APPLICATION OF DOUBLE POWER SERIES METHOD FOR SOLVING NONLINEAR BURGER'S EQUATION ARISING IN LONGITUDINAL DISPERSION PHENOMENA

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ABSTRACT
This article analyzes an approximate solution of the nonlinear Burger's equation arising in longitudinal dispersion phenomena. The solution has been obtained by using Double Power Series Method (DPSM) which represents the concentration C of fluid for any distance X and any time T. The numerical as well as graphical solutions are obtained by MATLAB coding and physically interpreted well.

KEYWORDS: Nonlinear Burger's Equation, Longitudinal Dispersion Phenomena, Double Power Series Method.

MSC: 40C15, 35C99, 76B99

RESUMEN
Este artículo analiza una solución aproximada de la ecuación de Burger no lineal el que surge en fenómenos de dispersión longitudinal. La solución se obtuvo utilizando el método de la Serie de Doble Potencia (DPSM), que representa la concentración C de fluido para cualquier distancia X y cualquier tiempo T. Las soluciones numéricas y gráficas se obtienen mediante la codificación de MATLAB y se interpretan bien físicamente.

PALABRAS CLAVE: Ecuación de Burger no lineal, Fenómenos de dispersión longitudinal, Método de Series de Doble Potencia.

1. INTRODUCTION

Burger's equation is the fundamental model equation appearing in different fields of applied mathematics like traffic flow, fluid mechanics, gas dynamics, etc. In 1948, Burger introduced the interaction of two physical transport phenomenon diffusion and convection. Navier-Stokes equation and Burger's equation are analogous in the form of nonlinearities and the occurrence of higher-order derivatives with small coefficients.
Dispersion is the process by which miscible fluids in laminar flow mix in the direction of the flow. Dispersion is the mixing of solute in a fluid flow through porous media which combines the diffusive and advective transport phenomena. Dispersion phenomena occur in oil reservoir engineering, in chemical engineering, in many problems of groundwater flow, etc.
The study of dispersion phenomena in fluid flow through porous media is of great importance for petroleum engineering, chemical engineering, soil science, hydrologists, etc. This problem has been discussed by many researchers with different viewpoints and used different methods such as Adomian Decomposition Method (ADM) **[7]**, Variational Iteration Method (VIM) **[8]**, Homotopy Analysis Method (HAM) **[11, 12, 14]**, Variational Homotopy Perturbation Method (VHPM) **[3]**, New Integral Transform Homotopy Perturbation Method (NITHPM) **[17]**, Sumudu Transform Homotopy Perturbation Method (STHPM) **[19]**, Finite Difference Method (FDM) **[2]**, Modified Variational Iteration Method (MVIM) **[10]**, Laplace Transform Method (LTM) **[13]** and Modified Homotopy Analysis Method (MHAM) **[18]** to obtain a numerical and analytical solutions.
The aim of this paper is to use the Double Power Series Method (DPSM) **[5]** to achieve an approximate solution of the nonlinear Burger's equation arising in longitudinal dispersion phenomena. Using DPSM, we get an infinite series solution. Other authors have studied DPSM **[9, 16]** to solve different kinds of NPDEs.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

In the case of incompressible fluids, the mass conservation law for the mixture **[1]** is given by

$$\rho_t + \nabla \cdot (\rho\, \vec{u}) = 0, \tag{1}$$

where ρ is the density and $\vec{u}$ is seepage velocity of the mixture.

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The diffusion equation for fluid dynamics in porous medium, without changing the dispersing material **[15]** is given by

$$C_t + \nabla \cdot (C\,\vec{u}) = \nabla \cdot [\rho\,\bar{D}\nabla(C\,\rho^{-1})], \tag{2}$$

where C is the concentration of the fluids and $\bar{D}$ is the tensor coefficients of dispersion with components D_{ij}.
If density ρ is constant in laminar flow for an incompressible fluid through homogeneous porous medium, then

$$\nabla \cdot \vec{u} = 0. \tag{3}$$

Therefore equation (2) can be rewritten as

$$C_t + \vec{u} \cdot (\nabla C) = \nabla \cdot [\bar{D}\nabla C]. \tag{4}$$

Assuming that the seepage velocity $\vec{u}$ is along x-axis, the coefficient of longitudinal dispersion $D_{11} \approx D_L$ and D_{ij} is zero for $i, j \neq 1$ **[15]**. Now equation (4) can be written as

