C_{12} = \frac{1}{\bar{X}_1\bar{X}_2}E$.

where A, K, B, C, D, E, F are same as defined in section 3.

Expressing $\hat{T}^{**}$ in terms of $\hat{e}_0^{**}$, e_1^{**} and e_2^{**}, we have

$$\hat{T}^{**} = \bar{Y}\left[w_0(1+\hat{e}_0^{**})(1+e_1^{**})^{-\alpha}\exp\left\{\frac{-\alpha_1}{2}e_1^{**}\left(1+\frac{1}{2}e_1^{*}\right)^{-1}\right\}\exp\left\{\frac{-\alpha_2}{2}e_2^{**}\left(1+\frac{1}{2}e_2^{*}\right)^{-1}\right\}+ w_1\frac{1}{R_1}e_1^{**}+w_2\frac{1}{R_2}e_2^{**}\right] \quad (4.2)$$

where $R_1 = \frac{\bar{Y}}{\bar{X}_1}$ and $R_2 = \frac{\bar{Y}}{\bar{X}_2}$.

We assume that $|e_1^{**}| << 1$, $|e_2^{**}| << 1$, $\left|\frac{1}{2}e_1^{**}\right| << 1$ and $\left|\frac{1}{2}e_2^{**}\right| << 1$ so that $(1+e_1^{**})^{-\alpha}$, $(1+e_2^{**})^{\eta}$, $\left(1+\frac{1}{2}e_1^{**}\right)^{-1}$ and $\left(1+\frac{1}{2}e_2^{**}\right)^{-1}$ are expandable. Now expanding the right hand side of (4.2), multiplying out and neglecting terms of $e^{**'}s$ having power greater than two we have

$$(\hat{T}^{**}-\bar{Y}) \cong \bar{Y}\left[w_0\left\{1+\hat{e}_0^{**}-\theta_1(e_1^{**}+\hat{e}_0^{**}e_1^{**})+\theta_2(e_2^{**}+\hat{e}_0^{**}e_2^{**})+\theta_1\theta_2 e_1^{**}e_2^{**}+\theta_3 e_1^{**2}+\theta_4 e_2^{**2}\right\}+ w_1\frac{1}{R_1}e_1^{**}+w_2\frac{1}{R_2}e_2^{**}-1\right] \quad (4.3)$$

where $\theta_1 = (\alpha+\frac{1}{2}\alpha_1)$, $\theta_2 = (\eta+\frac{1}{2}\alpha_2)$; $\theta_3 = \left[\frac{\alpha(\alpha+1)}{2}+\frac{\alpha\alpha_1}{2}+\frac{\alpha_1(\alpha_1+2)}{8}\right]$ and

$\theta_4 = \left[\frac{\eta(\eta-1)}{2}+\frac{\eta\alpha_2}{2}+\frac{\alpha_2(\alpha_2-2)}{8}\right]$.

Taking expectation of both sides of (4.3) we get the bias of $\hat{T}^{**}$ to the first degree of approximation as

$$B(\hat{T}^{**}) = \bar{Y}\left[w_0\left\{1-\theta_1 C_{01}+\theta_2 C_{02}-\theta_1\theta_2 C_{12}+\theta_3 C_1^2+\theta_4 C_2^2\right\}-1\right] \quad (4.4)$$

Squaring both sides of (4.3) and neglecting terms $e^{**'}s$ having power greater than two, we have

$$(\hat{T}^{**}-\bar{Y})^2=\bar{Y}^2\Big[1+w_0^2\{1+2\hat{e}_0^{**}-2\theta_1(e_1^{**}+2\theta_2(e_2^{**}-4\theta_1\hat{e}_0^{**}e_1^{**})+4\theta_2\hat{e}_0^{**}e_2^{**})-4\theta_1\theta_2e_1^{**}e_2^{**}+ (\theta_1^2+2\theta_3)e_1^{**2}+(\theta_2^2+2\theta_4)e_2^{**2}\}+w_1\left(\frac{1}{R_1^2}\right)e_1^{**2}+w_2\left(\frac{1}{R_2^2}\right)e_2^{**2}-2w_0\{1+\hat{e}_0^{**}- \theta_1(e_1^{**}+\hat{e}_0^{**}e_1^{**})+\theta_2(e_2^{**}+\hat{e}_0^{**}e_2^{**})-\theta_1\theta_2e_1^{**}e_2^{**}+\theta_3e_1^{**2}+\theta_4e_2^{**2}\}-2w_1\left(\frac{1}{R_1}\right)e_1^{**}- 2w_2\left(\frac{1}{R_2}\right)e_2^{**}+2w_1w_2\left(\frac{1}{R_1R_2}\right)e_1^{**}e_2^{**}+2w_0w_1\left(\frac{1}{R_1}\right)\{e_1^{**}+\hat{e}_0^{**}e_1^{**}-\theta_1e_1^{**2}+ \theta_2e_1^{**}e_2^{**}\}+2w_0w_2(\frac{1}{R_2})\Big\{e_2^{**}+\hat{e}_0^{**}e_2^{**}+\theta_2e_2^{**2}+\theta_1e_1^{**}e_2^{**}\Big\}\Big] \quad (4.5)$$

Taking expectation of both sides of (4.5), we get the MSE of $\hat{T}^{**}$ to the first degree of approximation as

$$MSE(\hat{T}^{**})=\bar{Y}^2\left[1+w_0^2H_0+w_1^2H_1+w_2^2H_2+2w_0w_1H_3+2w_0w_2H_4+2w_1w_2H_5-3w_0H_6\right] \quad (4.6)$$

where $H_0=\left[1+C_0^2-4\theta_1C_{01}+4\theta_2C_{02}-4\theta_1\theta_2C_{12}+(\theta_1^2+2\theta_3)C_1^2+(\theta_2^2+2\theta_4)C_2^2\right]$, $H_1=\frac{C_1^2}{R_1^2}$, $H_2=\frac{C_2^2}{R_2^2}$, $H_3=\frac{1}{R_1}\Big[C_{01}-\theta_1C_1^2+\theta_2C_{12}\Big]$, $H_4=\frac{1}{R_2}\Big[C_{02}-\theta_1C_{12}+\theta_2C_2^2\Big]$, $H_5=\frac{C_{12}}{R_1R_2}$ and $H_6=\Big[1-\theta_1C_{01}+\theta_2C_{02}+\theta_1\theta_2C_{12}+\theta_3C_1^2+\theta_4C_2^2\Big]$.

Setting $\frac{MSE(\hat{T}^{**})}{\partial w_i}=0$, $i=0,1,2$; we have

$$\begin{bmatrix}H_0 & H_3 & H_4\\ H_3 & H_1 & H_5\\ H_4 & H_5 & H_2\end{bmatrix}\begin{bmatrix}w_0\\ w_1\\ w_2\end{bmatrix}=\begin{bmatrix}H_6\\ 0\\ 0\end{bmatrix} \quad (4.7)$$

Solving (4.7) for (w_0,w_1,w_2) we get the optimum of (w_0,w_1,w_2) as

$$w_{0(opt.)}=\frac{\Delta_0}{\Delta},w_{1(opt.)}=\frac{\Delta_1}{\Delta},w_{2(opt.)}=\frac{\Delta_2}{\Delta}. \quad (4.8)$$

where $\Delta=\begin{vmatrix}H_0 & H_3 & H_4\\ H_3 & H_1 & H_5\\ H_4 & H_5 & H_2\end{vmatrix}=H_0\begin{vmatrix}H_1 & H_5\\ H_5 & H_2\end{vmatrix}-H_3\begin{vmatrix}H_3 & H_5\\ H_4 & H_2\end{vmatrix}+H_4\begin{vmatrix}H_3 & H_1\\ H_4 & H_5\end{vmatrix}$

$$=H_0(H_1H_2-H_5^2)-H_3(H_2H_3-H_4H_5)+H_4(H_3H_5-H_1H_4);$$

$$\Delta_0=\begin{vmatrix}H_6 & H_3 & H_4\\ 0 & H_1 & H_5\\ 0 & H_5 & H_2\end{vmatrix}=H_6(H_1H_2-H_5^2);$$

$$\Delta_1=\begin{vmatrix}H_0 & H_6 & H_4\\ H_3 & 0 & H_5\\ H_4 & 0 & H_2\end{vmatrix}=H_6(H_2H_3-H_4H_5) \text{ and}$$

$$\Delta_2 = \begin{vmatrix} H_0 & H_3 & H_6 \\ H_3 & H_1 & 0 \\ H_4 & H_5 & 0 \end{vmatrix} = H_6(H_3H_5 - H_1H_4).$$

Thus the resulting minimum MSE of $\hat{T}^{**}$ is given by

$$min.MSE(\hat{T}^{**}) = \bar{Y}^2\left[1 - \frac{H_6\Delta_0}{\Delta}\right] \tag{4.9}$$

Now we arrived at the following theorem

Theorem 4.1. *The $MSE(\hat{T}^{**})$ is greater than or equal to the minimum MSE of $\hat{T}^{**}$ i.e.*

$$\begin{aligned} MSE(\hat{T}^{**}) &\leq min.MSE(\hat{T}^{**}) \\ &= \bar{Y}^2\left[1 - \frac{H_6\Delta_0}{\Delta}\right] \end{aligned}$$

with equality holds if

$w_i = w_{i(opt.)}$; $i = 0, 1, 2$; where $w_{i(opt.)}$; $i = 0, 1, 2$; is given in (12).

