DIFFERENTIAL TRANSFORM METHOD APPLIED ON GROUNDWATER INFILTRATION PHENOMENON IN HORIZONTAL DIRECTION WITH EXTERNAL EFFECT (INPUT AND OUTPUT)

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ABSTRACT
In this paper, we have first discussed that groundwater is very important, especially as it is getting scarce. We see rain waters on the puddles but it is not going underground. That has to do with the increased saturated soil, which gives less space for water to move deeper. We studied the nonlinear problem that arises in infiltration phenomenon with External effect (Input and output). The infiltration phenomenon with External effect (Input and output) is defined by a non-linear partial differential equation, which gives the soil moisture content under suitable conditions. The moisture content was determined and was analyzed using the Differential transform method. We investigated groundwater infiltration with an external impact (Input and output) using the Differential Transform method. The analysis is also explained, as well as its numerical and graphical depiction. For this, we utilized MATLAB for that.

KEYWORDS: Moisture content, Differential Transform Method, Infiltration phenomenon, MATLAB, Non-linear partial differential equation, External effect (Input and output)

MSC: 76S05

RESUMEN
En este documento, hemos discutido primero que el agua subterránea es muy importante, especialmente porque se está volviendo escasa. Vemos agua de lluvia en los charcos pero no va bajo tierra. Eso tiene que ver con el aumento del suelo saturado, lo que da menos espacio para que el agua se mueva más profundo. Estudiamos el problema no lineal que surge en el fenómeno de infiltración con efecto Externo (Entrada y salida). El fenómeno de infiltración con efecto Externo (Entrada y salida) está definido por una ecuación diferencial parcial no lineal, que da el contenido de humedad del suelo en condiciones adecuadas. Se determinó el contenido de humedad y se analizó mediante el método de transformada diferencial. Investigamos la infiltración de agua subterránea con un impacto externo (Entrada y salida) usando el método de Transformación Diferencial. También se explica el análisis, así como su representación numérica y gráfica. Para esto, utilizamos MATLAB para eso.

PALABRAS CLAVE: contenido de humedad, método de transformada diferencial, fenómeno de infiltración, MATLAB, ecuación diferencial parcial no lineal, efecto externo (entrada y salida)

1. INTRODUCTION

Underground water i.e., Water located under the earth's surface in soil pore spaces and rock formation fractures is attributed to as groundwater. More than 90% of the liquid fresh water accessible on or near the earth's surface is derived from groundwater. Natural rains can recharge groundwater, which should be plenty but natural discharge is more common at springs and seeps, where it can form oases or wetlands. Extraction wells, which are made by digging groundwater are commonly used for irrigation. Fluid flows from one reservoir to another by different ways like evaporation, condensation, precipitation, erosion, infiltration, transpiration, and groundwater flow. Amongst all, for this paper Water infiltration is what we are going to see in depth. It is required for water salinity management, water pollution regulation, and agricultural uses. It is the process through which water on the ground surface reaches unsaturated soil or an oil reservoir. It is commanded by two forces: gravity and capillary action. Although smaller pores are more resistant to gravity, extremely small pores draw water by capillary action in addition to, and even against, gravity. The rate of infiltration is affected by soil properties such as ease of entry, storage capacity, and transmission rate through the soil.
In this phenomenon, water is infiltrated horizontally into a homogeneous unsaturated porous media via its vertical permeable surface, as seen in figure (1). Because the bottom is regarded impermeable, infiltrating water can only move horizontally. Boussinesq's equation, a nonlinear partial differential equation, is the result of the mathematical formulation. External effect (input and output) has been applied on homogeneous porous media in

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horizontal direction and the Mathematical equation has been obtained. Differential Transform Method has been applied to solve the equations.