$$C_t + u_1\,C_x = \mathrm{D_L}\,C_{xx}, \tag{5}$$

where u_1 is the component of seepage velocity in x direction which is time dependent as well as concentration along the x-axis in $x \geq 0$ direction, and $\mathrm{D_L} > 0$. According to **[6]**, we assume that $u_1 = C_0^{-1}C(x,t)$. Substituting the value of u_1 in equation (5), we obtain

$$C_t + C_0^{-1}C\,C_x = \mathrm{D_L}\,C_{xx}, \tag{6}$$

To convert equation (6) in dimensionless form, we choose

$$X = \frac{C_0 x}{L}, T = \frac{t}{L}, \gamma = \frac{D_L C_0^2}{L} \tag{7}$$

$$\Rightarrow X_x = \frac{C_0}{L}, T_t = \frac{1}{L}$$

Now,

$$C_t = C_T \cdot T_t = C_T \cdot \frac{1}{L}, C_x = C_X \cdot X_x = C_X \cdot \frac{C_0}{L}$$

and

$$C_{xx} = (C_x)_x = (C_x)_X \cdot X_x = \left(C_X \cdot \frac{C_0}{L}\right)_X \cdot \frac{C_0}{L} = \frac{C_0^2}{L^2} \cdot C_{XX}$$

Equation (6), reduces to

$$C_T \cdot \frac{1}{L} + C_0^{-1}C\,C_X \cdot \frac{C_0}{L} = \mathrm{D_L}\,\frac{C_0^2}{L^2} \cdot C_{XX}$$

$$\therefore C_T + C\,C_X = \frac{\mathrm{D_L} C_0^2}{L} \cdot C_{XX}$$

Hence,

$$C_T = \gamma\,C_{XX} - C\,C_X \tag{8}$$

with appropriate initial and boundary conditions are as follows

$$C(X, 0) = e^{-X}, X \geq 0 \tag{9}$$

$$C(0, T) = 1, T \geq 0 \tag{10}$$

Equation (8) is known as nonlinear Burger's equation arising in longitudinal dispersion phenomena.

3. IMPLEMENTATION OF DOUBLE POWER SERIES METHOD

Let double power series solution of equation (8) be of the form

$$C(X,T) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\,X^m\,T^n \tag{11}$$

Differentiating equation (11) partially with respect to X and T, we get double series expansion of C_X and C_T are as follows:

$$C_X = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+1)a_{(m+1)n}\,X^m\,T^n \tag{12}$$

$$C_T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(n+1)a_{m(n+1)}\,X^m\,T^n \tag{13}$$

Again differentiating equation (12) partially with respect to X, we get

$$C_{XX} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+1)(m+2)a_{(m+2)n}\,X^m\,T^n \tag{14}$$

Also the double series expansion of $C\,C_x$ as follow

$$C\, C_x = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{p=0}^{n}\sum_{q=0}^{m}(m-q+1)\, a_{qp}\, a_{(m-q+1)(n-p)}\right] X^m\, T^n \tag{15}$$

Putting equations (13), (14), and (15) into (8), we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(n+1)a_{m(n+1)}\, X^m\, T^n = \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+1)(m+2)a_{(m+2)n}\, X^m\, T^n - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{p=0}^{n}\sum_{q=0}^{m}(m-q+1)\, a_{qp}\, a_{(m-q+1)(n-p)}\right] X^m\, T^n$$

$$\therefore \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(n+1)a_{m(n+1)}\, X^m\, T^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\gamma(m+1)(m+2)a_{(m+2)n} - \sum_{p=0}^{n}\sum_{q=0}^{m}(m-q+1)\, a_{qp}\, a_{(m-q+1)(n-p)}\right] X^m\, T^n$$

Equating the coefficient both the sides, we obtain the following recursion formula

$$(n+1)a_{m(n+1)} = \gamma(m+1)(m+2) - \sum_{p=0}^{n}\sum_{q=0}^{m}(m-q+1)\, a_{qp}\, a_{(m-q+1)(n-p)}, \forall\, m,n \geq 0 \tag{16}$$