Some particular members of the class $\hat{T}^{**}$ are given below

- For (α_1, α_2)=$(0, 0)$; $\hat{T}^{**}$ reduces to:

$$\hat{T}^{**}_{(1)} = w_0\hat{\bar{y}}^{**}\left(\frac{\bar{X}_1}{\bar{x}_1^{**}}\right)^{\alpha}\left(\frac{\bar{x}_2^{**}}{\bar{X}_2^{**}}\right)^{\eta} + w_1(\bar{x}_1^{**} - \bar{X}_1) + w_2(\bar{x}_2^{**} - \bar{X}_2)$$

The Bias and MSE of $\hat{T}^{**}_{(1)}$ is given by

$$\begin{aligned} B(\hat{T}^{**}_{(1)}) &= \bar{Y}\left[w_0\left\{1 - \alpha C_{01} + \eta C_{02} - \alpha\eta C_{12} + \frac{\alpha(\alpha+1)}{2}C_1^2 + \frac{\eta(\eta-1)}{2}C_2^2\right\} - 1\right] \\ &= \bar{Y}\left[w_0 H_{6(1)} - 1\right], \end{aligned} \tag{4.10}$$

$$\begin{aligned} MSE(\hat{T}^{**}_{(1)}) = \bar{Y}^2\Big[&1 + w_0^2 H_{0(1)} + w_1^2 H_1 + w_2^2 H_2 + 2w_0w_1H_{3(1)} + 2w_0w_1H_{4(1)} + \\ &2w_1w_2H_5 - 2w_0H_{6(1)}\Big], \end{aligned} \tag{4.11}$$

where $H_{0(1)} = \left[1 + C_0^2 - 4\alpha C_{01} + 4\eta C_{02} - 4\alpha\eta C_{12} + \alpha(2\alpha+1)C_1^2 + \eta(2\eta-1)C_2^2\right]$, $H_{3(1)} = \frac{1}{R_1}\left[C_{01} - \alpha C_1^2 + \eta C_{12}\right]$, $H_{4(1)} = \frac{1}{R_2}\left[C_{02} - \alpha C_{12} + \eta C_2^2\right]$ and $H_{6(1)} = \left[1 - \alpha C_{01} + \eta C_{02} - \alpha\eta C_{12} + \frac{\alpha(\alpha+1)}{2}C_1^2 + \frac{\eta(\eta-1)}{2}C_2^2\right]$.

The optimum values of (w_0, w_1, w_2) and the minimum MSE of $\hat{T}^{**}_{(1)}$ are given by

$$w_{0(opt.)1} = \frac{\Delta_{0(1)}}{\Delta_{(1)}}, w_{1(opt.)1} = \frac{\Delta_{1(1)}}{\Delta_{(1)}}, w_{2(opt.)1} = \frac{\Delta_{2(1)}}{\Delta_{(1)}}.$$

where $\Delta_{(1)} = \big[H_{0(1)}(H_1H_2 - H_5^2) - H_{3(1)}(H_2H_{3(1)} - H_{4(1)}H_5) + H_{4(1)}(H_{3(1)}H_5 - H_1H_{4(1)})\big]$, $\Delta_{0(1)} = H_{6(1)}(H_1H_2 - H_5^2)$, $\Delta_{1(1)} = H_{6(1)}(H_2H_{3(1)} - H_{4(1)}H_5)$, $\Delta_{2(1)} = H_{6(1)}(H_{3(1)}H_5 - H_1H_{4(1)})$.

Thus the resulting minimum MSE of $\hat{T}^{**}_{(1)}$ is given by

$$min.MSE(\hat{T}^{**}_{(1)}) = \bar{Y}^2\left[1 - \frac{H_{6(1)}\Delta_{0(1)}}{\Delta_{(1)}}\right]. \tag{4.12}$$

- For different choices of (α, η) a large number of estimators can be generated from the class of estimators $\hat{T}^{**}_{(1)}$.

- For (α, η)=$(0, 0)$; $\hat{T}^{**}$ reduces to:

$$\hat{T}^{**}_{(2)} = w_0\hat{\bar{y}}^{**}\exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}_1^{**})}{\bar{X}_1 + \bar{x}_1^{**}}\right\}\exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}_2^{**})}{\bar{X}_2 + \bar{x}_2^{**}}\right\} + w_1(\bar{x}_1^{**} - \bar{X}_1) + w_2(\bar{x}_2^{**} - \bar{X}_2)$$

Putting $(\alpha, \eta) = (0, 0)$ in (4.4) and (4.6) we get the bias and MSE of $\hat{T}^{**}_{(2)}$ to the first degree of approximation, respectively as

$$B(\hat{T}^{**}_{(2)}) = \bar{Y}\big[w_0H_{6(2)} - 1\big], \tag{4.13}$$

$$MSE(\hat{T}^{**}_{(2)}) = \bar{Y}^2\Big[1 + w_0^2H_{0(2)} + w_1^2H_1 + w_2^2H_2 + 2w_0w_1H_{3(2)} + 2w_0w_1H_{4(2)} + 2w_1w_2H_5 - 2w_0H_{6(2)}\Big], \tag{4.14}$$

where $H_{0(2)} = \left[1 + C_0^2 - 2\alpha_1C_{01} + 2\alpha_2C_{02} - \alpha_1\alpha_2C_{12} + \frac{\alpha_1(\alpha_1+1)}{2}C_1^2 + \frac{\alpha_2(\alpha_2-1)}{2}C_2^2\right]$, $H_{3(2)} = \frac{1}{R_1}\left[C_{01} - \frac{\alpha_1}{2}C_1^2 + \frac{\alpha_2}{2}C_{12}\right]$, $H_{4(2)} = \frac{1}{R_2}\left[C_{02} - \alpha_1C_{12} + \alpha_2C_2^2\right]$ and $H_{6(2)} = \left[1 - \frac{\alpha_1}{2}C_{01} + \frac{\alpha_2}{2}C_{02} - \frac{\alpha_1\alpha_2}{4}C_{12} + \frac{\alpha_1(\alpha_1+2)}{8}C_1^2 + \frac{\alpha_2(\alpha_2-2)}{8}C_2^2\right]$.

The optimum values of (w_0, w_1, w_2) are given by

$$w_{0(opt.)2} = \frac{\Delta_{0(2)}}{\Delta_{(2)}}, w_{1(opt.)2} = \frac{\Delta_{1(2)}}{\Delta_{(2)}}, w_{2(opt.)2} = \frac{\Delta_{2(2)}}{\Delta_{(2)}},$$

where $\Delta_{(2)} = \big[H_{0(2)}(H_1H_2 - H_5^2) - H_{3(2)}(H_2H_{3(2)} - H_{4(2)}H_5) + H_{4(2)}(H_{3(2)}H_5 - H_1H_{4(2)})\big]$, $\Delta_{0(2)} = H_{6(2)}(H_1H_2 - H_5^2)$, $\Delta_{1(2)} = H_{6(2)}(H_2H_{3(2)} - H_{4(2)}H_5)$, $\Delta_{2(2)} = H_{6(2)}(H_{3(2)}H_5 - H_1H_{4(2)})$.

Thus the resulting minimum MSE of $\hat{T}^{**}_{(2)}$ is given by

$$min.MSE(\hat{T}^{**}_{(2)}) = \bar{Y}^2\left[1 - \frac{H_{6(2)}\Delta_{0(2)}}{\Delta_{(2)}}\right]. \tag{4.15}$$

- For (η, α_1)=$(0, 0)$; $\hat{T}^{**}$ boils down to:

$$\hat{T}^{**}_{(3)} = w_0\hat{\bar{y}}^{**}\left(\frac{\bar{X}_1}{\bar{x}_1^{**}}\right)^{\alpha}\exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}_2^{**})}{\bar{X}_2 + \bar{x}_2^{**}}\right\} + w_1(\bar{x}_1^{**} - \bar{X}_1) + w_2(\bar{x}_2^{**} - \bar{X}_2)$$

Putting (η, α_1) in (4.4) and (4.6) we get the bias and MSE of $\hat{T}^{**}_{(3)}$ to the first degree of approximation, respectively as

$$B(\hat{T}^{**}_{(3)}) = \bar{Y}\big[w_0 H_{6(3)} - 1\big], \tag{4.16}$$

$$MSE(\hat{T}^{**}_{(3)}) = \bar{Y}^2\Bigg[1 + w_0^2 H_{0(3)} + w_1^2 H_1 + w_2^2 H_2 + 2w_0 w_1 H_{3(3)} + 2w_0 w_1 H_{4(3)} + 2w_1 w_2 H_{5(3)} - 2w_0 H_{6(3)}\Bigg], \tag{4.17}$$

Here $(\eta, \alpha_1) = (0, 0)$, $\theta_1 = \alpha$, $\theta_2 = \frac{1}{2}\alpha_2$, $(\theta_2^2 + 2\theta_4) = \frac{\alpha_2(\alpha_2-1)}{2}$ $\theta_3 = \frac{\alpha(\alpha+1)}{2}$, $\theta_4 = \frac{\alpha_2(\alpha_2-2)}{8}$, $(\theta_1^2 + 2\theta_3) = \alpha(2\alpha + 1)$.

