NUMERICAL METHOD FOR SYSTEM OF NONLINEAR INITIAL VALUE PROBLEMS (ODE)

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ABSTRACT

In this study, a straightforward approach has been made to develop a numerical method for system of nonlinear initial value problems. With that motive, the second order variant method for the solution of initial value problems (VM) [5] is chosen and implemented suitably to develop a new scheme (VMS). The validity and performance of the proposed method are verified by conducting numerical tests on some nonlinear initial value problems of diverse nature. It is an efficient algorithm with less computational complexity that produces results with high accuracy. A comparative analysis is provided for the computed results and the results of some existing efficient algorithms. The scheme can also be applied successfully to the higher order differential equations.
KEYWORDS: Runge-Kutta method, order of convergence, Initial value problem, chaotic system, dynamical system, Lorentz system, HIV-1 infection problem
MSC: 65L05

RESUMEN

En este trabajo se presente un método numérico directo para resolver problemas de valor inicial no lineales. Se propone e implementa el esquema VMS que constituye una nueva variante del método de segundo orden para el problema de valor inicial (VM), ver [5]. Se resuelven problemas de distinta naturaleza para validar el comportamiento del nuevo enfoque. Se ilustra que el algoritmo logra una buena precision con una menor complejidad computacional. Se comparan los resultados de VMS con otros algortimos existentes. También se resuelven numéricamente ecuaciones diferenciales de orden superior.

PALABRAS CLAVE: Método de Runge-Kutta method, orden de convergencia, problema de valor inicial, sistemas caóticos, sistemas dinámicos, sistema de Lorentz, modelo para la propagación por VIH-1

1. INTRODUCTION

Several problems in science and engineering are inherently nonlinear. Their mathematical models are often found in the form of system of initial value problems.

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A nonlinear initial value problem in ordinary differential equation is of the form:

$$\begin{aligned} \frac{dx_1}{dt} &= g_1(t, x_1, x_2 \cdots, x_n), \\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2 \cdots, x_n), \\ &\vdots \\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2 \cdots, x_n), \end{aligned} \tag{1.1}$$

where $x_i(t_0) = x_{i0}, \forall i = 0, 1, \cdots, n$.
The solution of (1.1) is a column matrix $\boldsymbol{x} = [x(t_1), x(t_2), \cdots, x(t_n)]^T$.

Researchers have contributed valuable findings and suggested the exact and numerical solution of the system (1.1). The famous age-old Euler's method [9] and the Runge-Kutta method [6, 21] are handy to yield the numerical solution. One can find the recently developed multi-stage differential transformation methods in [1, 22], multi-stage variational iteration method in [18], multi-stage Adomain decomposition methods in [15, 17], multi-stage homotopy analysis methods in [[2]-[4]] and piecewise homotopy pertubation methods in [10, 16]. In the year 2012, Mosta [19] has suggested a modification of succesive linearization method (PSLM). In the year 2020, Adiyaman and Noyan [12] have developed a Residual method (RM) based on minimizing the residual function using Taylor's series.
The Lotka-Voltera system [19]

$$\begin{cases} \frac{du}{dt} = u(\alpha - \beta v), & u(t_0) = u_0, \\ \frac{dv}{dt} = v(\gamma - \delta u), & v(t_0) = v_0 \end{cases} \tag{1.2}$$

is a nonlinear system that describes the dynamics of biological system involving the interaction of the two species, the predator and the prey.
Here u is the density of the prey, v is the density of the predator and α, β, γ and δ are the parameters representing the interaction between the predator and the prey.
The Genisio system [13]

$$\begin{cases} \frac{du}{dt} = v, & u(t_0) = u_0, \\ \frac{dv}{dt} = w, & v(t_0) = v_0, \\ \frac{dw}{dt} = -cu - bv - aw + u^2, & w(t_0) = w_0 \end{cases} \tag{1.3}$$

is a nonlinear system of differential equations. It becomes chaotic for $a = 1.2$, $b = 2.92$ and $c = 6$. A chaotic system is a nonlinear deterministic system with complex and unpredictable behaviour.
The Lorentz system [8]

$$\begin{aligned} &\frac{dx}{dt} - \alpha(y - x) = 0, \\ &\frac{dy}{dt} + xz - \beta x + y = 0, \\ &\frac{dz}{dt} - xy + \gamma z = 0, \end{aligned} \tag{1.4}$$