2. LITERATURE REVIEW

J. Boussinesq (1903/04) [3] is recognized as being the first to propose the porous medium equation $\partial_i u = \Delta(u^m), m > 1$ as a mathematical model for a physical technique that accurately determines the height of the water mound in groundwater infiltration. He applied the limited gradient assumption and the simple flow rule established by H. Darcy (1856) [6]. It is worth mentioning that the exponent is m = 2. The development of the infiltration equation and its numerical solution was reported by Philip [11]. Srivastava and Yeh [13] utilized analytical methods to characterise one-dimensional vertical infiltration to the water table in homogeneous and layered soils. Witelski [14] expanded the Boltzmann similarity solution's applicability to include a time shift constant to explain the long-term effect of absorption in slightly wet soil layers. Wojnar [15] studied the Boussinesq equation for aquifer flow with time-dependent porosity. Borana et al. [2] utilised the finite difference method to solve the Boussinesq equation that arises in the one-dimensional infiltration phenomena. Patel et al. [9] used homotopy analysis to solve the Boussinesq equation for infiltration into unsaturated porous media. Chavan and Panchal [4] used the homotopy perturbation approach with the Elzaki transform to solve the porous medium equation that arises in fluid flow via porous media. Parikh [8] used the differential quadrature method to solve the Boussinesq equation for groundwater infiltration. Desai [7] discovered a solution that is comparable to the non-linear Boussinesq equation, which occurs when an incompressible fluid flow enters. Patel and Desai [10] used homotopy analysis to address the problem.

3. MATHEMATICAL FORMULATION

The reservoir field is surrounded by unsaturated homogenous soil and holds water of height OA = hmax = maximum height. Figure 1 depicts a cross section of a reservoir surrounded by unsaturated porous material, which clearly depicts the infiltration phenomen

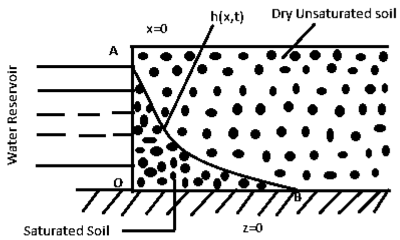

Figure 1: Groundwater infiltration scheme

When OB = x = 1, the free surface height is 0, the dotted arc below the curve is saturated by infiltrated groundwater, and the dry region of unsaturated soil is above the curve. Water cannot flow downward because the bottom is considered to be impervious. Infiltration is the process by which reservoir groundwater enters unsaturated soil via a vertical permeable wall. The infiltrating groundwater will enter unsaturated soil and form a water table or water mound between the saturated and unsaturated soils. Under the following simplifying assumptions, groundwater infiltrated through the adjacent vertical side:

1. The layer has a height of h_{max} and rests on a horizontal impermeable bed (z = 0);
2. The cross-sectional variable is ignored.
3. The mass of water that infiltrates the soil is in a region known as $\Omega = \{(x, z) \in R: z \leq h(x, t)\}$

We assume that there is no partial saturation area. This is a transformative model. Obviously $0 \leq h(x, t) \leq h_{max}$, h_{max} the maximum height of the free area and the free boundary function 'h' are unknown functions for this problem. We have a system of three equations in a domain variable, with the two velocity components 'u', 'w', and 'p' as unknowns: a conservation of mass equation for an incompressible fluid and two Navier-Stokes equations for momentum conservation. The resultant system is too complex, but it may be reduced for practical computation by inserting an appropriate assumption, the near-horizontal flow hypothesis, which says that a near-horizontal flow with the velocity u $(u, 0)$ has tiny gradients. As a result, the vertical component of the moment equation has the following form:

$$p\left(\frac{du_z}{dt} + u.\Delta u_z\right) = -\frac{\partial p}{\partial z} - \rho g \quad (1)$$

The inertia term was neglected (left side). The integration equation (1) with respect to z results in a first approximation p + ρgz = constant (2)

Now we calculate the free surface constant $z = h(x, t)$. If we apply the pressure continuity across the interface (assuming a constant atmospheric pressure in the air filling the pores of the dry zone we get p = 0.z > h (x, t)).

So, we get $p = \rho g (h - z)$ (3)

As a result, the pressure can be utilized using the hydrostatic approximation. We now move on to the law of conservation of mass, which gives us the equation. Here's how we do it: Let's look at a section $S = (x, x + a) \times (0, C)$.

