ANALYSIS OF COMPETITION BETWEEN TWO RETAILERS USING LOGISTIC PREY-PREDATOR MODEL WITH FUZZY INITIAL CONDITION

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ABSTRACT
The interaction of prey and predator species is a biological process with profound ecological implications that occurs all over the world. The dynamics of Prey-Predator is used for modelling competition in industry, finance and business etc. In this article, we employ a logistic Prey-Predator model to explain competition between the two retailers. While modelling such problem, collecting parameters and/or initial conditions may imprecise so, fuzzy theory gives more realistic depiction. We solve this nonlinear model taking initial condition fuzzy using Fuzzy Laplace transform technique and give closed form fuzzy solution. We also give the existence of fuzzy solution and condition for stability of the system. Lastly, a numerical illustration is solved and compared at core.

KEYWORDS: Logistic Prey-Predator model, Nonlinear differential equations, Triangular Fuzzy Number, Modified Hukuhara derivative, Stability analysis, Taylor's theorem

MSC: 34A07

RESUMEN
La interacción de las especies presa y depredadora es un proceso biológico con profundas implicaciones ecológicas que ocurre en todo el mundo. La dinámica Prey-Predator se utiliza para modelar la competencia en la industria, las finanzas y los negocios, etc. En este artículo, empleamos un modelo logístico Prey-Predator para explicar la competencia entre dos distribuidores minoristas. Al modelar dicho problema, la recopilación de parámetros y/o condiciones iniciales puede ser imprecisa, por lo que la teoría difusa brinda una descripción más realista. Resolvemos este modelo no lineal tomando la condición inicial difusa usando la técnica de transformada difusa de Laplace y damos una solución difusa de forma cerrada. También damos la existencia de solución difusa y condición de estabilidad del sistema. Por último, se resuelve una ilustración numérica y se compara en el núcleo.

PALABRAS CLAVE: modelo logístico presa-depredador, ecuaciones diferenciales no lineales, número borroso triangular, derivada de Hukuhara modificada, análisis de estabilidad, teorema de Taylor

1. INTRODUCTION

Prey-Predator model is used in modelling the real-life problems like ecological, finance, engineering etc. In this article, using the Prey-Predator model, we for the first time quantitatively simulate competition between two retailers in accordance with our study. The goal of this article is to use a logistic model to explain the competition between two retailers. The following important interesting and significant questions are examined in this article:

- How do the competitive system between two retailers eves over time under fuzzy initial condition?
- What are the parameter conditions under which, this real-life model is stable?
- Why this model is more appropriate in fuzzy environment than the crisp one?

This section contains two subsections.

- The development of Prey-Predator model.
- Use of Prey-Predator model to formulate competition between two retailers.

1.1. Development in the Prey-Predator Model

The Prey-Predator model is useful for simulating physical problems in a variety of fields, including finance, biology, environmental studies, and engineering. The following is a basic overview of the Prey-Predator model.

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When prey (x) and predator (y) populations are not from the same domain, prey increases with growth rate a in the absence of predator population, and predator population drops with decay rate c in the absence of prey population. This phenomenon is mathematically expressed as follows,

$$\frac{dx}{dt} = ax\,, \qquad \frac{dy}{dt} = -cy, \tag{1}$$

with the initial conditions, $x(0) = x_0, y(0) = y_0$.
The system in equation (1) is the simplest model that describes exponentially growth and decay of two species that contradicts the natural behaviour of any population.
Now, when these two populations interact then the Lotka–Volterra model (1910) [5] is used. It is often known as the predator–prey equations, are a pair of first-order nonlinear differential equations that are frequently used to explain the dynamics of biological systems in which two species interact as predator and prey. The populations vary throughout time as a result of the following equations:

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -cy + dxy, \tag{2}$$

with the initial conditions, $x(0) = x_0, y(0) = y_0$.
Where, b is the decay rate at which prey population decreases and d is the growth rate at which predator has enough food.
This is the classical model, where prey and predator behave periodically. This behaviour is also absurd. Because, population of any species must come to a saturation point after that population is neither increasing nor decreasing. So, taking care of this, the more appropriate behaviour of Prey Predator model is given in logistic form as follows,

$$\begin{aligned} \dot{x}(t) &= ax\left(1 - \frac{x}{N_1}\right) - bxy \\ y(t) &= -cy\left(1 + \frac{y}{N_2}\right) + dxy \end{aligned} \tag{3}$$

with initial condition, $x(0) = x_0$ & $y(0) = y_0$.

where, a is the growth rate of prey population, b is the decay rate of prey when it interacts with predator and N_1 is the carrying capacity of prey. Similarly, c, d and N_2 are decay rate, growth rate and carrying capacity of a predator respectively.
Letetia *et. al.* (2017) [4], Nikolaiva, *et. al.* (2019) [6], W. Jirong *et. al.* (2020) [13], P. Mishra *et.al* (2020) [10], N. K Raid *et.al* (2018) [3], V. Laxmi (2020) [12] and Pandit *et. al* (2021) [7] have solved the Prey-Predator model using different approaches in crisp environment. Akin *et. al.* (2012) [1], Pandit *et. al* (2014) [8], Ahmed *et. al* (2009) [2], Pandit *et. al.* (2020) [9] and S. Tudu *et. al*. (2021) [11] have solved the Prey-Predator model in a fuzzy environment.
The following section defines retail competition and provides a mathematical model based on it.