$H_{0(3)} = \Bigg[1 + C_0^2 - 4\alpha C_{01} + 2\alpha_2 C_{02} - 2\alpha\alpha_2 C_{12} + \alpha_1(2\alpha_1 + 1)C_1^2 + \frac{\alpha_2(\alpha_2-1)}{2}C_2^2\Bigg]$, $H_{3(3)} = \frac{1}{R_1}\Bigg[C_{01} - \frac{\alpha}{2}C_1^2 + \frac{\alpha_2}{2}C_{12}\Bigg]$, $H_{4(3)} = \frac{1}{R_2}\Bigg[C_{02} - \alpha C_{12} + \alpha_2 C_2^2\Bigg]$ and $H_{6(3)} = \Bigg[1 - \frac{\alpha}{2}C_{01} + \frac{\alpha_2}{2}C_{02} - \frac{\alpha\alpha_2}{2}C_{12} + \frac{\alpha(\alpha+1)}{2}C_1^2 + \frac{\alpha_2(\alpha_2-2)}{8}C_2^2\Bigg]$.

The optimum values of (w_0, w_1, w_2) are respectively given by

$$w_{0(opt.)3} = \frac{\Delta_{0(2)}}{\Delta_{(2)}}, w_{1(opt.)3} = \frac{\Delta_{1(2)}}{\Delta_{(2)}}, w_{2(opt.)3} = \frac{\Delta_{2(2)}}{\Delta_{(2)}},$$

where $\Delta_{(3)} = \big[H_{0(3)}(H_1H_2 - H_5^2) - H_{3(3)}(H_2H_{3(3)} - H_{4(3)}H_5) + H_{4(3)}(H_{3(3)}H_5 - H_1H_{4(3)})\big]$, $\Delta_{0(3)} = H_{6(3)}(H_1H_2 - H_5^2)$, $\Delta_{1(3)} = H_{6(3)}(H_2H_{3(3)} - H_{4(3)}H_5)$, $\Delta_{2(3)} = H_{6(3)}(H_{3(3)}H_5 - H_1H_{4(3)})$.

The minimum MSE of $\hat{T}^{**}_{(3)}$ is given by

$$min.MSE(\hat{T}^{**}_{(3)}) = \bar{Y}^2\left[1 - \frac{H_{6(3)}\Delta_{0(3)}}{\Delta_{(3)}}\right]. \tag{4.18}$$

- For (α, α_2)=(0,0); $\hat{T}^{**}$ reduces to:

$$\hat{T}^{**}_{(4)} = w_0\hat{\bar{y}}^{**}\left(\frac{\bar{X}_2}{\bar{x}_2^{**}}\right)^{\alpha} \exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}_1^{**})}{\bar{X}_1 + \bar{x}_1^{**}}\right\} + w_1(\bar{x}_1^{**} - \bar{X}_1) + w_2(\bar{x}_2^{**} - \bar{X}_2)$$

Inserting $(\alpha, \alpha_2) = (0, 0)$ in (4.4) and (4.6) we get the bias and MSE of the class of estimators

$$B(\hat{T}^{**}_{(4)}) = \bar{Y}\big[w_0 H_{6(4)} - 1\big], \tag{4.19}$$

$$MSE(\hat{T}^{**}_{(4)}) = \bar{Y}^2\Bigg[1 + w_0^2 H_{0(4)} + w_1^2 H_1 + w_2^2 H_2 + 2w_0 w_1 H_{3(4)} + 2w_0 w_1 H_{4(4)} + 2w_1 w_2 H_5 - 2w_0 H_{6(4)}\Bigg], \tag{4.20}$$

where $H_{0(4)} = \Bigg[1 + C_0^2 - 2\alpha_1 C_{01} + 4\eta C_{02} - 2\alpha_1\eta C_{12} + \frac{\alpha_1(\alpha_1+1)}{2}C_1^2 + \eta(2\eta-1)C_2^2\Bigg]$, $H_{3(4)} = \frac{1}{R_1}\Bigg[C_{01} - \frac{\alpha_1}{2}C_1^2 + \frac{\eta}{2}C_{12}\Bigg]$, $H_{4(4)} = \frac{1}{R_2}\Bigg[C_{02} - \alpha_1 C_{12} + \eta C_2^2\Bigg]$ and $H_{6(4)} = \Bigg[1 - \alpha_1 C_{01} + \eta C_{02} - \frac{\alpha_1\eta}{2}C_{12} + \frac{\alpha_1(\alpha_1+2)}{8}C_1^2 + \eta(\eta-1)C_2^2\Bigg]$.

The optimum values of (w_0, w_1, w_2) are given by

$$w_{0(opt.)4} = \frac{\Delta_{0(4)}}{\Delta_{(4)}}, w_{1(opt.)4} = \frac{\Delta_{1(4)}}{\Delta_{(4)}}, w_{2(opt.)4} = \frac{\Delta_{2(4)}}{\Delta_{(4)}},$$

where $\Delta_{(4)} = \big[H_{0(4)}(H_1H_2 - H_5^2) - H_{3(4)}(H_2H_{3(4)} - H_{4(4)}H_5) + H_{4(4)}(H_{3(4)}H_5 - H_1H_{4(4)})\big]$, $\Delta_{0(4)} = H_{6(4)}(H_1H_2 - H_5^2)$, $\Delta_{1(4)} = H_{6(4)}(H_2H_{3(4)} - H_{4(4)}H_5)$, $\Delta_{2(4)} = H_{6(4)}(H_{3(4)}H_5 - H_1H_{4(4)})$.

The minimum MSE of $\hat{T}^{**}_{(4)}$ is given by

$$min.MSE(\hat{T}^{**}_{(4)}) = \bar{Y}^2\left[1 - \frac{H_{6(4)}\Delta_{0(4)}}{\Delta_{(4)}}\right]. \tag{4.21}$$

- For $(\alpha, \eta, \alpha_1, \alpha_2)$=$(0, 0, 0, 0)$; $\hat{T}^{**}$ turns out to be:

$$\hat{T}^{**}_{(5)} = w_0\hat{\bar{y}}^{**} + w_1(\bar{x}^{**}_1 - \bar{X}_1) + w_2(\bar{x}^{**}_2 - \bar{X}_2)$$

The bias and MSE of the class of estimators $\hat{T}^{**}_{(5)}$ are given by

$$B(\hat{T}^{**}_{(5)}) = \bar{Y}\big[w_0 - 1\big], \tag{4.22}$$

$$MSE(\hat{T}^{**}_{(5)}) = \bar{Y}^2\Big[1 + w_0^2H_{0(5)} + w_1^2H_1 + w_2^2H_2 + 2w_0w_1H_{3(5)} + 2w_0w_2H_{4(5)} + 2w_1w_2H_5 - 2w_0H_{6(5)}\Big], \tag{4.23}$$

where $H_{0(5)} = \Big[1 + C_0^2\Big]$, $H_{3(5)} = C_{01} - \frac{\alpha_1}{2}C_1^2$, $H_{4(5)} = \frac{C_{02}}{R_2^2}$ and $H_{6(6)} = 1$.

The optimum values of (w_0, w_1, w_2) that minimizes the MSE of $\hat{T}^{**}_{(5)}$ are given by

$$w_{0(opt.)5} = \frac{\Delta_{0(5)}}{\Delta_{(5)}}, w_{1(opt.)5} = \frac{\Delta_{1(5)}}{\Delta_{(5)}}, w_{2(opt.)5} = \frac{\Delta_{2(5)}}{\Delta_{(5)}},$$

where $\Delta_{(5)} = \big[H_{0(5)}(H_1H_2 - H_5^2) - H_{3(5)}(H_2H_{3(5)} - H_{4(5)}H_5) + H_{4(5)}(H_{3(5)}H_5 - H_1H_{4(5)})\big]$, $\Delta_{0(5)} = H_{6(5)}(H_1H_2 - H_5^2)$, $\Delta_{1(5)} = H_{6(5)}(H_2H_{3(5)} - H_{4(5)}H_5)$, $\Delta_{2(5)} = H_{6(5)}(H_{3(5)}H_5 - H_1H_{4(5)})$.