with $x(t_0) = x_0$, $y(t_0) = y_0$, $z(t_0) = z_0$ and α, β, γ are all positive.
This is a chaotic problem. This system was derived by Lorentz while modelling the two dimensional

fluid cell at different temperatures between two parallel plates.
We can find numerical solutions of the system (1.4) in [11, 14, 17, 19, 20].
The primary HIV-1 infection problem [7]

$$\begin{aligned}
\frac{dT}{dt} &= a - bT - cVT, \quad T(t_0) = T_0 \\
\frac{dI}{dt} &= cVT - \delta I, \quad I(t_0) = I_0 \\
\frac{dV}{dt} &= dI - eV, \quad V(t_0) = V_0
\end{aligned} \tag{1.5}$$

where T is the concentration of target cells, I is the concentration of infected cells and V is the serum viral concentration with a as the constant influx rate of target cells, b as the loss rate of target cell, c as the infection rate constant of the target cells, $\delta =$ as the loss rate constant of the infected cells, d as the viral production rate constant and e as the virus concentration rate constant.
This system (1.5) is a dynamical system and the equations in system express the dynamics of the target cells, the dynamics of the infected cells and the viral dynamics respectively. Burg et.al. [7] and Adiyaman and Noyan [12] have suggested numerical solution to this model.
Mishra et. al. [5] have suggested a second order method by evaluating the function at some special points

$$y_{n+1} = y_n + \frac{h}{2}\left[V_1 + V_2\right] \tag{1.6}$$

where $V_1 = f\left(x_n + \frac{h}{4}, y(x_n) + \frac{h}{4} f_n\right)$ and $V_2 = f\left(x_n + \frac{3h}{4}, y(x_n) + \frac{3h}{4} f_n\right)$.
to solve the initial value problems.
Now we use the method (1.6) to suggest a numerical method for nonlinear system of initial value problems.
The sections in the paper are managed in the following order. In the subsequent sections, the new scheme of iteration is introduced. The nonlinear problems discussed in this section are solved using the proposed scheme, a comparative study is also displayed to investigate the validity and accuracy. Finally, a conclusion is drawn about the application and validity of the presented method.

2. MAIN RESULTS

2.1. The New Method (Variant Method for System of Nonlinear Initial Value Problems (VMS))

Let the system be

$$\begin{aligned}
\frac{dx}{dt} &= g_1(t, x, y, z), \quad x(t_0) = x_0 \\
\frac{dy}{dt} &= g_2(t, x, y, z), \quad y(t_0) = y_0 \\
\frac{dz}{dt} &= g_3(t, x, y, z), \quad z(t_0) = z_0
\end{aligned} \tag{2.1}$$

Let

$$
\begin{aligned}
k_1 &= g_1\Bigg[t_n + \frac{h}{4}, x_n + \frac{h}{4}g_1\bigg(t_n, x_n, y_n, z_n\bigg), \\
&\qquad y_n + \frac{h}{4}g_2\bigg(t_n, x_n, y_n, z_n\bigg), z_n + \frac{h}{4}g_3\bigg(t_n, x_n, y_n, z_n\bigg)\Bigg] \\
l_1 &= g_2\Bigg[t_n + \frac{h}{4}, x_n + \frac{h}{4}g_1\bigg(t_n, x_n, y_n, z_n\bigg), \\
&\qquad y_n + \frac{h}{4}g_2\bigg(t_n, x_n, y_n, z_n\bigg), z_n + \frac{h}{4}g_3\bigg(t_n, x_n, y_n, z_n\bigg)\Bigg] \\
m_1 &= g_3\Bigg[t_n + \frac{h}{4}, x_n + \frac{h}{4}g_1\bigg(t_n, x_n, y_n, z_n\bigg), \\
&\qquad y_n + \frac{h}{4}g_2\bigg(t_n, x_n, y_n, z_n\bigg), z_n + \frac{h}{4}g_3\bigg(t_n, x_n, y_n, z_n\bigg)\Bigg] \\
k_2 &= g_1\bigg(t_n + \frac{3h}{4}, x_n + \frac{3h}{4}k_1, y_n + \frac{3h}{4}l_1, z_n + \frac{3h}{4}m_1\bigg) \\
l_2 &= g_2\bigg(t_n + \frac{3h}{4}, x_n + \frac{3h}{4}k_1, y_n + \frac{3h}{4}l_1, z_n + \frac{3h}{4}m_1\bigg) \\
m_2 &= f_3\bigg(t_n + \frac{3h}{4}, x_n + \frac{3h}{4}k_1, y_n + \frac{3h}{4}l_1, z_n + \frac{3h}{4}m_1\bigg)
\end{aligned} \tag{2.2}
$$