1.2. Mathematical Model of Retail Competition under Fuzzy Environment

The competition between two retailers is the purchase of comparable items from the same supplier at the lowest possible cost, with each retailer attracting an increasing number of customers for their products.
In this article, we extend the model of retail competition using logistic prey-predator model in Yali Yang *et.al.* (2019) [14] under fuzzy environment. The description of model is given as follows.
Let x and y be the similar kind of products of retailers 1 and 2 respectively. The parameters a and c denotes probability of scaling demand of retailer 1 and declined of retailer 2 respectively, b and d denotes the probability of loss in demand of retailer 1 and increasing demand of retailer 2 respectively when both retailers interact. N_1 and N_2 are carrying capacity (maximum number of products in certain time period). When we model such a problem, we can differ the number of products based on demand and other seasonal variations. So, the fuzzy set theory provides a more appropriate explanation for such problems.
Thus, we model this problem in fuzzy environment, taking initial conditions i.e., number of products as fuzzy triangular number. Due to fuzzy initial conditions, the model is given as follows,

$$\dot{\tilde{x}}(t) = a\tilde{x}\left(1 - \frac{\tilde{x}}{N_1}\right) - b\tilde{x}\tilde{y} \tag{4}$$

$$\dot{\tilde{y}}(t) = -c\tilde{y}\left(1+\frac{\tilde{y}}{N_2}\right) + d\tilde{x}\tilde{y}$$

with fuzzy initial conditions, $\tilde{x}(0) = \tilde{x}_0$ & $\tilde{y}(0) = \tilde{y}_0$.
where, $\tilde{x}$ and $\tilde{y}$ are number the products for retailers 1 and 2 respectively.
In this article, first we linearise the equation (4) around its equilibrium point using Taylor's expansion. After that, we use Fuzzy Laplace Transform technique on equation (4) to obtain fuzzy solution in closed form. We also give the stability condition for equation (4).
The next section explains the objective of research.
This article contains 6 sections, introduction, basic concepts, main result, numerical illustration, result and discussion followed by conclusion.
The next section contains some basic concepts, which is used in establishing the related results.

2. FUZZY THEORY BASIC CONCEPTS

Let $E = \{\tilde{u}: R \to [0\ 1] \text{ such that } \tilde{u} \text{ satisfies following properties}\}$

- $\tilde{u}$ is normal.
- $\tilde{u}$ is a fuzzy convex.
- $\tilde{u}$ is upper semicontinuous.
- $\overline{\text{supp } (\tilde{u}) = \{x \in R/\tilde{u}(x) \geq 0\}}$ is compact.

  $\alpha - cut$ is an important tool to convert fuzzy number into the crisp one and it is defined for $0 < \alpha \leq 1$, as ${}^{\alpha}\tilde{u} = \{ x \in R/u(x) \geq \alpha\}$. E be the collection of fuzzy numbers.
  A fuzzy number in parametric form, obtained by performing $\alpha - cut$, is an ordered pair of the form ${}^{\alpha}\tilde{A} = [\underline{A}, \overline{A}]$, satisfying the following conditions:

- $\underline{A}$ is bounded left continuous increasing function in $[0\ 1]$.
  - $\overline{A}$ is bounded right continuous decreasing function in $[0\ 1]$.
  - $\underline{A} \leq \overline{A}$.

The triangular fuzzy number is denoted as (l, m, n), and its membership function $\tilde{A}(x)$, is given as

$$\tilde{A}(x) = \begin{cases} \dfrac{x-l}{m-l}, & l < x \leq m \\ \dfrac{n-x}{n-m}, & m < x \leq n \\ 0 & otherwise \end{cases}$$

and it's $\alpha - cut$, is given as, ${}^{\alpha}\tilde{A} = [\underline{A}, \overline{A}] = [l + (m-l)\alpha, n - (n-m)\alpha]$.
The operations on fuzzy numbers are defined as follows,

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers and λ be any scalar. The scalar multiplication of fuzzy number is given as,

- $\lambda\, {}^{\alpha}\tilde{A} = \lambda[\underline{A}, \overline{A}] = [\lambda\underline{A}, \lambda\overline{A}]$.

  Also, the arithmetic operations between $\tilde{A}$ and $\tilde{B}$ is defined using their parametric form, as follows.

- ${}^{\alpha}\tilde{A} + {}^{\alpha}\tilde{B} = [\underline{A}, \overline{A}] + [\underline{B}, \overline{B}] = [\underline{A} + \underline{B}, \overline{A} + \overline{B}]$
- ${}^{\alpha}\tilde{A}\ {}^{\alpha}\tilde{B} = [\underline{A}, \overline{A}][\underline{B}, \overline{B}] = [\min(\underline{A}\,\underline{B}, \underline{A}\,\overline{B}, \overline{A}\,\underline{B}, \overline{A}\,\overline{B}), \max(\underline{A}\,\underline{B}, \underline{A}\,\overline{B}, \overline{A}\,\underline{B}, \overline{A}\,\overline{B})]$

The derivative is used in establishing the result is given as follows,

A function $\tilde{f}: (a, b) \to E$ is said to be modified Hukuhara differentiable (Pandit et. al. 2019) at $t_0 \in (a, b)$ $\exists$ an element $\dot{\tilde{f}}(t_0) \in E$ such that for all $h > 0$ sufficiently small $\exists$ $\tilde{f}(t_0 + h) - \tilde{f}(t_0)$, $\tilde{f}(t_0) - \tilde{f}(t_0 - h)$ should exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0 + h) - \tilde{f}(t_0)}{h} = \lim_{h \to 0-} \frac{\tilde{f}(t_0) - \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

Its equivalent parametric form is given as follows,

$$\lim_{h \to 0+} \frac{{}^{\alpha}\tilde{f}(t_0 + h) \ominus {}^{\alpha}\tilde{f}(t_0)}{h}$$

$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\underline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}\right\},\right.$$
$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\underline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}\right\}\right]$$

$$\lim_{h\to 0-}\frac{{}^{\alpha}\tilde{f}(t_0)\ominus\ {}^{\alpha}\tilde{f}(t_0-h)}{h}=$$
$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\underline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\overline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\overline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\underline{f}(t_0-h)\right)}{h}\right\},\right.$$
$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\underline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\overline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\overline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\underline{f}(t_0-h)\right)}{h}\right\}\right]$$