The minimum MSE of $\hat{T}^{**}_{(5)}$ is given by

$$min.MSE(\hat{T}^{**}_{(5)}) = \bar{Y}^2\left[1 - \frac{H_{6(5)}\Delta_{0(5)}}{\Delta_{(5)}}\right]. \tag{4.24}$$

- For $(w_0, \alpha, \eta, \alpha_1, \alpha_2)$=$(1, 0, 0, 0, 0)$; $\hat{T}^{**}$ reduces to:

$$\hat{T}^{**}_{(6)} = \hat{\bar{y}}^{**} + w_1(\bar{x}^{**}_1 - \bar{X}_1) + w_2(\bar{x}^{**}_2 - \bar{X}_2)$$

The bias and MSE of the class of estimator $\hat{T}^{**}_{(6)}$ is given by

$$B(\hat{T}^{**}_{(6)}) = 0, \tag{4.25}$$

$$MSE(\hat{T}^{**}_{(6)}) = \bar{Y}^2\Bigg[1 + H_{0(6)} - 2H_{6(6)} + w_1^2 H_1 + +w_2^2 H_2 + 2w_1 H_{3(6)} + 2w_2 H_{4(6)} + 2w_1 w_2 H_5\Bigg],$$

$$MSE(\hat{T}^{**}_{(6)}) = \Bigg[(A+K) + w_1^2 B + w_2^2 C + 2w_1 w_2 E + 2w_1 D + 2w_2 F\Bigg] \tag{4.26}$$

where $H_{0(6)} = \left[1 + C_0^2\right]$, $H_{3(6)} = C_{01} - \frac{\alpha_1 C_1^2}{2}$, $H_{4(6)} = \frac{C_{02}}{R_2^2}$ and $H_{6(6)} = 0$.

The $MSE(T_{(6)})$ is minimized for

$$w_{1(opt.)6} = \frac{(EF - CD)}{(BC - E^2)}, w_{2(opt.)6} = \frac{(DE - BF)}{(BC - E^2)},$$

Thus, the minimum MSE of $\hat{T}^{**}_{(6)}$ is given by

$$min.MSE(\hat{T}^{**}_{(6)}) = \Bigg[(A+K) - \frac{(CD^2 - 2DEF + BF^2)}{(BC - E^2)}\Bigg]. \tag{4.27}$$

- For $(w_0, w_2, \alpha, \eta, \alpha_1, \alpha_2)$=$(1,0,0,0,0,0)$; $\hat{T}^{**}$ reduces to:

$$\hat{T}^{**}_{(7)} = \hat{\bar{y}}^{**} + w_1(\bar{x}_1^{**} - \bar{X}_1)$$

The bias and MSE of the class of $\hat{T}^{**}_{(7)}$ estimator for population mean $\bar{Y}$ as

$$B(\hat{T}^{**}_{(7)}) = 0, \tag{4.28}$$

$$MSE(\hat{T}^{**}_{(7)}) = \Bigg[(A+K) + w_1^2 B + 2w_1 D\Bigg], \tag{4.29}$$

which is minimum when for

$$w_{1(opt.)7} = -\frac{D}{B},$$

Thus, the minimum MSE of $\hat{T}^{**}_{(7)}$ is given by

$$min.MSE(\hat{T}^{**}_{(7)}) = \Bigg[(A+K) - \frac{D}{B}\Bigg]. \tag{4.30}$$

- For $(w_0, w_1, \alpha, \eta, \alpha_1, \alpha_2)$=$(1,0,0,0,0,0)$; $\hat{T}^{**}$ reduces to:

$$\hat{T}^{**}_{(8)} = \hat{\bar{y}}^{**} + w_2(\bar{x}_2^{**} - \bar{X}_2)$$

The bias and MSE of $\hat{T}^{**}_{(8)}$ are respectively given by

$$B(\hat{T}^{**}_{(8)}) = 0, \tag{4.31}$$

$$MSE(\hat{T}^{**}_{(8)}) = \Bigg[(A+K) + w_2^2 C + 2w_2 F\Bigg], \tag{4.32}$$

The MSE($\hat{T}^{**}_{(8)}$) is minimum when for

$$w_{2(opt.)8} = -\frac{F}{C},$$

Thus, the minimum MSE of $\hat{T}^{**}_{(8)}$ is given by

$$min.MSE(\hat{T}^{**}_{(8)}) = \left[(A+K) - \frac{F^2}{C}\right]. \tag{4.33}$$

- For $(w_2, \alpha, \eta, \alpha_1, \alpha_2)$=$(0,0,0,0,0,0)$; $\hat{T}^{**}$ boils down to:

$$\hat{T}^{**}_{(9)} = w_0\hat{\bar{y}}^{**} + w_1(\bar{x}^{**}_1 - \bar{X}_1)$$

The bias and MSE of $\hat{T}^{**}_{(9)}$ are respectively given by

$$B(\hat{T}^{**}_{(9)}) = (w_0 - 1)\bar{Y}, \tag{4.34}$$

$$MSE(\hat{T}^{**}_{(9)}) = \bar{Y}^2\left[1 + w_0^2 H_{0(9)} + w_1^2 H_1 + 2w_0w_1H_3(9) - 2w_0H_{6(9)}\right] \tag{4.35}$$

where $H_{0(9)} = \left[1 + C_0^2\right]$, $H_{3(9)} = \frac{D}{R_1^2\bar{X}_1^2}$ and $H_{6(6)} = 1$.

The MSE($\hat{T}^{**}_{(9)}$) is minimized for

$$w_{0(opt.)9} = \frac{H_1H_{6(9)}}{(H_{0(9)}H_1 - H^2_{3(9)})}, w_{1(opt.)9} = \frac{H_{3(9)}H_{6(9)}}{(H_{0(9)}H_1 - H^2_{3(9)})},$$

Thus, the minimum MSE of $\hat{T}^{**}_{(9)}$ is given by

$$min.MSE(\hat{T}^{**}_{(9)}) = \bar{Y}^2\left[1 - \frac{H^2_{6(9)}H_1}{(H_{0(9)}H_1 - H^2_{3(9)})}\right] = \frac{\bar{Y}^2\left[(A+K)B - D^2\right]}{B\bar{Y}^2 + \{(A+K)B - D^2\}}. \tag{4.36}$$

- For $(w_1, \alpha, \eta, \alpha_1, \alpha_2)$=$(0,0,0,0,0,0)$; $\hat{T}^{**}$ reduces to:

$$\hat{T}^{**}_{(10)} = w_0\hat{\bar{y}}^{**} + w_2(\bar{x}^{**}_2 - \bar{X}_2)$$

The bias and MSE of $\hat{T}^{**}_{(10)}$ are respectively given by

$$B(\hat{T}^{**}_{(10)}) = (w_0 - 1)\bar{Y}, \tag{4.37}$$

$$MSE(\hat{T}^{**}_{(10)}) = \bar{Y}^2\left[1 + w_0^2 H_{0(10)} + w_1^2 H_{1(1)} + 2w_0w_1H_3(10) - 2w_0H_{6(10)}\right], \tag{4.38}$$

The MSE($\hat{T}^{**}_{(10)}$) is minimized for

$$w_{0(opt.)10} = \frac{H_{6(10)}H_{2(1)}}{(H_{0(10)}H_{2(1)} - H^2_{4(10)})}, w_{1(opt.)10} = \frac{H_{4(10)}H_{6(10)}}{(H_{0(10)}H_{2(1)} - H^2_{4(10)})},$$

Thus, the minimum MSE of $\hat{T}^{**}_{(10)}$ is given by

$$min.MSE(\hat{T}^{**}_{(10)}) = \bar{Y}^2\left[1 - \frac{H^2_{6(10)}H_2}{(H_{0(10)}H_2 - H^2_{4(10)})}\right]. \tag{4.39}$$

Many more estimators can be identified from the class of estimators $\hat{T}^{**}$.

4.1. Members of the Proposed Estimator '$\hat{T}^{**}$'

Putting $w_1 = 0, w_2 = 0$ in $\hat{T}^{**}$, we get the following class of estimator for population mean $\bar{Y}$ as

$$\hat{t}^{**} = w_0 \hat{\bar{y}}^{**} \left(\frac{\bar{X}_1}{\bar{x}_1^{**}}\right)^{\alpha} \left(\frac{\bar{x}_2^{**}}{\bar{X}_2^{**}}\right)^{\eta} \exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}_1^{**})}{\bar{X}_1 + \bar{x}_1^{**}}\right\} \exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}_2^{**})}{\bar{X}_2 + \bar{x}_2^{**}}\right\} \tag{4.40}$$

Putting $(w_1, w_2) = (0, 0)$ in (4.4) and (4.6) we get the bias and MSE of $\hat{t}^{**}$ to the first degree of approximation as

$$B(\hat{t}^{**}) = \bar{Y}\left[w_0 H_6 - 1\right], \tag{4.41}$$

$$MSE(\hat{t}^{**}) = \bar{Y}\left[1 + w_0^2 H_0 - 2w_0 H_6\right], \tag{4.42}$$

The MSE is minimum when

$$w_0(opt.) = \frac{H_6}{H_0}, \tag{4.43}$$

Thus the minimum MSE of $\hat{t}^{**}$ is given by

$$min.MSE(\hat{t}^{**}) = \bar{Y}^2\left(1 - \frac{H_6^2}{H_0}\right). \tag{4.44}$$

- For different values of $(\alpha, \eta, \alpha_1, \alpha_2)$ a large number of estimators for population mean can be generated from '$\hat{t}^{**}$'.