Now we suggest a new method, Variant Method for System of Nonlinear Initial Value Problems (VMS) for the solution of the system (2.1) using the formula (1.6) as follows

$$
\begin{aligned}
x_{n+1} &= x_n + \frac{h}{2}(k_1 + k_2) \\
y_{n+1} &= y_n + \frac{h}{2}(l_1 + l_2) \\
z_{n+1} &= z_n + \frac{h}{2}(m_1 + m_2), \qquad n = 0, 1, 2, 3, \cdots
\end{aligned} \tag{2.3}
$$

The order of the method is already determined to be two and the region of stability is same as RK 2nd order method [5] .

3. NUMERICAL EXPERIMENTS

Problem-1 (Lotka-Voltera system):
We solve the Lotka-Voltera system

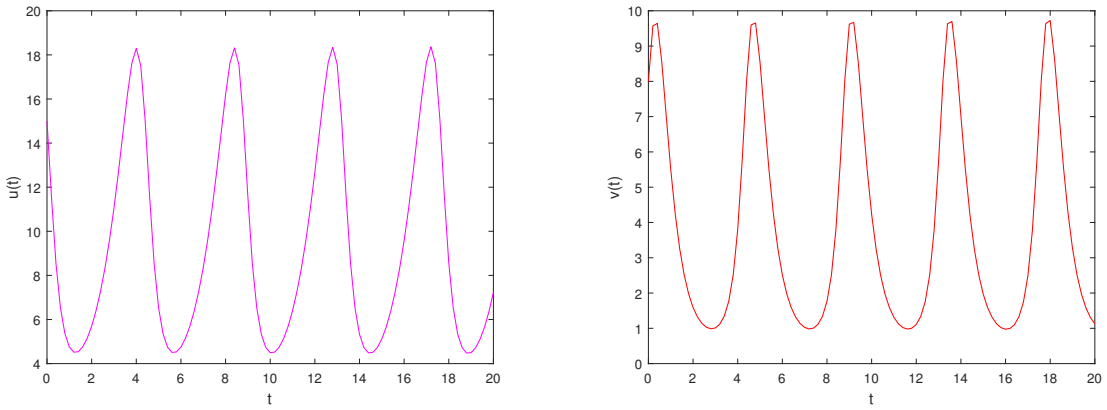
$$
\begin{cases}
\frac{du}{dt} = u(0.95 - 0.25v), & u(0) = 15; \\
\frac{dv}{dt} = v(2.45 - 0.25u), & v(0) = 8
\end{cases} \tag{3.1}
$$

using (VMS) (2.3) with $h = 0.1$. Simulated results are compared with PSLM [19] and displayed in the Table-1 for some selected values of t in $0 \leq t \leq 70$. The results are also graphically displayed for $u(t)$ and $v(t)$ in $0 \leq t \leq 20$ in Figure-1.

Table 1: Comparison of computed results between VMS(2.3) and PSLM [19] for Problem-1(Lotka-Voltera system).

t	$u(t)$ (PSLM)[19]	$u(t)$ (VMS)(2.3)	$v(t)$ (PSLM)[19]	$v(t)$ (VMS)(2.3)
10	4.5020591	4.5019712	4.6035969	4.5073657
20	6.7184208	6.8567729	1.1994501	1.1796220
30	14.7855711	15.1464383	1.3712822	1.4505687
40	11.9439152	11.0468241	9.4606105	9.7008126
50	4.4435049	4.4720300	3.6107490	3.3092895
60	7.5867316	7.9874010	1.0628491	1.0254131
70	16.3709596	17.0306219	1.7840713	2.0532463

Figure 1: Solution curve of Lotka-Voltera system using VMS(2.3)