We apply fuzzy Laplace transform [9] (Pandit *et. al.* 2020) theory on the system as in equation (4) after linearization.
The Taylor's theorem is used for linearization and it's statement is given below,
Let $k \geq 1$ be an integer and let the function $f : R \to R$ be k times differentiable at the point $c \in R$. Then there exists a function $h_k : R \to R$ such that

$$f(x) = f(c) + (x-c)\dot{f}(c) + \frac{(x-c)^2}{2!}\ddot{f}(c) + \frac{(x-c)^3}{3!}\dddot{f}(c) + \cdots + h_k(x)(x-a)^n$$

where, $\lim_{x\to a} h_k(x) = 0$.
The fuzzy Laplace Transform is defined as follows,
Fuzzy valued function $\tilde{f}(t) = [\underline{f}(t), \overline{f}(t)]$in parametric form, is bounded and piecewise continuous on the interval $[0,\infty)$ and suppose that $\tilde{f}(t)e^{-st}$ is improper fuzzy Riemann integrable, then $\int_0^\infty e^{-st}\tilde{f}(t)dt$ is called fuzzy Laplace Transform and it is defined as,

$$\tilde{F}(s) = \mathcal{L}\left(\tilde{f}(t)\right) = \int_0^\infty e^{-st}\tilde{f}(t)dt$$

Taking $\alpha - cut$ on both sides,

$${}^{\alpha}\tilde{F}(s) = \int_0^\infty e^{-st}\ {}^{\alpha}\tilde{f}(t)dt$$

$$\mathcal{L}\left(\left[\underline{f}(t), \overline{f}(t)\right]\right) = \lim_{t\to\infty}\left[\int_0^t e^{-st}\underline{f}(t)\,dt, \int_0^t e^{-st}\overline{f}(t)dt\right]$$

Fuzzy inverse Laplace Transform is defined as,

$$\mathcal{L}^{-1}\left[\underline{F}(s), \overline{F}(s)\right] = \left[\underline{f}(t), \overline{f}(t)\right].$$

The derivative of fuzzy Laplace Transform is given as below,

If $\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ be continuous fuzzy valued function, $\lim_{t\to\infty} e^{-st}\underline{f}(t) \to 0$ and $\lim_{t\to\infty} e^{-st}\overline{f}(t) \to 0$ for large value of s and $\dot{\tilde{f}}(t)$ is piecewise continuous then $\mathcal{L}\left(\dot{\tilde{f}}(t)\right)$ exist, and is given by,

$$\mathcal{L}\left(\dot{\tilde{f}}(t)\right) = s\mathcal{L}\left(\tilde{f}(t)\right) - \tilde{f_0}.$$

In next section, we have given the Main result that describes solving technique for system (2).
In next section, we propose and establish the technique for solving the equation (4).

3. MAIN RESULT

For solving system as in equation (4), we linearise the nonlinear term in equation (4) using Taylor's theorem, about the equilibrium point. After that we apply Fuzzy Laplace Transform on linearised system.
We first obtain the equilibrium point by solving, $\dot{\tilde{x}}(t) = 0$, $\dot{\tilde{y}}(t) = 0$ in equation (4) and we obtain two equilibrium points as follows, one is $\begin{pmatrix} x_e \\ y_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the other is$\begin{pmatrix} x_e \\ y_e \end{pmatrix} = \begin{pmatrix} \frac{cN_1(a-bcN_2)}{dbN_1N_2+ac} \\ \frac{aN_2(N_1d-c)}{dbN_1N_2+ac} \end{pmatrix}$.
Now, first we linearise the equation (4) about (0,0) and give the solution using fuzzy Laplace Transform in following sub section.

3.1. Solution of equation (4) about $(\mathbf{0}, \mathbf{0})$

When we consider first equilibrium point $(0,0)$ then system (1) gets converted into following system,

$$\dot{\tilde{x}}(t) = ax \qquad (5)$$
$$\dot{\tilde{y}}(t) = -cy$$

with fuzzy initial condition, $\tilde{x}(0) = \tilde{x}_0, \tilde{y}(0) = \tilde{y}_0$.
Equation (5) is the basic mathematical representation when prey and predator do not interact.
The compact form of equation (5) is given as follows,

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{y}}(t) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \qquad (6)$$

with fuzzy initial condition, $\tilde{x}(0) = \tilde{x}_0, \tilde{y}(0) = \tilde{y}_0$.
The fuzzy solution of the equation (6) using Fuzzy Laplace derivative as in, Section 2.7.

$$\tilde{x}(t) = \tilde{x}_0 e^{at},$$
$$\tilde{y}(t) = \tilde{y}_0 e^{-ct}.$$

Taking $\alpha - cut$ on both sides,

$${}^{\alpha}\tilde{x}(t) = {}^{\alpha}\tilde{x}_0 e^{at},$$
$${}^{\alpha}\tilde{y}(t) = {}^{\alpha}\tilde{y}_0 e^{-ct}.$$

Which gives,

$$[\underline{x}(t), \overline{x}(t)] = [\underline{x}_0, \overline{x}_0] e^{at},$$
$$[\underline{y}(t), \overline{y}(t)] = [\underline{y}_0, \overline{y}_0] e^{-ct}$$

3.2. The solution of equation (4) about $\left(\boldsymbol{x_e} = \frac{\boldsymbol{cN_1(a-bcN_2)}}{\boldsymbol{dbN_1N_2+ac}}, \boldsymbol{y_e} = \frac{\boldsymbol{aN_2(N_1d-c)}}{\boldsymbol{dbN_1N_2+ac}}\right)$

Using Taylor's theorem, we linearise system (2) about equilibrium points. The linearized form of system (2) inving fuzzy initial conditions is given below,

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{y}}(t) \end{bmatrix} = \begin{bmatrix} \left(a - \frac{2a}{N_1}x_e - by_e\right) & -bx_e \\ dy_e & \left(-c - \frac{2y_e c}{N_2} + dx_e\right) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{pmatrix} bx_e y_e - \frac{ax_e^2}{N_1} \\ \frac{cy_e^2}{N_2} - dx_e y_e \end{pmatrix} \qquad (7)$$

with fuzzy initial condition, $\tilde{x}(0) = \tilde{x}_0, \tilde{y}(0) = \tilde{y}_0$.
The existence of fuzzy solution in parametric form of system in equation (7) is given as follows.