The initial condition (9) and boundary condition (10) are transformed as follows

$$\sum_{m=0}^{\infty} a_{m0}\, X^m = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\, X^m,$$

which gives

$$a_{m0} = \frac{(-1)^m}{m!}, \forall\, m \geq 0 \tag{17}$$

and

$$\sum_{n=0}^{\infty} a_{0n}\, T^n = 1 \Rightarrow a_{00} + \sum_{m=0}^{\infty} a_{0n}\, T^n = 1 + \sum_{m=0}^{\infty} 0\, T^n,$$

which gives

$$a_{0n} = \begin{cases} 1, & n = 0, \\ 0, & n \geq 0. \end{cases} \tag{18}$$

Substituting equations (17) and (18) into equation (16) for several values of m and n, and by recursive method, we get the following coefficients

$$a_{11} = 6\gamma a_{30} - a_{00}a_{20} - a_{10}a_{10} = -(\gamma+2),$$

$$a_{21} = 12\gamma a_{40} - 3a_{00}a_{30} - 3a_{10}a_{20} = \frac{1}{2!}(\gamma+4),$$

$$a_{21} = 20\gamma a_{50} - 4a_{00}a_{40} - 4a_{10}a_{30} - 2a_{20}a_{20} = -\frac{1}{3!}(\gamma+8),$$

$$a_{21} = 30\gamma a_{60} - 5a_{00}a_{50} - 5a_{10}a_{40} - 5a_{20}a_{30} = \frac{1}{4!}(\gamma+16),$$

and so on.

Substituting all a_{mn} into equation (11), we obtain the following infinite series solution

$$C(X,T) = 1 - X + \frac{1}{2!}\, X^2 - \frac{1}{3!}\, X^3 + \frac{1}{4!}\, X^4 - \cdots - (\gamma+2)\, X\, T + \frac{1}{2!}(\gamma+4)\, X^2\, T - \frac{1}{3!}(\gamma+8)\, X^3\, T + \frac{1}{4!}(\gamma+16)\, X^4\, T - \cdots \tag{19}$$

which is the solution of equation (8). The numerical solution of equation (19) is shown in Table 1 and its graphical representation are shown in Figures 1 and 2. Table 2 and Figure 3 indicate the comparison between the results acquired by various standard methods such as DPSM, VHPM and NITHPM.

According to a convergence of double power series given by Ghorpade and Limaye **[4]**, substituting $T = 0$ in (19), we get

$$C(X,0) = 1 - X + \frac{1}{2!}\, X^2 - \frac{1}{3!}\, X^3 + \frac{1}{4!}\, X^4 - \cdots, \tag{24}$$

which is convergent. Hence we can say that the double power series obtained on right hand side of (19) is also convergent.

X	$T = 0.001$	$T = 0.002$	$T = 0.003$	$T = 0.004$	$T = 0.005$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.9046	0.9043	0.9040	0.9037	0.9035
0.2	0.8182	0.8177	0.8172	0.8167	0.8162
0.3	0.7401	0.7394	0.7387	0.7380	0.7373
0.4	0.6695	0.6686	0.6678	0.6669	0.6660
0.5	0.6058	0.6047	0.6037	0.6027	0.6017
0.6	0.5483	0.5471	0.5460	0.5449	0.5437
0.7	0.4966	0.4954	0.4942	0.4930	0.4917
0.8	0.4505	0.4492	0.4479	0.4466	0.4453
0.9	0.4095	0.4082	0.4069	0.4056	0.4043
1.0	0.3737	0.3724	0.3711	0.3698	0.3685
X	$T = 0.006$	$T = 0.007$	$T = 0.008$	$T = 0.009$	$T = 0.010$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.9032	0.9029	0.9026	0.9023	0.9021
0.2	0.8157	0.8152	0.8146	0.8141	0.8136
0.3	0.7366	0.7359	0.7352	0.7344	0.7337
0.4	0.6651	0.6643	0.6634	0.6625	0.6616
0.5	0.6007	0.5996	0.5986	0.5976	0.5966
0.6	0.5426	0.5415	0.5403	0.5392	0.5381
0.7	0.4905	0.4893	0.4881	0.4869	0.4856
0.8	0.4441	0.4428	0.4415	0.4402	0.4390
0.9	0.4030	0.4017	0.4004	0.3991	0.3978
1.0	0.3673	0.3660	0.3647	0.3634	0.3621

Table 1: Numerical values of the concentration to distance X and time T, taking $\gamma = 1$ is fixed.