Problem-2 (Genesi-Tesi system):
We solve the chaotic Genesi-Tesi system

$$\begin{cases} \frac{du}{dt} = v, \quad u(0) = 0.2 & ; \\ \frac{dv}{dt} = w, \quad v(0) = -0.3 & ; \\ \frac{dw}{dt} = -6u - 2.92v - 1.2w + u^2, \quad w(0) = 0.1 \end{cases} \tag{3.2}$$

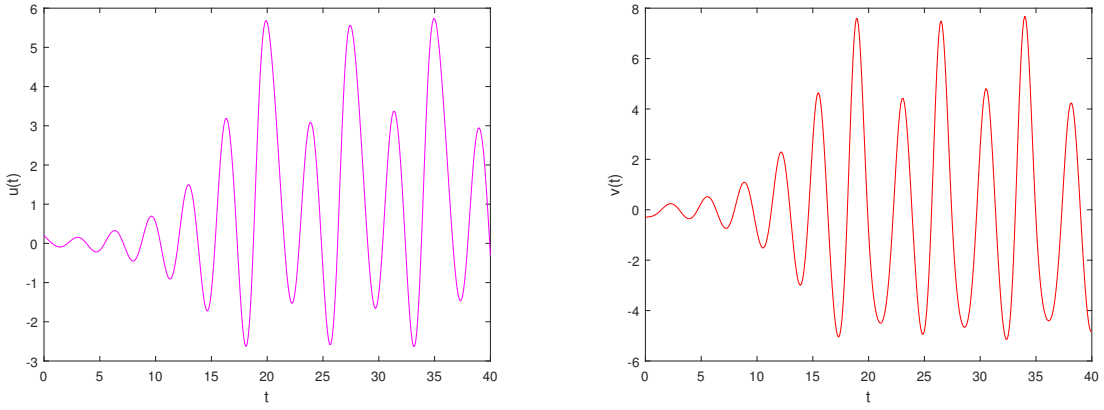
using (VMS) (2.3) with $h = 0.1$ and computed values of $u(t)$, $v(t)$ and $w(t)$ are compared with the results obtained using PSLM [19] in the interval $0 \leq t \leq 64$ and presented in Table-2. The solution curve is displayed for $0 \leq t \leq 40$ in Figure-2.

Problem-3 (Lorentz system):

Table 2: Comparison of computed results between VMS (2.3) and PSLM [19] for Problem-2(Genesi-Tesi system).

t	$u(t)$(PSLM)[19]	$u(t)$ (VMS)	$v(t)$(PSLM)[19]	$v(t)$ (VMS)	$w(t)$(PSLM)[19]	$w(t)$ (VMS)
8	-0.4462003	-0.4514075	-0.0033784	0.0295703	1.6908599	1.7280999
16	2.5905165	2.7559039	2.8024002	2.6094241	-6.7054130	-7.1446820
32	1.9642789	1.8556763	-3.8390800	-4.3409260	-4.9875967	-4.3313841
48	-2.0621034	-2.3888379	-3.0539975	-2.3854037	7.7517806	9.8813372
64	-0.5055765	0.4141096	6.3163584	7.3984916	6.6943032	3.6929157

Figure 2: Solution curve of Genesi-Tesi system using VMS(2.3).



We solve the chaotic Lorentz system

$$\begin{aligned} &\frac{dx}{dt} - 10(y - x) = 0, \\ &\frac{dy}{dt} + xz - 28x + y = 0, \\ &\frac{dz}{dt} - xy + \frac{8}{3}z = 0, \end{aligned} \tag{3.3}$$

with $x(0) = 1$, $y(0) = 5$ and $z(0) = 10$.

To implement the proposed method VMS (2.3), we have taken $h = 0.01$. The simulated values of $x(t)$, $y(t)$ and $z(t)$ are compared with the results obtained using PSLM [19] in Table-3 in the interval $0 \leq t \leq 10$ and the corresponding solution curve of $x(t)$, $y(t)$ and $z(t)$ are depicted in Figure-3.