3.3. Theorem 1

If $\tilde{x}(t)$ and $\tilde{y}(t)$ are continuous in equation (7) then the solution of equation (7) is given below,

$$[\underline{x}(t), \overline{x}(t)] = \left[\frac{\beta' bx_e}{\gamma_1} + bx_e[\underline{y}_0, \overline{y}_0] + \frac{\beta(\gamma_1 - \alpha')}{\gamma_1} + [\underline{x}_0, \overline{x}_0](\gamma_1 - \alpha')\right]\frac{e^{\gamma_1 t}}{\gamma_1 - \gamma_2} + \left[\frac{\beta' bx_e}{\gamma_2} + bx_e[\underline{y}_0, \overline{y}_0] + \frac{\beta(\gamma_2 - \alpha')}{\gamma_2} + [\underline{x}_0, \overline{x}_0](\gamma_2 - \alpha')\right]\frac{e^{\gamma_2 t}}{\gamma_2 - \gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta' bx_e - \beta\,\alpha']$$

$$[\underline{y}(t), \overline{y}(t)] = \left[\frac{\beta dy_e}{\gamma_1} + dy_e[\underline{x}_0, \overline{x}_0] + \frac{\beta'(\gamma_1 - \alpha)}{\gamma_1} + [\underline{y}_0, \overline{y}_0](\gamma_1 - \alpha)\right]\frac{e^{\gamma_1 t}}{\gamma_1 - \gamma_2} + \left[\frac{\beta dy_e}{\gamma_2} + dy_e[\underline{x}_0, \overline{x}_0] + \frac{\beta'(\gamma_2 - \alpha)}{\gamma_2} + [\underline{y}_0, \overline{y}_0](\gamma_2 - \alpha)\right]\frac{e^{\gamma_2 t}}{\gamma_2 - \gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta dy_e - \beta'\alpha]$$

where,

$$\alpha = \left(a - \frac{2a}{N_1}x_e - by_e\right), \beta = \left(bx_e y_e - \frac{ax_e^2}{N_1}\right)$$
$$\alpha' = \left(-c - \frac{2y_e c}{N_2} + dx_e\right), \beta' = \left(\frac{cy_e^2}{N_2} - dx_e y_e\right)$$
$$\gamma_1 = \frac{(\alpha + \alpha') + \sqrt{(\alpha + \alpha')^2 - 4(\alpha\alpha' - bdx_e y_e)}}{2}$$
$$\gamma_2 = \frac{(\alpha + \alpha') - \sqrt{(\alpha + \alpha')^2 - 4(\alpha\alpha' - bdx_e y_e)}}{2}$$

Proof:

For our convenience, we take following notations,

$$\delta = \left(a - \frac{2a}{N_1}x_e - by_e\right), \beta = \left(bx_e y_e - \frac{ax_e^2}{N_1}\right)$$

$$\delta' = \left(-c - \frac{2y_e c}{N_2} + dx_e\right), \beta' = \left(\frac{cy_e^2}{N_2} - dx_e y_e\right)$$

Using these notations, equation (7) gets converted into following system,

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{y}}(t) \end{bmatrix} = \begin{bmatrix} \delta & -bx_e \\ dy_e & \delta' \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{pmatrix} \beta \\ \beta' \end{pmatrix} \tag{8}$$

with fuzzy initial condition, $\tilde{x}(0) = \tilde{x}_0, \tilde{y}(0) = \tilde{y}_0$.

To solve equation (8), Take fuzzy Laplace Transform on equation (8),

$$\mathcal{L}\left(\dot{\tilde{x}}(t)\right) = \delta\mathcal{L}(\tilde{x}) - bx_e\mathcal{L}(\tilde{y}) + \mathcal{L}(\beta)$$

$$L(\dot{\tilde{y}}(t)) = \delta'\mathcal{L}(\tilde{y}) + dy_e\mathcal{L}(\tilde{x}) + \mathcal{L}(\beta')$$

Using fuzzy Laplace Transform derivative as in, Section 2.7,

$$s\tilde{x}(s) - \tilde{x}_0 = \delta\tilde{x}(s) - bx_e\tilde{y}(s) + \frac{\beta}{s} \tag{9}$$

$$s\tilde{y}(s) - \tilde{y}_0 = \delta'\tilde{y}(s) + dy_e\tilde{x}(s) + \frac{\beta'}{s} \tag{10}$$

Now using elimination method on equations (11) and (12), and taking inverse fuzzy Laplace Transform we have,

$$\tilde{x}(t) = \left[\frac{\beta' bx_e}{\gamma_1} + bx_e\tilde{y}_0 + \frac{\beta(\gamma_1-\delta')}{\gamma_1} + \tilde{x}_0(\gamma_1 - \delta')\right]\frac{e^{\gamma_1 t}}{\gamma_1-\gamma_2} + \left[\frac{\beta' bx_e}{\gamma_2} + bx_e\tilde{y}_0 + \frac{\beta(\gamma_2-\delta')}{\gamma_2} + \tilde{x}_0(\gamma_2 - \delta')\right]\frac{e^{\gamma_2 t}}{\gamma_2-\gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta' bx_e - \beta\,\delta'], \tag{11}$$