1. Bias and MSE of the Class of Estimators $\hat{\mathbf{t}}^{**}_{(\mathbf{1})}$:

Inserting $(\alpha_1, \alpha_2) = (0, 0)$ in $\hat{t}^{**}$ at (4.40) we get the estimator for the population mean $\bar{Y}$ as

$$\hat{t}^{**}_{(1)} = w_0 \hat{\bar{y}}^{**} \left(\frac{\bar{X}_1}{\bar{x}_1^{**}}\right)^{\alpha} \left(\frac{\bar{x}_2^{**}}{\bar{X}_2^{**}}\right)^{\eta} \tag{4.45}$$

The bias and MSE of $\hat{t}^{**}_{(1)}$ to the first degree of approximation respectively as

$$B(\hat{t}^{**}_{(1)}) = \bar{Y}\left[w_0 H_{6(1)} - 1\right], \tag{4.46}$$

$$MSE(\hat{t}^{**}_{(1)}) = \bar{Y}\left[1 + w_0^2 H_{0(1)} - 2w_0 H_{6(1)}\right], \tag{4.47}$$

The MSE($\hat{t}_1^{**}$) is minimized for

$$w_{0(opt.)} = \frac{H_{6(1)}}{H_{0(1)}},$$

Thus, the minimum MSE of $\hat{t}^{**}_{(1)}$ is given by

$$min.MSE(\hat{t}^{**}_{(1)}) = \bar{Y}^2\left(1 - \frac{H_{6(1)}^2}{H_{0(1)}}\right). \tag{4.48}$$

2. Bias and MSE of the Class of Estimators $\hat{\mathbf{t}}^{**}_{(\mathbf{2})}$:

Inserting $(\alpha, \eta) = (0, 0)$ we get the estimator for the population mean $\bar{Y}$ as

$$\hat{t}^{**}_{(2)} = w_0 \hat{\bar{y}}^{**} \exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}_1^{**})}{\bar{X}_1 + \bar{x}_1^{**}}\right\} \exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}_2^{**})}{\bar{X}_2 + \bar{x}_2^{**}}\right\} \tag{4.49}$$

To the first degree of approximation, the bias and MSE of $\hat{t}^{**}_{(2)}$ are respectively given by

$$B(\hat{t}^{**}_{(2)}) = \bar{Y}\big[w_0 H_{6(2)} - 1\big], \tag{4.50}$$

$$MSE(\hat{t}^{**}_{(2)}) = \bar{Y}\big[1 + w_0^2 H_{0(2)} - 2w_0 H_{6(2)}\big], \tag{4.51}$$

The MSE($\hat{t}^{**}_{(2)}$) is minimum when

$$w_{0(opt.)} = \frac{H_{6(2)}}{H_{0(2)}},$$

Thus, the minimum MSE of $\hat{t}^{**}_{(2)}$ is given by

$$min.MSE(\hat{t}^{**}_{(2)}) = \bar{Y}^2\left(1 - \frac{H^2_{6(2)}}{H_{0(2)}}\right). \tag{4.52}$$

3. Bias and MSE of the Class of Estimators $\hat{\mathbf{t}}^{**}_{(\mathbf{3})}$:

Inserting $(\eta, \alpha_1) = (0, 0)$, the class of estimators '$\hat{t}^{**}$' reduces to the class of estimators for population mean $\bar{Y}$ as for the population mean $\bar{Y}$ as

$$\hat{t}^{**}_{(3)} = w_0 \hat{\bar{y}}^{**}\left(\frac{\bar{X}_1}{\bar{x}^{**}_1}\right)^{\alpha} \exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}^{**}_2)}{\bar{X}_2 + \bar{x}^{**}_2}\right\} \tag{4.53}$$

The bias and MSE of $\hat{t}^{**}_{(3)}$ to the first degree of approximation, are respectively given by

$$B(\hat{t}^{**}_{(3)}) = \bar{Y}\big[w_0 H_{6(3)} - 1\big], \tag{4.54}$$

$$MSE(\hat{t}^{**}_{(3)}) = \bar{Y}\big[1 + w_0^2 H_{0(3)} - 2w_0 H_{6(3)}\big], \tag{4.55}$$

The MSE($\hat{t}^{**}_{(3)}$) is minimum for

$$w_{0(opt.)} = \frac{H_{6(3)}}{H_{0(3)}},$$

Thus, the minimum MSE of $\hat{t}^{**}_{(3)}$ is given by

$$min.MSE(\hat{t}^{**}_{(3)}) = \bar{Y}^2\left(1 - \frac{H^2_{6(3)}}{H_{0(3)}}\right). \tag{4.56}$$

4. Bias and MSE of the Class of Estimators $\hat{\mathbf{t}}^{**}_{(\mathbf{4})}$:

Inserting $(\alpha, \alpha_2) = (0, 0)$, the class of estimators '$\hat{t}^{**}$' boils down to the estimator for population mean $\bar{Y}$ as

$$\hat{t}^{**}_{(4)} = w_0 \hat{\bar{y}}^{**}\left(\frac{\bar{X}_2}{\bar{x}^{**}_2}\right)^{\eta} \exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}^{**}_1)}{\bar{X}_1 + \bar{x}^{**}_1}\right\} \tag{4.57}$$

The bias and MSE of $\hat{t}^{**}_{(4)}$ to the first degree of approximation, are respectively given by

$$B(\hat{t}^{**}_{(4)}) = \bar{Y}\big[w_0 H_{6(4)} - 1\big], \tag{4.58}$$

$$MSE(\hat{t}^{**}_{(4)}) = \bar{Y}\big[1 + w_0^2 H_{0(4)} - 2w_0 H_{6(4)}\big], \tag{4.59}$$

The MSE($\hat{t}_4^{**}$) is minimum for

$$w_{0(opt.)} = \frac{H_{6(4)}}{H_{0(4)}},$$

Thus, the minimum MSE of $\hat{t}_{(4)}^{**}$ is given by

$$min.MSE(\hat{t}_{(4)}^{**}) = \bar{Y}^2\left(1 - \frac{H_{6(4)}^2}{H_{0(4)}}\right). \quad (4.60)$$

- For $w_0 = 1$, the estimator(s) $\hat{t}_0^{**}$, $\hat{t}_{(1)}^{**}$, $\hat{t}_{(2)}^{**}$, $\hat{t}_{(3)}^{**}$ and $\hat{t}_{(4)}^{**}$ respectively reduce to the estimators for population mean $\bar{Y}$:

$$t_0^* = \hat{\bar{y}}^{**}\left(\frac{\bar{X}_1}{\bar{x}_1^{**}}\right)^{\alpha}\left(\frac{\bar{x}_2^{**}}{\bar{X}_2^{**}}\right)^{\eta}\exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}_1^{**})}{\bar{X}_1 + \bar{x}_1^{**}}\right\}\exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}_2^{**})}{\bar{X}_2 + \bar{x}_2^{**}}\right\}, \quad (4.61)$$

$$t_{(1)}^* = \hat{\bar{y}}^{**}\left(\frac{\bar{X}_1}{\bar{x}_1^{**}}\right)^{\alpha}\left(\frac{\bar{x}_2^{**}}{\bar{X}_2^{**}}\right)^{\eta}, \quad (4.62)$$

$$t_{(2)}^* = \hat{\bar{y}}^{**}\exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}_1^{**})}{\bar{X}_1 + \bar{x}_1^{**}}\right\}\exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}_2^{**})}{\bar{X}_2 + \bar{x}_2^{**}}\right\}, \quad (4.63)$$

$$t_{(3)}^* = \hat{\bar{y}}^{**}\left(\frac{\bar{X}_1}{\bar{x}_1^{**}}\right)^{\alpha}\exp\left\{\frac{\alpha_2(\bar{X}_2 - \bar{x}_2^{**})}{\bar{X}_2 + \bar{x}_2^{**}}\right\}, \quad (4.64)$$

$$t_{(4)}^* = \hat{\bar{y}}^{**}\left(\frac{\bar{X}_2}{\bar{x}_2^{**}}\right)^{\eta}\exp\left\{\frac{\alpha_1(\bar{X}_1 - \bar{x}_1^{**})}{\bar{X}_1 + \bar{x}_1^{**}}\right\}, \quad (4.65)$$