Problem-4 (HIV-1 infection problem):

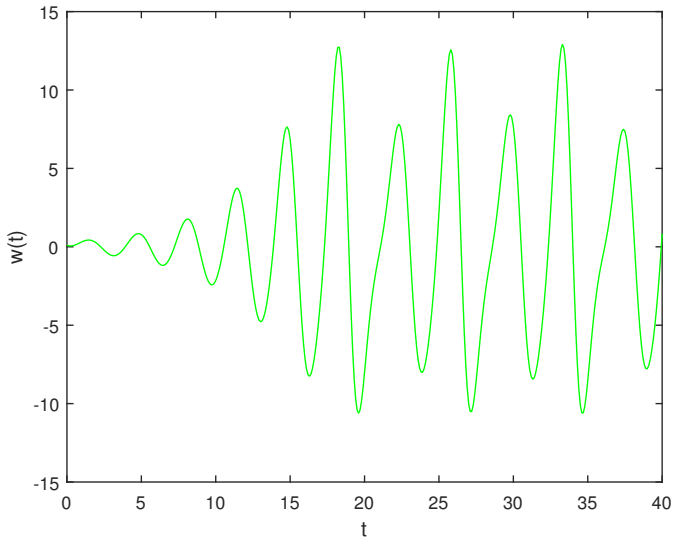


Table 3: Comparison of computed results between VMS (2.3) and PSLM [19] for Problem-3(Lorentz system).

t	$x(t)$ (PSLM)[19]	$x(t)$ (VMS)	$y(t)$ (PSLM)[19]	$y(t)$ (VMS)	$z(t)$ (PSLM)[19]	$z(t)$ (VMS)
2	-1.444359	-1.4406253	-1.074977	-1.0942211	19.517057	19.4372324
4	-14.675080	-14.8377774	-20.189107	-20.0766368	29.063361	29.7126317
6	-2.883028	-2.9542570	-4.763557	-4.8419533	20.355558	20.2874559
8	-2.679253	-2.3750202	1.429476	1.2921206	27.105659	26.3908126
10	-12.026645	-12.6840214	-17.520281	-16.8201759	24.300154	27.4677491

We solve the dynamical system, the HIV-1 infection problem

$$\begin{aligned}
\frac{dT}{dt} &= 10^2 - \frac{1}{10^2}T - 1.3 \times \frac{1}{10^6}TV, \quad T(0) = 10^4 \\
\frac{dI}{dt} &= 1.3 \times \frac{1}{10^6}TV - \delta I, \quad I(0) = 0 \\
\frac{dV}{dt} &= 10^3 I - 3V, \quad V(0) = \frac{1}{10^6}
\end{aligned} \tag{3.4}$$

where δ = death rate of the infected cell.

We solve the problem using our proposed method VMS (2.3) taking $h = 0.25$.

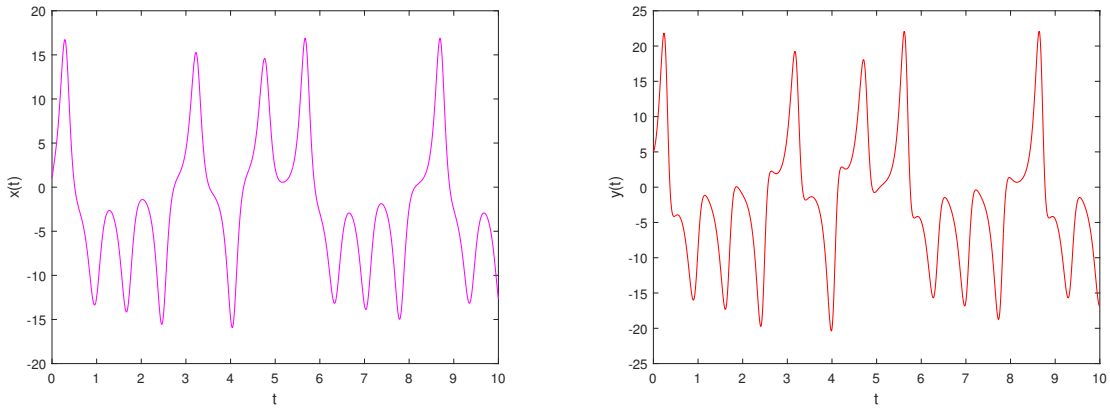
Here T = the number of target cells,

I = the number of infected cells and

V = the number of viruses.