$$\tilde{y}(t) = \left[\frac{\beta dy_e}{\gamma_1} + dy_e\tilde{x}_0 + \frac{\beta'(\gamma_1-\delta)}{\gamma_1} + \tilde{y}_0(\gamma_1 - \delta)\right]\frac{e^{\gamma_1 t}}{\gamma_1-\gamma_2} + \left[\frac{\beta dy_e}{\gamma_2} + dy_e\tilde{x}_0 + \frac{\beta'(\gamma_2-\delta)}{\gamma_2} + \tilde{y}_0(\gamma_2 - \delta)\right]\frac{e^{\gamma_2 t}}{\gamma_2-\gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta dy_e - \beta'\delta] \tag{12}$$

where,

$$\gamma_1 = \frac{(\delta + \delta') + \sqrt{(\delta + \delta')^2 - 4(\delta\delta' - bdx_e y_e)}}{2}$$

$$\gamma_2 = \frac{(\delta + \delta') - \sqrt{(\delta + \delta')^2 - 4(\delta\delta' - bdx_e y_e)}}{2}$$

Taking $\alpha - cut$ of equations (11) and (12), we get,

$$^{\alpha}\tilde{x}(t) = \left[\frac{\beta' bx_e}{\gamma_1} + bx_e\,^{\alpha}\tilde{y}_0 + \frac{\beta(\gamma_1-\delta')}{\gamma_1} + \,^{\alpha}\tilde{x}_0(\gamma_1 - \delta')\right]\frac{e^{\gamma_1 t}}{\gamma_1-\gamma_2} + \left[\frac{\beta' bx_e}{\gamma_2} + bx_e\,^{\alpha}\tilde{y}_0 + \frac{\beta(\gamma_2-\delta')}{\gamma_2} + \,^{\alpha}\tilde{x}_0(\gamma_2 - \delta')\right]\frac{e^{\gamma_2 t}}{\gamma_2-\gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta' bx_e - \beta\,\delta'],$$

$$^{\alpha}\tilde{y}(t) = \left[\frac{\beta dy_e}{\gamma_1} + dy_e\,^{\alpha}\tilde{x}_0 + \frac{\beta'(\gamma_1-\delta)}{\gamma_1} + \,^{\alpha}\tilde{y}_0(\gamma_1 - \delta)\right]\frac{e^{\gamma_1 t}}{\gamma_1-\gamma_2} + \left[\frac{\beta dy_e}{\gamma_2} + dy_e\,^{\alpha}\tilde{x}_0 + \frac{\beta'(\gamma_2-\delta)}{\gamma_2} + \,^{\alpha}\tilde{y}_0(\gamma_2 - \delta)\right]\frac{e^{\gamma_2 t}}{\gamma_2-\gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta dy_e - \beta'\delta].$$

Which gives the parametric form as follows,

$$\left[\underline{x}(t), \overline{x}(t)\right] = \left[\frac{\beta' bx_e}{\gamma_1} + bx_e\left[\underline{y}_0, \overline{y}_0\right] + \frac{\beta(\gamma_1-\delta')}{\gamma_1} + \left[\underline{x}_0, \overline{x}_0\right](\gamma_1 - \delta')\right]\frac{e^{\gamma_1 t}}{\gamma_1-\gamma_2} + \left[\frac{\beta' bx_e}{\gamma_2} + bx_e\left[\underline{y}_0, \overline{y}_0\right] + \frac{\beta(\gamma_2-\delta')}{\gamma_2} + \left[\underline{x}_0, \overline{x}_0\right](\gamma_2 - \delta')\right]\frac{e^{\gamma_2 t}}{\gamma_2-\gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta' bx_e - \beta\,\delta'],$$

$$\left[\underline{y}(t), \overline{y}(t)\right] = \left[\frac{\beta dy_e}{\gamma_1} + dy_e\left[\underline{x}_0, \overline{x}_0\right] + \frac{\beta'(\gamma_1-\delta)}{\gamma_1} + \left[\underline{y}_0, \overline{y}_0\right](\gamma_1 - \delta)\right]\frac{e^{\gamma_1 t}}{\gamma_1-\gamma_2} + \left[\frac{\beta dy_e}{\gamma_2} + dy_e\left[\underline{x}_0, \overline{x}_0\right] + \frac{\beta'(\gamma_2-\delta)}{\gamma_2} + \left[\underline{y}_0, \overline{y}_0\right](\gamma_2 - \delta)\right]\frac{e^{\gamma_2 t}}{\gamma_2-\gamma_1} + \frac{1}{\gamma_1\gamma_2}[\beta dy_e - \beta'\delta].$$

As, we know that $\underline{x}_0 \leq \overline{x}_0$ and $\underline{y}_0 \leq \overline{y}_0$ and the parameters are used in above equations are positive.

Thus, the obtained fuzzy solution of equation (7) exists.

In next section, we give stability condition for equation (4) about equilibrium points.

4. STABILITY ANALYSIS

For stability, we linearize the equation (4), around the both equilibrium points (0,0) and $\left(\frac{cN_1(a-bcN_2)}{dbN_1N_2+ac}, \frac{aN_2(N_1d-c)}{dbN_1N_2+ac}\right)$ and obtain linearized equations (6) and (7) respectively.

In equation (6), for stability, we find eigenvalues of $\begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$.

i.e., $\lambda_{1,2} = a, -c$

Thus, at (0,0), equation (6) is unstable.

The system in equation (7), is stable if the roots of $\begin{bmatrix} \delta & -bx_e \\ dy_e & \delta' \end{bmatrix}$ are imaginary and real part is negative. To establish stability condition, we take some assumptions which is stated in following theorem.