The MSE's of t_0^*, $t_{(1)}^*$, $t_{(2)}^*$, $t_{(3)}^*$ and $t_{(4)}^*$ to the first degree of approximation are respectively given by

$$MSE(t_0^*) = \left[(A+K) + \theta_1^2R_1^2B + \theta_2^2R_2^2C - 2\theta_1R_1D + 2\theta_2R_2F - 2\theta_1\theta_2R_1R_2E\right], \quad (4.66)$$

$$MSE(t_{(1)}^*) = \left[(A+K) + \alpha^2R_1^2B + \eta^2R_2^2C - 2\alpha R_1D + 2\eta R_2F - 2\alpha\eta R_1R_2E\right], \quad (4.67)$$

$$MSE(t_{(2)}^*) = \left[(A+K) + (\alpha_1^2/4)R_1^2B + (\alpha_2^2/4)R_2^2C - (\alpha_1\alpha_2/2)R_1R_2E - \alpha_1R_1D + \alpha_2R_2F\right], \quad (4.68)$$

$$MSE(t_{(3)}^*) = \left[(A+K) + \alpha^2R_1^2B + (\alpha_2^2/4)R_2^2C - 2\alpha R_1D + \alpha_2R_2F - 2\alpha\alpha_2R_1R_2E\right], \quad (4.69)$$

$$MSE(t_{(4)}^*) = \left[(A+K) + (\alpha_1^2/4)R_1^2B + \eta^2R_2^2C - \alpha_1\eta R_1R_2E - \alpha_1R_1D + 2\eta R_2F\right], \quad (4.70)$$

The MSE's of t_0^*, $t_{(1)}^*$, $t_{(2)}^*$, $t_{(3)}^*$ and $t_{(4)}^*$ are respectively minimized for

$$\theta_1 = \frac{(CD - EF)}{\alpha_1(BC - E^2)}, \theta_2 = \frac{(ED - FB)}{R_2(BC - E^2)}$$

$$\alpha_{opt.} = \frac{(CD - EF)}{R_1(BC - E^2)}, \eta_{opt.} = \frac{(ED - FB)}{R_2(BC - E^2)}$$

$$\alpha_{1(opt.)} = \frac{2(CD-EF)}{R_1(BC-E^2)}, \eta_{1(opt.)} = \frac{2(ED-FB)}{R_2(BC-E^2)}$$

$$\alpha_{opt.} = \frac{(CD-EF)}{R_1(BC-E^2)}, \alpha_{2(opt.)} = \frac{2(ED-FB)}{R_2(BC-E^2)}$$

$$\alpha_{1(opt.)} = \frac{2(CD-EF)}{R_1(BC-E^2)}, \eta_{(opt.)} = \frac{(DE-BF)}{R_2(BC-E^2)}$$

Substitution of respective optimum values of constants in MSE of ($t^*_{(0)}$, $t^*_{(1)}$, $t^*_{(2)}$, $t^*_{(3)}$, $t^*_{(4)}$) as

$$min.MSE(\hat{t}^{**}_j) = \left[A + k - \frac{CD^2 - 2DEF + BF^2}{(BC-E^2)}\right]; j = 0, 1, 2, 3, 4. \tag{4.71}$$

5. EFFICIENCY COMPARISONS

To evaluate the mean squared error of the proposed estimator $\hat{T}^{**}$ over the mean squared error of class of estimator(s) and Kumar et al [12], we obtain the efficiency condition as follows

(i) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(1)})$

$if \quad \Delta_{(1)}H_6\Delta_0 - \Delta H_{6(1)}\Delta_{0(1)} < 0,$

(ii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(2)})$

$if \quad \Delta_{(2)}H_6\Delta_0 - \Delta H_{6(2)}\Delta_{0(2)} < 0,$

(iii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(3)})$

$if \quad \Delta_{(3)}H_6\Delta_0 - \Delta H_{6(3)}\Delta_{0(3)} < 0,$

(iv) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(4)})$

$if \quad \Delta_{(4)}H_6\Delta_0 - \Delta H_{6(4)}\Delta_{0(4)} < 0,$

(v) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(5)})$

$if \quad \Delta_{(5)}H_6\Delta_0 - \Delta H_{6(5)}\Delta_{0(5)} < 0,$

(vi) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(6)})$

$$if \quad \bar{Y}^2\left[1 - \frac{H_6\Delta_0}{\Delta}\right] - \left[(A+K) - \frac{(CD^2 - 2DEF + BF^2)}{(BC-E^2)}\right] < 0,$$

(vii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(7)})$

$$if \quad \bar{Y}^2\left[1 - \frac{H_6\Delta_0}{\Delta}\right] - \left[(A+K) - \frac{D}{B}\right] < 0,$$

(viii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(8)})$

$$if \quad \bar{Y}^2\left[1 - \frac{H_6\Delta_0}{\Delta}\right] - \left[(A+K) - \frac{F^2}{C}\right] < 0,$$

(ix) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(9)})$

$$if \quad \left[1 - \frac{H_6\Delta_0}{\Delta}\right] - \frac{\left[(A+K)B - D^2\right]}{B\bar{Y}^2 + \{(A+K)B - D^2\}} < 0,$$

(x) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(10)})$

$$if \quad \Delta_{(10)}H_6\Delta_0 - \Delta H_{6(10)}\Delta_{0(10)} < 0,$$

(xi) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{t}^{**})$

$$if \quad \Delta_0 H_0 - \Delta H_6 < 0,$$

(xii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{t}^{**})$

$$if \quad \Delta_0 H_0 - \Delta H_6 < 0,$$

(xiii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{t}^{**}_{(1)})$

$$if \quad \Delta_{0(1)}H_0 - \Delta H_{6(1)} < 0,$$

(xiv) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{t}^{**}_{(2)})$

$$if \quad \Delta_{0(2)}H_0 - \Delta H_{6(2)} < 0,$$

(xv) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{t}^{**}_{(3)})$

$$if \quad \Delta_{0(3)}H_0 - \Delta H_{6(3)} < 0,$$

(xvi) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{t}^{**}_{(4)})$

$$if \quad \Delta_{0(4)}H_0 - \Delta H_{6(4)} < 0,$$

(xvii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{t}^{**}_{(j)}); j = 0, 1, 2, 3, 4$

$$if \quad \bar{Y}^2\left[1 - \frac{H_6^2}{H_0}\right] - \left[A + k - \frac{CD^2 - 2DEF + BF^2}{(BC - E^2)}\right] < 0,$$

(xviii) $min.MSE(\hat{T}^{**}) < min.MSE(\hat{T}^{**}_{(ssz)})$

$$if \quad \bar{Y}^2\left[1 - \frac{H_6^2}{H_0}\right] - \bar{Y}^2(4\hat{\alpha}^*_{opt.} - 1)^2 + 16\hat{\alpha}^{*2}_{opt.}A + 16\beta^2_{yx_1}B + 16\beta^2_{yx_2}C - 32\hat{\alpha}^*_{opt.}\beta_{yx_1}D +$$
$$32\beta_{yx_1}\beta_{yx_2}E - 32\hat{\alpha}^*_{opt.}\beta_{yx_2}F + 16\hat{\alpha}^{*2}_{opt.}\kappa < 0.$$

If the above conditions from (i) to (xviii) are met then it is clear that the proposed class of estimator is more efficient than the Kumar et al. [12] estimator that are used for comparison purpose.

In the next section, we conducted a simulation study by using R software to verify the above results.

6. SIMULATION STUDY

In this section, a simulation study is carried out to compare the mean squared error (MSE) of the proposed class of estimator by using R software. We have generated a hypothetical population in section 6.1 and considered a real population in section 6.2. The descriptions of the variables with parametric values are given in section 6.1 and 6.2, respectively. The results are given in Tables 1 and 2, respectively.

6.1. Hypothetical Population generated from Normal distribution

In this subsection, we analyzed the efficiency of our estimator with the help of a hypothetical finite population of size $N = 10,000$ generated from a normal distribution and from N we take a sample of size $n' = 7000$ and of n' we picked a sample of size $n = 4000$ is taken using SRSWOR. In the first phase, only 1600 (n_1) provide a response to the survey question and $n_2 = n - n_1$ i.e. 2400 of them do not respond. In the second phase, we take another sample $\left(n_s = \frac{n_2}{k}\right)$ from the non-respondent group by using different values of $k = 2, 3, 4, 5$. A variable $X_1 \sim N(0.2, 1)$, $X_2 \sim N(0.2, 1)$ and variable Y which is defined as $Y = a * X_1 + a * X_2 + N(0, 1)$ also generated from a normal distribution where $a = 0.02$. The scrambling variable(s) S_1 is taken from a normal distribution with a mean of 1 and a variance of 0.5, whereas the scrambling variable S_2 is drawn from a normal distribution with a mean of 0 and a variance of 1, respectively. The MSE's of the considered estimators for different levels of π (i.e. 0.2 to 1) and k (i.e. $2, 3, 4, 5$) are shown in Table 1. Further, to give more clarity to the readers, results of Table 1 are represented graphically in Figure 1.