$\log_{10} T$, $\log_{10} I$ and $\log_{10} V$ in 250 days are computed for $\delta = 0.1, 1.0$. The results are illustrated in the Figure-4 and Figure-5 respectively.

Figure 3: Solution curve of Lorentz system using VMS (2.3).



It is observed that the results of HIV-1 infection problem, displayed in Figure-4 and Figure-5 are exactly same as the solution given in [7, 12].

4. CONCLUSION

A simple approach is initiated by applying a second order R-K type method [5] suitably for solving the nonlinear system of initial value problems. The toughest job was to handle the rapidly oscilating systems and the dynamical systems. The proposed method came out successfully and produced better approximations as compared to the discussed methods. It is also observed that the new method has less computational complexity with better accuracy. The method used the functional values at the intermediate points and that helped the method to be a stable one. Varieties of problems are solved with better accuracy to establish the validity and performance of the method. Additionally, we can use the method VMS to solve higher order differential equations with given initial conditions by transfoming it to a system of initial value problems using the technique of reduction of order.

RECEIVED: JUNE, 2023.
REVISED: NOVEMBER, 2023.

REFERENCES

[1] ALOMARI, A.K.(2011):A new analytic solution for fractional chaotic dynamical systems using the differential transform method.**Computers and Mathematics with Applications**, 9,2528-2534.

[2] ALOMARI, A. K., NOORANI, M.S.M., NAZAR, R. (2010):Homotopy approach for the hyperchaotic Chen system.**Phys. Scr.**, 81, 045005.

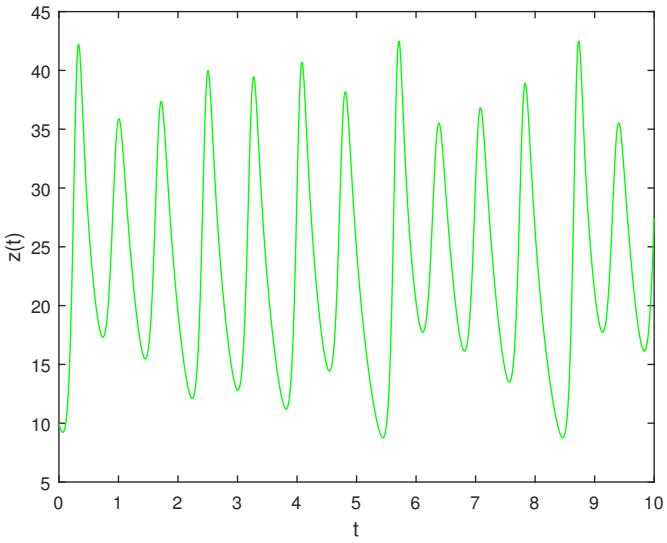
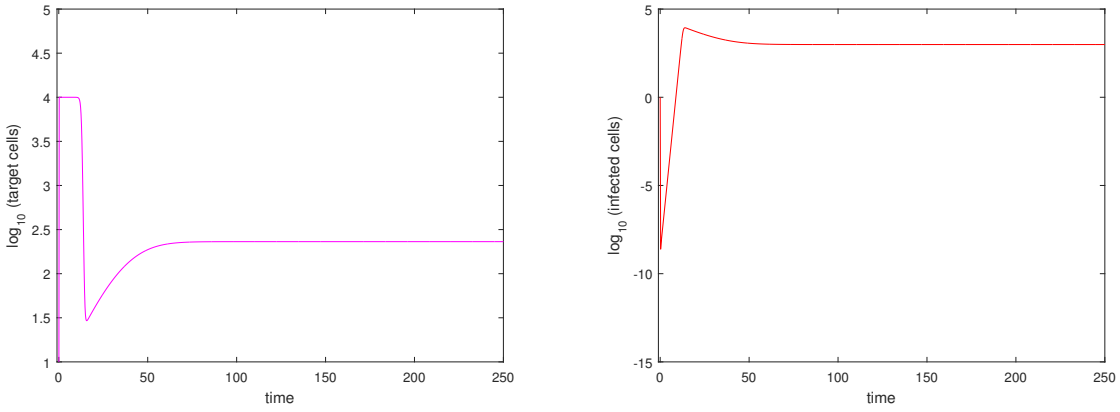


Figure 4: Logarithm (with base 10) of number of target cells, number of infected cells, number of viruses using VMS (2.3) for $\delta = 0.1$.