Then$\emptyset \frac{\partial}{\partial t}\int_{x}^{x+a}\int_{0}^{h} dydx = -\int_{\partial s} u.ndl$ (4)

Where ' u ' is the porosity of the medium, i.e., the fraction of volume accessible for flow circulation, and 'u' is the seepage velocity, which obeys Darcy's law in the form that incorporates gravity effects,

$u = -\frac{k}{\mu}\Delta(\text{p} + \text{ρgz})$ (5)

$u.\text{n} \approx (u,0).(1,0) = u$ i.e.,$\left(\frac{k}{\mu}\right)p_x$ on the right-hand side.

On the left, the velocity is indicated as '-u.' We get the following result by applying the 'p' formula and differentiating with respect to 'x.'

$\emptyset \frac{\partial h}{\partial t} = \frac{pgK}{\mu}\frac{\partial}{\partial x}\int_{0}^{h}\frac{\partial}{\partial x} hdz$ (6)

We obtained Boussinesq's equation as

$\frac{\partial h}{\partial t} = \frac{pgK}{2\mu\emptyset}\frac{\partial^2}{\partial x^2}(h^2)$ (7)

With constant $\beta = \frac{pgK}{2\mu\emptyset}$, which reflects the porous medium equation, is the essential equation in ground water infiltration. Taking into account a new dimensionless variable

$T = \frac{pgK}{\mu\emptyset}t$ and $X = \frac{x}{L}$ in Equation (7), it gives,

$\frac{\partial h}{\partial T} = \left\{h\frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial X}\right)^2\right\}$ (8)

Equation (8) gives the height of the water mound, using the following initial and boundary conditions:

h(X,0) = $h_0 e^{-X}$ X>0 and T=0 (9)

h(0, T) = h_{max} X=0 and T> 0 (10)

4. DIFFERENTIAL TRANSFORM METHOD

The Differential Transform Method, which addressed linear and nonlinear challenges in electrical circuit problems, was initially proposed by Zhou [16]. Chen and Ho [5] developed this methodology for partial differential equations, and Ayaz [1] applied it to differential equations. Several scholars have used this approach to solve different sorts of equations in recent years.

4.1 Two-dimensional differential transform method

Consider a function of two variables *d(x, t)* and suppose that it can be represented as product of two single-variable functions, i.e., d(x, t) = *f*(*x*)*g*(*t*)*:* Based on the properties of differential transform, function *d(x, t)* can be represented as

$d(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} D(i,j)x^i t^j$ (11)

The spectrum of d(x, t) is defined as D(i,j) = F(i)G(j).

Let d(x, t) be analytic and continuously differentiated with respect to time t and space x in the domain of interest.

$D(m\,,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n} d(x,t)\right]_{\substack{x=0\\t=0}}$ (12)

where t is the dimension of the spectrum function The transformed function, often known as the T-function, is D(m,n). D(m,n) has a differential inverse transform that is defined as follows:

$d(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D(m,n)x^m t^n$ (13)

Resulting in combination of eq (12) and (13)

$d(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n} d(x,t)\right]_{\substack{x=0\\t=0}} x^m t^n$ (14)

Table 1: Transformed form of DTM method

Function Form	Transformed Form
$d(x,t) = u(x,t) \pm v(x,t)$	$D(m,n) = U(m,n) \pm V(m,n)$
$d(x,t) = cu(x,t)$	$D(m,n) = cU(m,n)$
$d(x,t) = \frac{\partial}{\partial x}u(x,t)$	$D(m,n) = (m+1)U(m+1,n)$
$d(x,t) = \frac{\partial}{\partial t}u(x,t)$	$D(m,n) = (n+1)U(m,n+1)$
$d(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s}u(x,t)$	$D(m,n) = \frac{(m+r)!\,(n+s)!}{m!\,n!}U(m+r,n+s)$

$D(x,t)=u(x,t)v(x,t)$	$D(m,n)=\sum_{r=0}^{m}\sum_{s=0}^{n}U(r,n-s)V(m-r,s)$
$d(x,t)=x^{\alpha}t^{\beta}$	$D(m,n)=\delta(m-\alpha,n-\beta)=\begin{cases}1 & m=\alpha,n=\beta\\ 0 & otherwise\end{cases}$

5. EXTERNAL EFFECT OF WATER INPUT ON GROUNDWATER INFILTRATION PHENOMENON IN HORIZONTAL DIRECTION

When water is injected into the porous stratum with external input, the equation becomes

$$\frac{\partial h}{\partial T}=\frac{\partial^2}{\partial X^2}(h^2)+f \qquad (15)$$

The function f(X, Z, T) represents External effect. If we suppose that such external impacts occur at specific space positions above the free surface h(X, T), then the effects of f(X, Z, T) will raise the height of the free surface h(X,T) in the case of water input into porous medium, as represented by the downward arrow in figure 2.