Table 1: Mean Squared Error of estimator(s) at varying values of k and π

Estimator(s)	$\pi = 0.2$				$\pi = 0.6$				$\pi = 1$			
	k											
	2	3	4	5	2	3	4	5	2	3	4	5
$\hat{T}^{**}$	**0.000187**	**0.000254**	**0.000295**	**0.000314**	**0.000013**	**0.000020**	**0.000442**	**0.000101**	**0.000394**	**0.000574**	0.000764	0.000877
$\hat{T}^{**}_{(1)}$	0.001037	0.001406	0.002009	0.001874	0.001265	0.001712	0.001643	0.001137	0.001194	0.000987	**0.000231**	**0.000535**
$\hat{T}^{**}_{(2)}$	0.000505	0.000513	0.000858	0.000782	0.000705	0.000776	0.001167	0.001429	0.001141	0.001505	0.001917	0.002335
$\hat{T}^{**}_{(3)}$	0.007779	0.008916	0.010759	0.011465	0.008013	0.009204	0.011074	0.011827	0.008446	0.010104	0.011573	0.013052
$\hat{T}^{**}_{(4)}$	0.002682	0.003297	0.004055	0.004567	0.002881	0.003558	0.004368	0.004945	0.003254	0.004143	0.005047	0.006009
$\hat{T}^{**}_{(5)}$	0.000653	0.000824	0.000986	0.001142	0.000845	0.001084	0.001309	0.001537	0.001208	0.001583	0.001990	0.002385
$\hat{T}^{**}_{(6)}$	0.000655	0.000828	0.000992	0.001150	0.000849	0.001091	0.001319	0.001552	0.001217	0.001599	0.002014	0.002419
$\hat{T}^{**}_{(7)}$	0.000758	0.000961	0.001145	0.001319	0.000952	0.001222	0.001468	0.001716	0.001319	0.001729	0.002176	0.002625
$\hat{T}^{**}_{(8)}$	0.000757	0.000960	0.001143	0.001329	0.000951	0.001223	0.001471	0.001732	0.001313	0.001714	0.002148	0.002578
$\hat{T}^{**}_{(9)}$	0.000755	0.000955	0.001137	0.001308	0.000945	0.001213	0.001455	0.001699	0.001308	0.001711	0.002149	0.002585
$\hat{T}^{**}_{(10)}$	0.000754	0.000954	0.001135	0.001318	0.000946	0.001213	0.001458	0.001714	0.001303	0.001697	0.002120	0.002538
$\hat{t}^{**}$	0.000856	0.001086	0.001284	0.001483	0.001046	0.001343	0.001603	0.001874	0.001403	0.001824	0.002277	0.002737
$\hat{t}^{**}_{(1)}$	0.012851	0.015051	0.017701	0.019323	0.013092	0.015347	0.018035	0.019705	0.013422	0.016040	0.018322	0.020638
$\hat{t}^{**}_{(2)}$	0.003221	0.003879	0.004808	0.005331	0.003424	0.004141	0.005109	0.005691	0.003822	0.004835	0.005847	0.006872
$\hat{t}^{**}_{(3)}$	0.010052	0.011644	0.013824	0.014904	0.010289	0.011931	0.014135	0.015260	0.010693	0.012795	0.014649	0.016481
$\hat{t}^{**}_{(4)}$	0.005406	0.006672	0.008003	0.009109	0.005608	0.006932	0.008309	0.009475	0.005943	0.007481	0.008979	0.010549
$\hat{t}^{**}_{j}$	0.000655	0.000828	0.000992	0.001150	0.000849	0.001091	0.001319	0.001552	0.001217	0.001599	0.002014	0.002419
$\hat{T}^{**}_{ssz}$	0.010718	0.013196	0.016145	0.018698	0.010829	0.013360	0.016351	0.01895	0.010971	0.013855	0.016506	0.019128

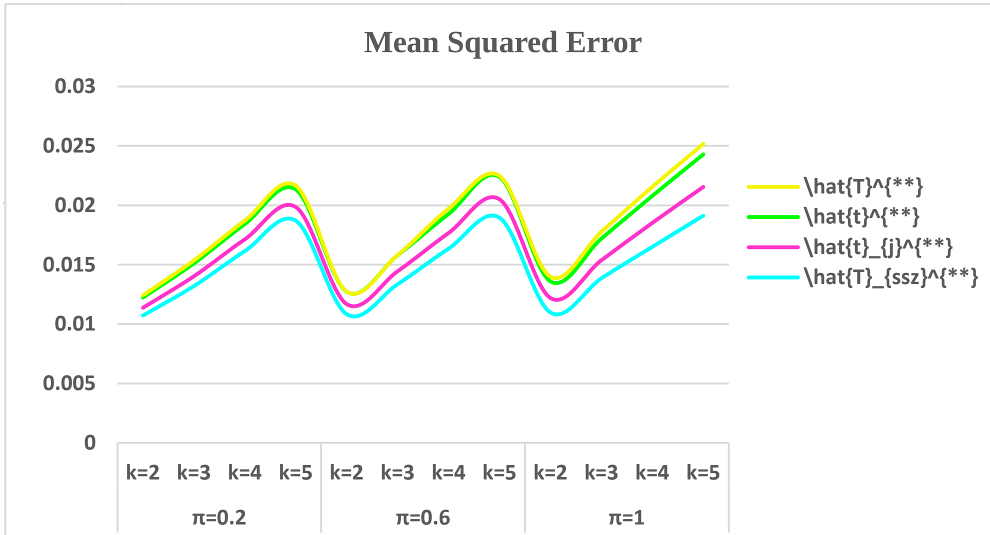

Figure 1: Mean Squared Error of estimator(s) at varying values of k and π.

Tables 1 represents the comparison of mean squared error of the different class of proposed estimator and Kumar et al. [12] estimator for different values of 'k' and π in simple random sampling. From Table 1, when the value of k increases from 2 to 5, the mean squared error of $\hat{T}^{**}_{(3)}$, $\hat{T}^{**}_{(4)}$, $\hat{T}^{**}_{(5)}$, $\hat{T}^{**}_{(6)}$, $\hat{T}^{**}_{(7)}$, $\hat{T}^{**}_{(8)}$, $\hat{T}^{**}_{(9)}$, $\hat{T}^{**}_{(10)}$, $\hat{t}^{**}$, $\hat{t}^{**}_{(1)}$, $\hat{t}^{**}_{(2)}$, $\hat{t}^{**}_{(3)}$, $\hat{t}^{**}_{(4)}$, $\hat{t}^{**}_{(j)}$, $\hat{T}^{**}_{ssz}$ also increases but the mean squared error of $\hat{T}^{**}$, $\hat{T}^{**}_{(1)}$ and $\hat{T}^{**}_{(2)}$ first increases for $k = 2$ to $k = 4$ and then decreases when $k = 5$. Also, for the value of π when it tends to increase, the mean squared error of each estimator also rises but the mean squared error of the proposed estimator $\hat{T}^{**}$ first increases and then decreases.

It is also illustrated from Table 1 that the mean squared error of proposed class of estimators i.e. $\hat{T}^{**}$, $\hat{T}^{**}_{(1)}$, $\hat{T}^{**}_{(2)}$ $\hat{T}^{**}_{(3)}$, $\hat{T}^{**}_{(4)}$, $\hat{T}^{**}_{(5)}$, $\hat{T}^{**}_{(6)}$, $\hat{T}^{**}_{(7)}$, $\hat{T}^{**}_{(8)}$, $\hat{T}^{**}_{(9)}$, $\hat{T}^{**}_{(10)}$, $\hat{t}^{**}$, $\hat{t}^{**}_{(1)}$, $\hat{t}^{**}_{(2)}$, $\hat{t}^{**}_{(3)}$, $\hat{t}^{**}_{(4)}$, $\hat{t}^{**}_{(j)}$ is lowest among Kumar et al. [12] estimator i.e. ($\hat{T}^{**}_{ssz}$). In the end, we obtain that the proposed estimator $\hat{T}^{**}$ is more efficient than the other classes of proposed estimators and Kumar et al. [12] estimator.

To show the results of Table 1 graphically, we have considered the MSE values of estimators $\hat{T}^{**}$, $\hat{t}^{**}$, $\hat{t}^{**}_{(j)}$ and $\hat{T}^{**}_{ssz}$ as other estimators are the special cases of these estimators.

6.2. Natural Population Based on Census 2011 Literacy rates in India

The dataset is based on Census 2011 literacy rates in India. The data is of $N = 35$ Indian states and union territories then a random sample is drawn from the population i.e., $n = 10$. The literacy rate is spread across the major parameters-Overall, Rural and Urban. Let y, x_1 and x_2 denotes the number of literates (people) in 2001, 2011, and the total literacy rate (2001), respectively. The results are shown in Table 2 and Figure 2 for different probability levels of sensitive variables, i.e. $\pi = 0.2, 0.4, 0.6, 0.8, 1$ and different non-response rates i.e., $k = 2, 3, 4, 5$ are used.