4.1. Theorem 2

The system in equation (7) is stable, if following conditions are satisfied,

1) $\frac{x_e}{y_e} < \frac{\left(b+\frac{2c}{N_2}\right)}{\left(d-\frac{2a}{N_1}\right)}$ for $c > a$.
2) Either, $\frac{a}{bc} > N_2, N_1 > \frac{c}{d}$ or $\frac{a}{bc} < N_2, N_1 < \frac{c}{d}$ condition satisfies.

Proof:

For the stability analysis, first we obtain eigen values $\lambda_{1,2}$ of equation (7) as follows,

$$\begin{bmatrix} \lambda-\alpha & -bx_e \\ dy_e & \lambda-\alpha' \end{bmatrix} = 0$$

$$\therefore (\lambda-\delta)(\lambda-\delta') + bdx_ey_e = 0$$

$$\therefore \lambda^2 - (\delta+\delta')\lambda + \delta\delta' + bdx_ey_e = 0$$

$$\therefore \lambda_{1,2} = \frac{(\delta+\delta') \pm \sqrt{(\delta+\delta')^2 - 4(bdx_ey_e+\delta\delta')}}{2}$$

The equation (7) is stable if roots are imaginary and real part is negative.

i.e., $(\delta+\delta')^2 - 4(bdx_ey_e+\delta\delta') < 0$ and $(\delta+\delta') < 0$.

Now take another inequality, $(\delta+\delta') < 0$.

Put these values of $\delta = \left(a - \frac{2a}{N_1}x_e - by_e\right)$ and $\delta' = \left(-c - \frac{2y_ec}{N_2} + dx_e\right)$ in $(\delta+\delta') < 0$.

Then, we have

$$a - \frac{2a}{N_1}x_e - by_e - c - \frac{2y_ec}{N_2} + dx_e < 0$$

According to first condition in theorem 2, $c > a$ which gives,

$$x_e\left(d - \frac{2a}{N_1}\right) - y_e\left(b + \frac{2c}{N_2}\right) < c - a,$$

$$\therefore x_e\left(d - \frac{2a}{N_1}\right) - y_e\left(b + \frac{2c}{N_2}\right) < 0.$$

Which gives,

$$\frac{x_e}{y_e} < \frac{\left(b+\frac{2c}{N_2}\right)}{\left(d-\frac{2a}{N_1}\right)}. \quad (13)$$

Now for second condition, we take following inequality,

$$\therefore \quad (\delta-\delta')^2 - 4\,bdx_ey_e < 0$$

$$\therefore \quad (\delta-\delta')^2 < 4\,bdx_ey_e$$

Since, $(\delta-\delta')^2$ is positive term due to square.

$$\therefore bdx_ey_e > 0 \Rightarrow x_ey_e > 0.$$

Now, putting value of $x_e = \frac{cN_1(a-bcN_2)}{dbN_1N_2+ac}, y_e = \frac{aN_2(N_1d-c)}{dbN_1N_2+ac}$ in $x_ey_e > 0$.

$$\therefore \frac{cN_1(a-bcN_2)}{(dbN_1N_2+ac)} \frac{aN_2(N_1d-c)}{(dbN_1N_2+ac)} > 0$$

$$\therefore cN_1(a-bcN_2)\,aN_2(N_1d-c) > 0$$

From above inequality, we have two possibilities,

Either, $(a-bcN_2) > 0$ and $(N_1d-c) > 0$ or $(a-bcN_2) < 0$ and $(N_1d-c) < 0$.

Which gives,

$$\frac{a}{bc} > N_2, N_1 > \frac{c}{d} \text{ or } \frac{a}{bc} < N_2, N_1 < \frac{c}{d}. \quad (14)$$

Equations (13) and (14) together gives stability condition for equation (7).

In next section, we solve a numerical illustration using proposed technique.

5. NUMERICAL ILLUSTRATION

We solve following illustration from [14] by proposed technique as in, Section 3.

$$\dot{\tilde{x}}(t) = \tilde{x}\left(1 - \frac{\tilde{x}}{100}\right) - 0.2\tilde{x}\tilde{y}$$
$$\dot{\tilde{y}}(t) = -0.8\,\tilde{y}\left(1 + \frac{\tilde{y}}{25}\right) + 0.04\tilde{x}\tilde{y}$$

with fuzzy initial condition,

$${}^{\alpha}\tilde{x}(0) = {}^{\alpha}\widetilde{40} = [35 + 5\alpha\,, 45 - 5\alpha] \text{ and } {}^{\alpha}\tilde{y}(0) = {}^{\alpha}\widetilde{15} = [10 + 5\alpha,\ 20 - 5\alpha].$$

First, we linearize the system using Taylor's theorem,
The above system gets converted as follows,

$$\dot{\tilde{x}}(t) = -0.2294\,\tilde{x} \ominus 4.614\tilde{y} \oplus 23.04 \tag{15}$$

$$\dot{\tilde{y}}(t) = -0.768\,\tilde{x} \ominus 5.65976\tilde{y} \oplus 17.71778 \tag{16}$$

with fuzzy initial condition,

$${}^{\alpha}\tilde{x}(0) = {}^{\alpha}\widetilde{40} = [35 + 5\alpha\,, 45 - 5\alpha] \text{ and } {}^{\alpha}\tilde{y}(0) = {}^{\alpha}\widetilde{15} = [10 + 5\alpha,\ 20 - 5\alpha].$$

Taking Fuzzy Laplace transform on equations (15),

$$\mathcal{L}\left(\dot{\tilde{x}}(t)\right) = -0.2294\,\mathcal{L}(\tilde{x}) \ominus 4.614\mathcal{L}(\tilde{y}) \oplus \mathcal{L}(23.04)$$
$$s\tilde{x}(s) \ominus \tilde{x}(0) = -0.2294\,\tilde{x}(s) \ominus 4.614\tilde{y}(s) \oplus \frac{(23.04)}{s}$$
$$s\tilde{x}(s) \oplus 0.2294\,\tilde{x}(s) = \ominus\, 4.614\tilde{y}(s) \oplus \frac{(23.04)}{s} \tag{17}$$