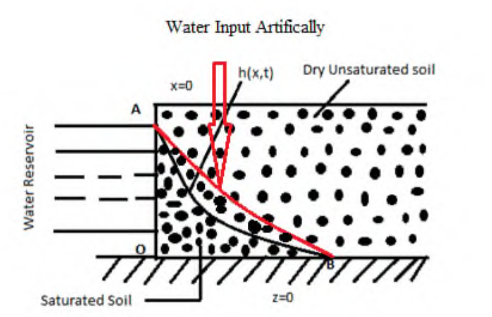

Figure 2: Ground water infiltration when water is injected into the porous stratum by external input.

As a result, we suppose that f(X, Z, T) is directly proportional to height increase with regard to X increase.Then $f \propto \frac{\partial h}{\partial X}$

$$\Rightarrow f = B\frac{\partial h}{\partial X} \qquad (16)$$

where B > 0 is constant. Hence equation (8) takes form

$$\frac{\partial h_i}{\partial T}=\frac{\partial^2}{\partial X^2}(h^2)+B\frac{\partial h}{\partial X} \qquad (17)$$

Where B = Constant of proportionality.
Appropriate boundary conditions to solve this problem are chosen from figure 2 as follows:
h(0,T) = h_1 when X=0, any T> 0 h(1,T) = 0 when T> 0, any X=l

Equation (8) can be rewritten as $\frac{\partial h_i}{\partial T}=\left[h\frac{\partial^2 h}{\partial X^2}+\left(\frac{\partial h}{\partial X}\right)^2\right]+B\frac{\partial h}{\partial X}$ (18)

5.1 Solution by differential transform method

According to the DTM and table 1, we can construct the following transformation of Equation (18)

$$\frac{\partial h_i}{\partial T}=\left[h\frac{\partial^2 h}{\partial X^2}+\left(\frac{\partial h}{\partial X}\right)^2\right]+B\frac{\partial h}{\partial X}$$

With Initial condition $h_i(X,0)=0.8e^{-X}$ as:

$(n+1)H(m,n+1)=2\sum_{r=0}^{m}\sum_{s=0}^{n}(r+1)(m-r+1)H(r+1,n-s)H(m-r+1,s)+2\sum_{r=0}^{m}\sum_{s=0}^{n}(m-r+1)(m-r+2)H(r,n-s)H(m-r+2,s)+B(m+1)H(m+1,n)$ (19)

By definition of DTM

$$H_i(m,n)=\frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial X^m\partial T^n}h_i(X,T)\right]_{\substack{X=0\\ t=0}} \qquad (20)$$

To Apply initial condition in equation (18) putting n=T=0 we get,

$$H_i(m,0)=\frac{1}{m!0!}\left[\frac{\partial^m}{\partial X^m}h_i(X,0)\right]_{X=0} \qquad (21)$$

Applying initial condition in equation (18) and putting different values of m , i.e m=0,1,2,3,4,5,… we get

$$H_i(0,0)=\frac{1}{0!}\left[\frac{\partial^0}{\partial X^0}h_i(X,0)\right]_{X=0}$$

So we get H_i (0,0) =0.8

$H_i(1,0)=\frac{1}{1!}\left[\frac{\partial^1}{\partial X^1}h_i(X,0)\right]_{X=0}$ $\therefore H_i(1,0)=[-0.8e^{-X}]_{X=0}$ So, we get H_i (1,0) = -0.8