[3] ALOMARI, A. K., NOORANI, M.S.M., NAZAR, R. (2009):Adaptation of homotopy analysis method for the numeric–analytic solution of Chen system, **Commun. Nonlinear Sci.**, 14, 2336-2346.

[4] ALOMARI, A. K., NOORANI, M.S.M., NAZAR R. LI, C. P. (2010): Homotopy analysis method for solving fractional Lorenz system. **Commun. Nonlinear Sci.**, 15, 1864-1872.

[5] MISHRA, B., DUTTA, S., and PANY, A. (2020): Variant Second order method for the numerical solution of Initial Value Problem in Ordinary Differential Equations. **Test Engineering and Management**, 83, 9309–9317.

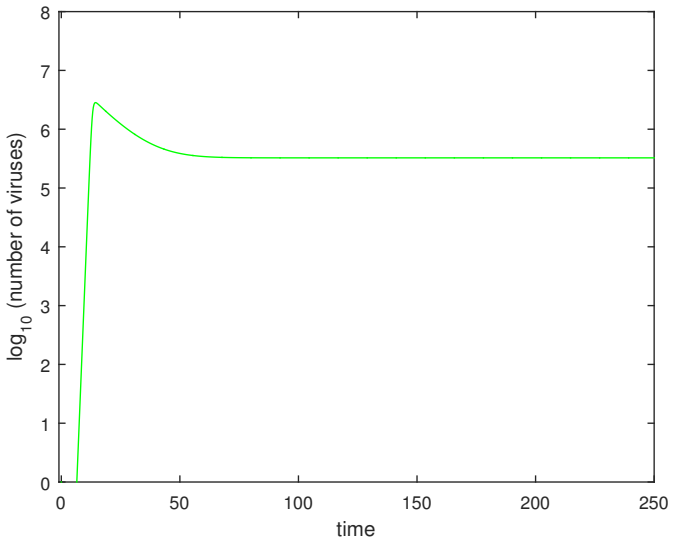
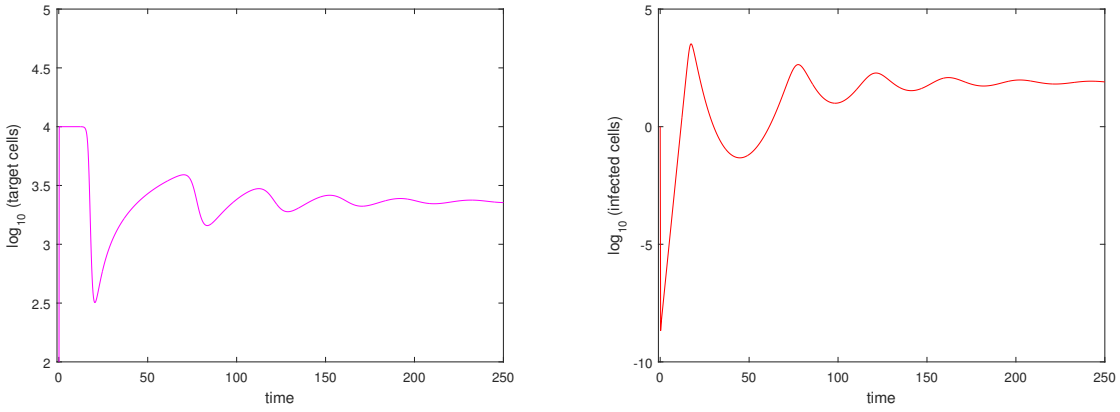


Figure 5: Logarithm (with base 10) of number of target cells, number of infected cells, number of viruses using VMS (2.3) for $\delta = 1.0$.