Taking Fuzzy Laplace transform on equations (16),

$$\mathcal{L}\left(\dot{\tilde{y}}(t)\right) = -0.768\,\mathcal{L}(\tilde{x}) \ominus 5.65976\,\mathcal{L}(\tilde{y}) \oplus \mathcal{L}(17.71778)$$
$$s\tilde{y}(s) \ominus \tilde{y}(0) = \ominus\, 4.614\tilde{y}(s) \oplus \frac{17.71778}{s}$$
$$s\tilde{y}(s) \oplus 4.614\tilde{y}(s) = \tilde{y}(0) \oplus \frac{17.71778}{s} \tag{18}$$

After solving equations (17) and (18), we have the graph of above system at fuzzy initial conditions as follows,

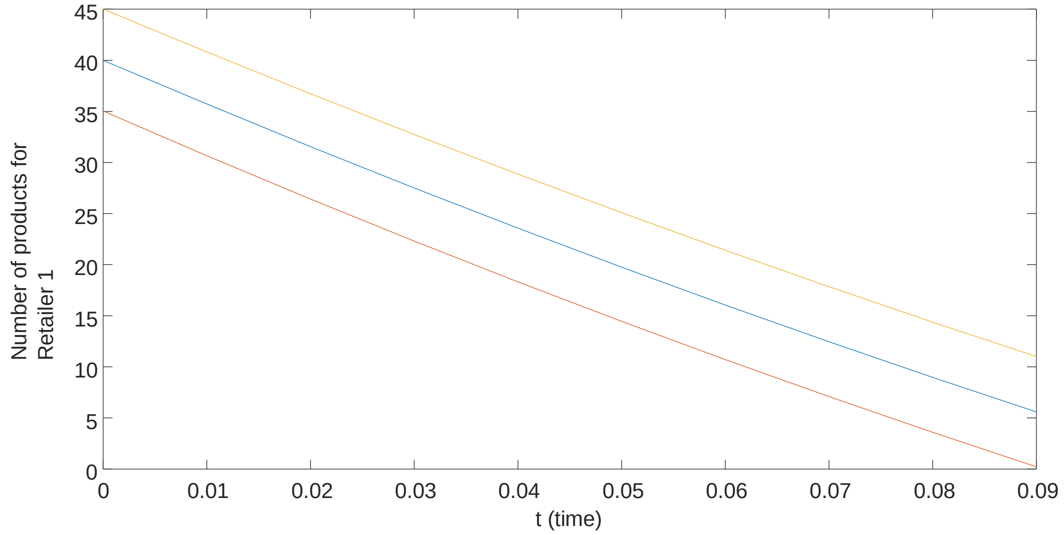

Fig.1: Variation in products for Retailer 1 at fuzzy initial conditions (35, 40, 45)

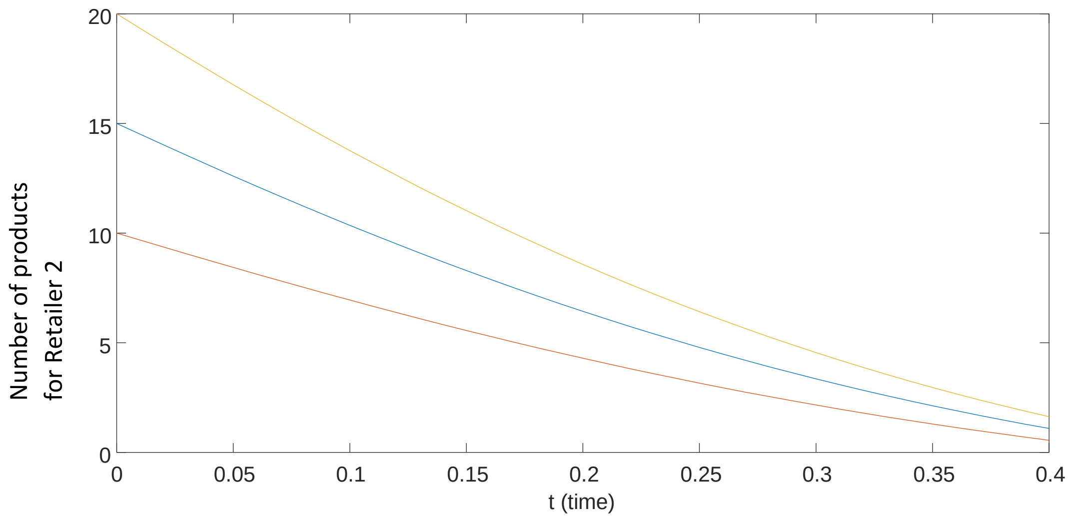

Fig.2: Variation in products for Retailer 2 at fuzzy initial conditions (10, 15, 20)

The following figures 3 and 4 shows the behaviour of equations (15) and (16) at small time interval and large time interval in 3-dimension respectively.