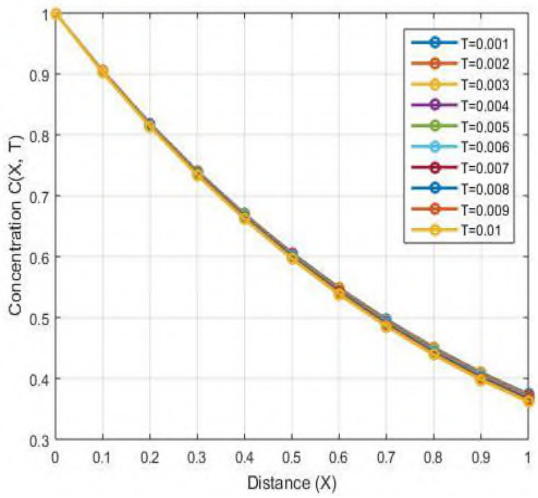

Figure 1: The graph of Concentration $C(X,T)$ Vs. Distance (X)

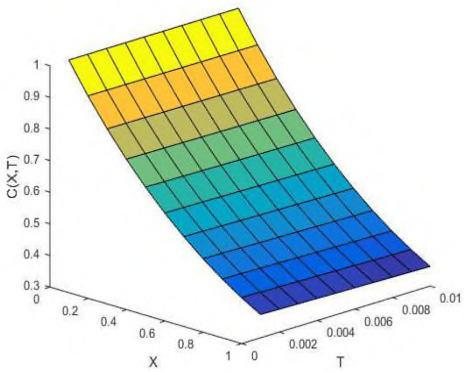

Figure 2: 3D of Concentration $C(X,T)$ Vs. Distance (X)

Table 2: Comparison of numerical values of the concentration by DPSM, VHPM and NITHPM for fixed time $T = 0.005$

X	$T = 0.005$		
	DPSM	VHPM **[3]**	NITHPM **[17]**
0.0	1.0000	1.0000	1.0000
0.1	0.9035	0.9136	0.9136
0.2	0.8162	0.8263	0.8263

0.3	0.7373	0.7473	0.7473
0.4	0.6660	0.6760	0.6760
0.5	0.6017	0.6114	0.6114
0.6	0.5437	0.5531	0.5531
0.7	0.4917	0.5003	0.5003
0.8	0.4453	0.4526	0.4526
0.9	0.4043	0.4094	0.4094
1.0	0.3685	0.3704	0.3704

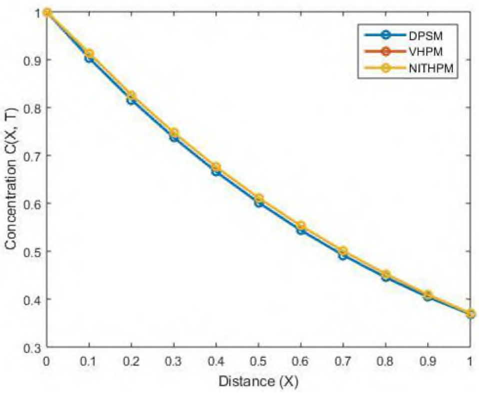

Figure 3: Comparison of solutions by DPSM, VHPM and NITHPM for fixed time $T = 0.005$

4. CONCLUSION

Here we have studied the phenomena of longitudinal dispersion, which leads to a nonlinear partial differential equation known as Burger's equation (8). Using appropriate initial and boundary conditions, the solution is obtained in the form of double power series in (19). The numerical solution and graphical representation have been demonstrated by MATLAB in Table 1 and Figure 1 respectively. It can be observed that form Table 1 and Figure 1, the concentration is decreasing with respect to distance as well as time which is consistent with the physical nature of the phenomenon.

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