Table 2: Mean Squared Error of estimator(s) at varying values of k and π

Estimator(s)	$\pi = 0.2$				$\pi = 0.6$				$\pi = 1$			
	k											
	2	3	4	5	2	3	4	5	2	3	4	5
$\hat{T}^{**}$	**0.1794**	**0.4487**	**0.2654**	**0.3303**	**4.4336**	**4.8277**	**4.7926**	**4.9698**	**12.8400**	**13.3691**	**15.0179**	**15.8489**
$\hat{T}^{**}_{(1)}$	1.0252	1.3336	1.2603	1.3832	5.1079	5.5157	5.5799	5.8036	13.3445	13.9060	15.4262	16.2332
$\hat{T}^{**}_{(2)}$	0.1794	0.9776	1.0434	1.1743	4.8133	5.1612	5.3659	5.6025	12.9784	13.5788	15.1027	15.8903
$\hat{T}^{**}_{(3)}$	0.6181	0.8551	0.8396	0.9521	4.6492	5.0124	5.1250	5.6025	12.8137	13.3439	14.7987	15.5692
$\hat{T}^{**}_{(4)}$	0.7675	0.9777	1.0435	1.1745	4.8134	5.1614	5.3660	5.3439	12.9785	13.5789	15.1028	15.8901
$\hat{T}^{**}_{(5)}$	0.7671	0.9772	1.0432	1.1742	4.8132	5.1611	5.3658	5.6026	12.9784	13.5787	15.1026	15.8908
$\hat{T}^{**}_{(6)}$	0.7673	0.9778	1.0436	1.1746	4.8180	5.1666	5.3717	5.6024	13.0121	13.6157	15.1484	15.9409
$\hat{T}^{**}_{(7)}$	0.7871	0.9937	1.1393	1.3128	4.8572	5.2123	5.5530	5.6089	13.0610	13.7971	15.5360	16.4647
$\hat{T}^{**}_{(8)}$	0.9834	1.3160	1.2389	1.3678	5.0188	5.4608	5.5145	5.8563	13.2184	13.7438	15.2154	15.9993
$\hat{T}^{**}_{(9)}$	0.7869	0.9935	1.1390	1.3124	4.8524	5.2068	5.5468	5.7470	13.0270	13.7592	15.4879	16.4108
$\hat{T}^{**}_{(10)}$	0.9832	1.3157	1.2386	1.3674	5.0137	5.4548	5.5083	5.8493	13.1836	13.7062	15.1693	15.9483
$\hat{t}^{**}$	0.2193	0.4884	0.3028	0.3669	4.4806	4.8744	4.8368	5.7403	12.8944	13.4252	15.0774	15.9102
$\hat{t}^{**}_{(1)}$	1.0279	1.3360	1.2624	1.3852	5.1124	5.5199	5.5835	5.0133	13.3513	13.9135	15.4347	16.2423
$\hat{t}^{**}_{(2)}$	3.3100	3.5879	3.4996	3.6169	7.6877	8.1015	8.1671	5.8070	16.2097	16.9030	18.6695	19.6090
$\hat{t}^{**}_{(3)}$	3.8352	4.1124	4.0195	4.1376	8.2723	8.6928	8.7572	8.4026	16.8523	17.5721	19.3903	20.3578
$\hat{t}^{**}_{(4)}$	3.3100	3.5879	3.4996	3.6169	7.6877	8.1015	8.1671	8.9969	16.2097	16.9030	18.6695	19.6090
$\hat{t}^{**}_{j}$	0.7673	0.9778	1.0436	1.1746	4.8180	5.1666	5.3717	8.4026	13.0121	13.6170	15.1484	15.9409
$\hat{T}^{**}_{ssz}$	92.3390	94.7140	96.0640	97.6450	41.0250	42.2470	43.1120	43.9760	27.6433	28.4353	29.8919	30.7958

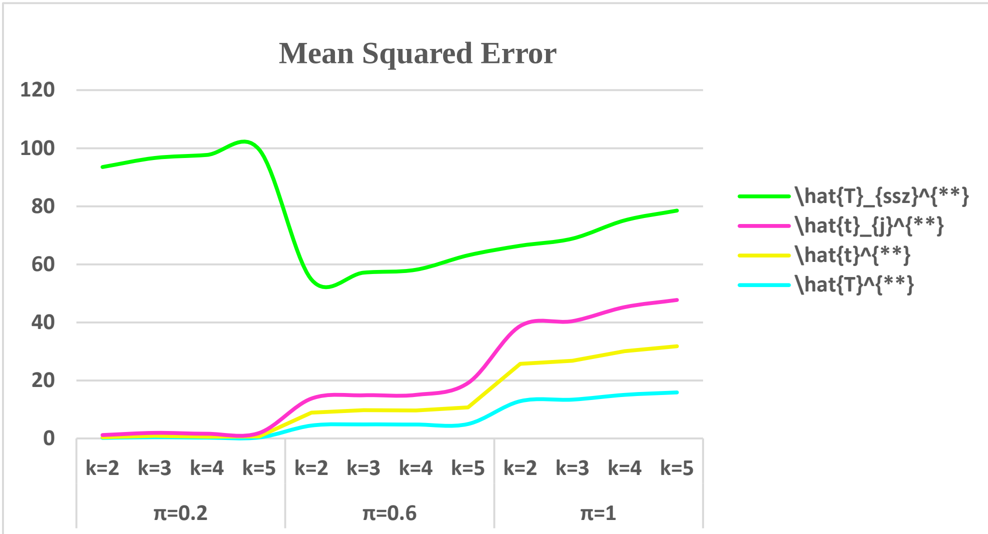

Figure 2: Mean Squared Error of estimator(s) at varying values of k and π.

Tables 2 represents the comparison of mean squared error of the proposed class of estimator over Kumar et al. [12] estimator for different values of 'k' and π in simple random sampling by using dataset which is based on Census 2011 literacy rates in India. It is envisaged from Table 2 that for increase in the value of π from 0.2 to 1, the mean squared error of the proposed class of estimator increases whereas the mean squared error of Kumar et al. [12] estimator decreases. So, it is evident that the mean squared error of proposed class of estimator i.e. $\hat{T}^{**}$, $\hat{T}^{**}_{(1)}$, $\hat{T}^{**}_{(2)}$ $\hat{T}^{**}_{(3)}$, $\hat{T}^{**}_{(4)}$, $\hat{T}^{**}_{(5)}$, $\hat{T}^{**}_{(6)}$, $\hat{T}^{**}_{(7)}$, $\hat{T}^{**}_{(8)}$, $\hat{T}^{**}_{(9)}$, $\hat{T}^{**}_{(10)}$, $\hat{t}^{**}$, $\hat{t}^{**}_{(1)}$, $\hat{t}^{**}_{(2)}$, $\hat{t}^{**}_{(3)}$, $\hat{t}^{**}_{(4)}$, $\hat{t}^{**}_{(j)}$ is lowest among the Kumar et al. [12] estimator i.e. ($\hat{T}^{**}_{ssz}$). Overall, the proposed estimator $\hat{T}^{**}$ performs well as compared to the other classes of proposed estimators and Kumar et al. [12] estimator.

To show the results of Table 2 graphically, we have considered the MSE values of estimators $\hat{T}^{**}$, $\hat{t}^{**}$, $\hat{t}^{**}_{(j)}$ and $\hat{T}^{**}_{ssz}$ as other estimators are the special cases of these estimators.

7. CONCLUSION

In conclusion, paper has addressed the challenging task of estimating the population mean of a sensitive variable under non-response and measurement error using ORRT models in SRSWOR. The efficiency of the proposed class of estimator are evaluated up to the first order of approximation in comparison to Kumar et al. [12] estimator and the conditions are also determined. Through the simulation study for both hypothetical and real population, the effectiveness and performance of the proposed class of estimator is evaluated and found that the proposed estimator $\hat{T}^{**}$ and different classes of proposed estimator i.e. $\hat{T}^{**}_{(1)}$, $\hat{T}^{**}_{(2)}$ $\hat{T}^{**}_{(3)}$, $\hat{T}^{**}_{(4)}$, $\hat{T}^{**}_{(5)}$, $\hat{T}^{**}_{(6)}$, $\hat{T}^{**}_{(7)}$, $\hat{T}^{**}_{(8)}$, $\hat{T}^{**}_{(9)}$, $\hat{T}^{**}_{(10)}$, $\hat{t}^{**}$, $\hat{t}^{**}_{(1)}$, $\hat{t}^{**}_{(2)}$, $\hat{t}^{**}_{(3)}$, $\hat{t}^{**}_{(4)}$, $\hat{t}^{**}_{(j)}$ obtain the lowest mean squared error among the Kumar et al. [12] ($\hat{T}^{**}_{ssz}$) estimator. As a result, we favour the use of proposed class of estimator for future studies by the researchers in practice.

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CONFLICT OF INTEREST

All authors have no conflict of interest to declare.

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