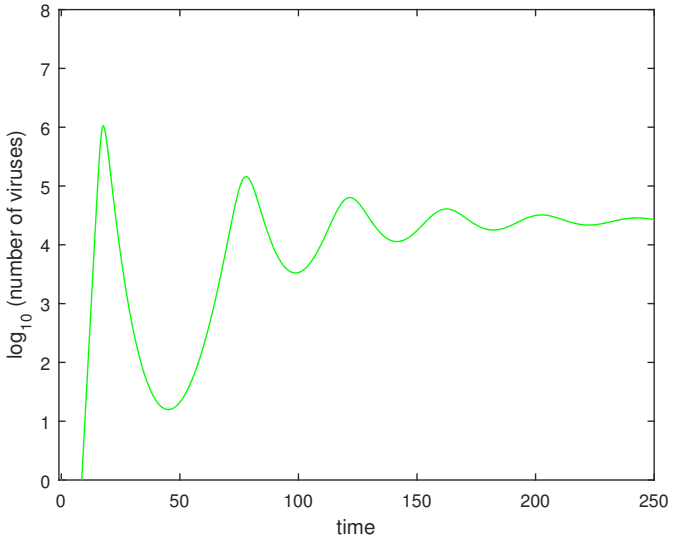


[6] RUNGE, C. (1895): On the numerical solution of differential equations. **Math. Ann.**, 46,167–178.

[7] BURG, D., RONG, L., NEUMANN, A. U. and DAHARI, H. (2009): Mathematical modeling of viral kinetics under immune control during primary HIV-1 infection. **Journal of Theoretical Biology**, 259, 751–759.

[8] LORENZ, E. N. (1963): Deterministic nonperiodic flow. **J. Atmos. Sci.**, 20, 130-141.

[9] EULER, H. (1768): Institutiones calculi integralis. **Volumen Primum (1768), Opera Omnia, Vol. XI**, B. G. Teubneri Lipsiae et Berolini MCMXIII.



[10] RAMOS, J. I.(2008): Piecewise homotopy methods for nonlinear ordinary differential equations. **Appl. Math. Comput.**, 198, 92-116.

[11] CHOWDHURY, M., HASHIM, I. and MOMANI, S. (2009): The multistage homotopyperturbation method: a powerful scheme for handling the Lorenz system. **Nonlinear Dynamics**, 40, 1929–1937.

[12] ADIYAMAN, M.E., NOYAN, B. (2020) : Residual Method for Nonlinear System of Initial Value Problems. **Computational Methods for Differential Equations**, 8 (4),733-744,.

[13] GENESIO, R., TESI, A. (1992): Harmonic Balance Methods for the Analysis of Chaotic Dynamics in Nonlinear Systems. **Automatica**, 28, 531-548.

[14] RODRIGUES, M. J. and MATOS, J. (2013): A tau method for nonlinear dynamical systems. **Central Numerical Algorithms**, 62, 583–600.

[15] MOSSA AL-SAWALHA, M., NOORANI, M.S.M., HASHIM, I. (2009): On accuracy of Adomian decomposition method for hyperchaotic Rössler system. **Chaos Soliton. Fract.**, 40, 1801-1807.

[16] CHOWDHURY, M. S. H., HASHIM, I. (2009): Application of multistage homotopy-perturbation method for the solutions of the Chen system. **Nonlinear Anal. Real.**, 10, 381-391.

[17] ABDULAZIZ, O., NOOR, N. F. M., HASHIM, I., NOORANI, M.S.M.(2008): Further accuracy tests on Adomian decomposition method for chaotic systems. **Chaos Soliton. Fract.**, 36, 1405-1411.

[18] GOH, S.M., NOORANI, M.S.M., HASHIM, I. (2010): On solving the chaotic Chen system: a new time marching design for the variational iteration method using Adomian's polynomial.**Numerical Algorithms**, 54,245-260.

[19] MOSTA, S. S.(2012): A new piecewise-quasilinearization method for solving chaotic systems of initial value problems. **Cent. Eur. J. Phys.**,10(4),936-946.

[20] MOSTA, S., DLAMINI, P. and KHUMALO, M.(2013): A new multistage spectral relaxation method for solving chaotic initial value systems. **Nonlinear Dynamics**, 72, 265–283.

[21] KUTTA, W. (1901): Beitrag Zurn noherungsweisen Integration totaler Differentialgleichungen. **Z. Math. Phys.**, 46, 435–453.

[22] ODIBAT, Z. M., BERTELLE, C., AZIZ-ALAOUI, M. A., DUCHAMP, G. H. E. (2010): A multi-step differential transform method and application to non-chaotic or chaotic systems. **Comput. Math. Appl.**, 59, 1462-1472.