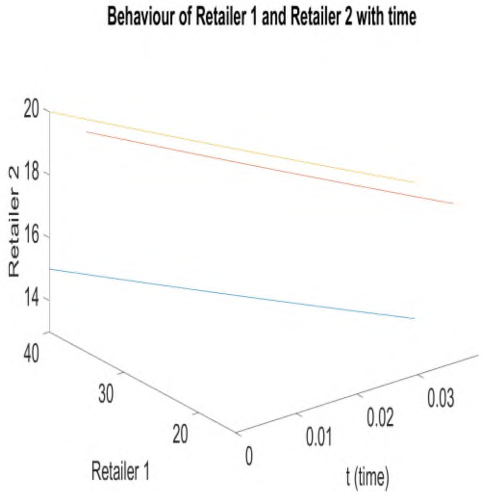

Fig.3 Behaviour of Retailer 1 and Retailer 2 at fuzzy initial conditions

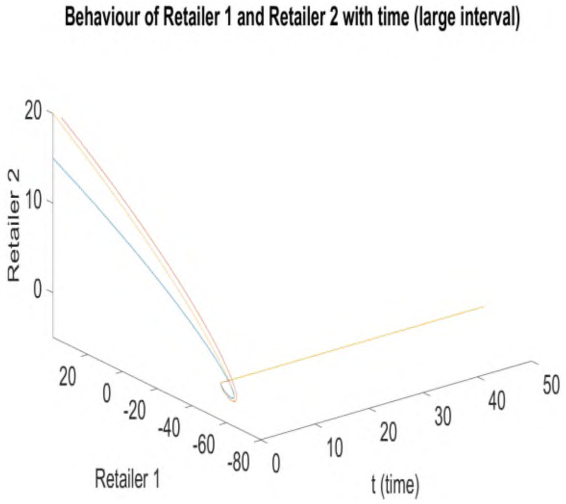

Fig: 4 Behaviour of Retailer 1 and Retailer 2 at different fuzzy initial conditions

6. RESULT AND DISCUSSION

From the fig.1 and fig.2, we observe that number of purchases of similar products of retailer 1 and retailer 2 decrease as time increases and after sometime this change becomes constant. The proposed method is only applicable for a small time, interval. Fig.3 represents linear behaviour between retailer 1 and 2 in 3-dimension. In a large time, interval, the products become negative as in fig.4. The fuzzy solution of the system matches with solution of its crisp system at the core, that validates the proposed technique.

7. CONCLUSION

In this article, the competition between two retailers for similar products, is studied in the fuzzy environment. The interaction between two retailers affects each other's business.We have solved system of nonlinear fuzzy differential equations, using fuzzy Laplace transform. We linearised the nonlinear term about equilibrium point then applied the technique which gives us fuzzy solution in compact form. We have also given the condition for stability analysis of the model. Lastly, one numerical illustration is solved which is compared at the core. Our proposed technique has limitation that it is valid only for small time interval. For future scope, one can take all parameters fuzzy and add more parameters like trade credit and demand to give optimise condition for both retailers.

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REFERENCES

[1] AKIN, O. and ORUC, O. (2012): A Prey Predator Model with Fuzzy Initial Values, **Hacettepe Journal of Mathematics and Statistics**. pkpandit@yahoo.compkpandit@yahoo.com 41 , 387-395.
[2] AHMAD M. Z. and BAEST, B. DE. (2009): A Prey Predator Model with Fuzzy Initial Populations. ***IFSA-EUSFLAT,*** Editor, Paulo Carvahlo, Didier Dubois, Ujay Kaymak and Miguel da Costa Sousa**,** 1311-1314.
[3] AHMED A. and RAID, N.K. (2018): A Study of a Diseased Prey-Predator Model with Refuge in Prey and Harvesting from Predator, **Journal of Applied Mathematics**, 2, 1-17.
[4] LETETIA ADDISON, B. BISHWAROOP and O. DAVID, (2017). A financial Prey-Predator model with infection in predator, **Journal of Advances in Mathematics and Computer Science** 25 (6),1-16.
[5] LOTKA, A. J. (1910): Contribution to the Theory of Periodic Reaction**, J. Phys. Chem.** 14, 271-274.doi:10.1021/j150111a004.
[6] NIKOLAIEVA, O. and BOCHKO, Y. (2019): Application of the “Predator-Prey” Model for Analysis and Forecasting the share of the Market of Mobile Operating Systems, **International Journal of Innovative Technologies in Economy**, 4. DOI: https://doi.org/10.31435/rsglobal_ijite/30062019/6527
[7] PANDIT, P., SINGH, P. and PATEL, T. (2021): Analysis of Prey-Predator model, **Differential Equations in Engineering,** 1st edition, eBook ISBN 9781003105145, 107-124.
[8] PANDIT, P. and SINGH, P. (20.14): Prey-Predator model and fuzzy initial condition, **International Journal of Engineering and Innovative Technology (IJEIT):** 3, 65-68.
[9] PANDIT, P. and SINGH, P (2020): Fully Fuzzy Semi-Linear Dynamical System Solved by Fuzzy Laplace Transform Under Modified Hukuhara Derivative. Ed. Das K., Bansal J., Deep K., Nagar A., Pathipooranam P., Naidu R. (Soft Computing for Problem Solving**) Advances in Intelligent Systems and Computing,** Springer, Singapore. 1048.
[10] RAW, S.N., MISHRA, P. and TIWARI, B. (2020): Mathematical Study About a Predator–Prey Model with Anti-predator Behavior. **Int. J. Appl. Comput. Math**, 6, 68.
[11] TUDU, S., MONDAL N. and ALAM, S. (2020): Dynamics of the Logistic Prey Predator Model in Crisp and Fuzzy Environment. In: Roy, P., Cao, X., Li, XZ., Das, P., Deo, S. (eds) **Mathematical Analysis and Applications in Modeling.** ICMAAM 2018. Springer Proceedings in Mathematics & Statistics, 302. Springer, Singapore. https://doi.org/10.1007/978-981-15-0422-8_37.
[12] VIJYA LAXMI, G. M, (2020): Effect of herd behaviour prey-predator model with competition in predator, **Materials today proceedings,** Volume 33, Part 7,2020, Pages 3197-3200, ISSN 2214-7853, https://doi.org/10.1016/j.matpr.2020.04.166.
[13] JINRONG, W. and MICHEL J. F., (2020): Dynamics of a Discrete Nonlinear Prey-Predator model, **International Journal of Bifurcation and Chaos**, 30, 2050055-1 to 2050055-15.
[14] YANG YALI, ZHIWEI LI, FEI WANG (2019): Global Stability of a Predator-Prey Model with the Logistic Term, **IOP Conf. Ser.: Mater. Sci. Eng.** 611 012027.