$H_i(2,0)=\frac{1}{2!}\left[\frac{\partial^2}{\partial X^2}h_i(X,0)\right]_{X=0}$ So, we get H_i (2,0) = $\frac{0.8}{2}$

In similar manner, we will get H_i (3,0) = $-\frac{0.8}{6}$, H_i (4,0) = $\frac{0.8}{24}$, H_i (5,0) = $-\frac{0.8}{120}$ (22)

Putting different values of m and n and B=0.4 in equation (15) we got different H(X,T) which are as follows:

Table 2: Value of H_i at different m and n's

H_i (0,0) = 0.8	H_i (1,0) = -0.8	H_i (2,0) = 0.4	H_i (3,0) = -0.133
H_i (0,1) = 0.98	H_i (1,1) = -2.24	H_i (2,1) = 2.606	H_i (3,1) = -2.024
H_i (0,2) = 3.56	H_i (1,2) = -13.1404	H_i (2,2) =24.24	H_i (3,2) = -29.81
H_i (0,3) = 22.47	H_i (1,3) = -107.036	H_i (2,3) = 254.558	H_i (3,3) = -403.90

and so on...
Now by equation (13) we get,
$h_i(X,T) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H_i(m,n)X^mT^n$
so, $h(X,T) = H_i(0,0)X^0T^0 + H_i(1,0)X^1T^0 + H_i(2,0)X^2T^0 + H_i(3,0)X^3T^0 + H_i(0,1)X^0T^1 + H_i(1,1)X^1T^1 + H_i(2,1)X^2T^1 + H_i(3,1)X^3T^1 + H_i(0,2)X^0T^2 + H_i(1,2)X^1T^2 + H_i(2,2)X^2T^2 + H_i(3,2)X^3T^2 + H_i(0,3)X^0T^3 + H_i(1,3)X^1T^3 + H_i(2,3)X^2T^3 + H_i(3,3)X^3T^3 + \cdots$
From above table, p=0.02, $\emptyset = 0.218$, L=1 and values of X and T we have,

$$h_i(x,t) = 0.8 - 0.8x + 0.4x^2 - 0.133x^3 - 0.098t + 0.224xt - 0.2606x^2t + 0.2024x^3t + 0.0356t^2 - 0.131404xt^2 + 0.2424x^2t^2 - 0.2981x^3t^2 - 0.02247t^3 + 0.107036xt^3 - 0.254558x^2t^3 + 0.4039x^3t^3 + \cdots \quad (23)$$

5.2 Numerical and graphical presentation

We used MATLAB coding to generate numerical and graphical representations of Equation (23). The graph of height h_i vs. x for fixed time t=0.1 to 1.0, with the difference of 0.1 is shown in **Figure 3**. The table below shows the numerical values for height for various distances x at fixed times t=0.1 to 1.0, with the difference of 0.1.

Table 3: Tabular value of water mound when external input is applied

	h_i (x, t)									
x	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5	t=0.6	t=0.7	t=0.8	t=0.9	t=1.0
0.1	0.7165	0.7096	0.7029	0.6965	0.6903	0.6841	0.6780	0.6717	0.6652	0.6585
0.2	0.6492	0.6436	0.6382	0.6332	0.6282	0.6232	0.6184	0.6134	0.6085	0.6033
0.3	0.5880	0.5835	0.5792	0.5750	0.5710	0.5670	0.5630	0.5591	0.5551	0.5511
0.4	0.5326	0.5290	0.5255	0.5222	0.5189	0.5157	0.5125	0.5094	0.5062	0.5030
0.5	0.4823	0.4794	0.4766	0.4740	0.4713	0.4688	0.4662	0.4637	0.4612	0.4587
0.6	0.4366	0.4343	0.4321	0.4299	0.4278	0.4257	0.4237	0.4217	0.4197	0.4177
0.7	0.3954	0.3936	0.3918	0.3901	0.3884	0.3867	0.3850	0.3834	0.3818	0.3802
0.8	0.3580	0.3566	0.3552	0.3538	0.3524	0.3511	0.3498	0.3485	0.3472	0.3459
0.9	0.3244	0.3232	0.3221	0.3210	0.3199	0.3188	0.3177	0.3167	0.3156	0.3146
1.0	0.2934	0.2925	0.2916	0.2907	0.2898	0.2890	0.2882	0.2873	0.2865	0.2857

6. WATER OUTPUT FROM THE POROUS STRATUM BY SINK OR PUMPING

If we assume that such external effects occur at precise space locations above the free surface h(X, T) in a dry region, then the effects of f(X, Z, T) will naturally or artificially decrease the height of the free surface h(X, T) in the case of water output from the porous medium shown by the upward arrow in the figure. As a result, we suppose that (X, Z, T) is directly proportional to the decrease in height h(X, T) as X increases. Appropriate boundary conditions to solve this problem are chosen from figure 4 as follows:

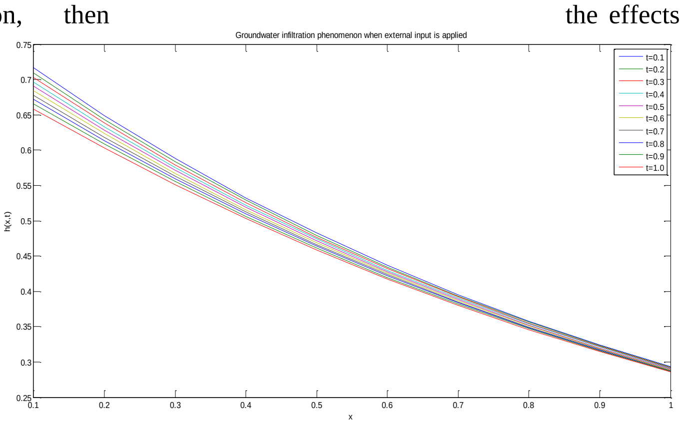

Figure 3: Groundwater infiltration phenomenon when input is applied

h(0,T) = h_1 when X=0, any T$>$ 0

$$\frac{\partial h_o}{\partial T} = \left[h\frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial X}\right)^2\right] - B\frac{\partial h}{\partial X} \quad (26)$$

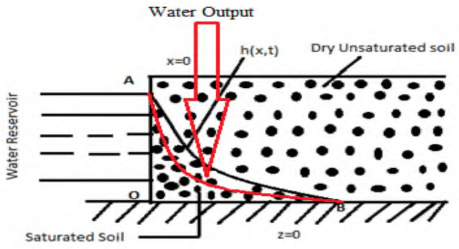

Figure 4: Schematic figure of ground water infiltration when water output from the porous stratum by Sink or Pumping

6.1. Solution by differential transform method

According to the DTM and table 1, we can construct the following transformation of Equation (26)

$$\frac{\partial h_o}{\partial T} = \left[h\frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial X}\right)^2\right] - B\frac{\partial h}{\partial X}$$

With Initial condition $h_o(\mathrm{X},0) = 0.8e^{-X}$ as:

$$(\mathrm{n}+1)\mathrm{H}(\mathrm{m},\mathrm{n}+1) = 2\sum_{\mathrm{r}=0}^{\mathrm{m}}\sum_{\mathrm{s}=0}^{\mathrm{n}}(\mathrm{r}+1)(\mathrm{m}-\mathrm{r}+1)\mathrm{H}(\mathrm{r}+1,\mathrm{n}-\mathrm{s})\mathrm{H}(\mathrm{m}-\mathrm{r}+1,\mathrm{s}) + 2\sum_{\mathrm{r}=0}^{\mathrm{m}}\sum_{\mathrm{s}=0}^{\mathrm{n}}(\mathrm{m}-\mathrm{r}+1)(\mathrm{m}-\mathrm{r}+2)\mathrm{H}(\mathrm{r},\mathrm{n}-\mathrm{s})\mathrm{H}(\mathrm{m}-\mathrm{r}+2,\mathrm{s}) - B(\mathrm{m}+1)\mathrm{H}(\mathrm{m}+1,\mathrm{n}) \quad (27)$$

By definition of DTM

$$H_o(m\,,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial X^m \partial T^n} h_o(X,T)\right]_{\substack{X=0\\t=0}} \quad (28)$$

To Apply initial condition in equation (28) putting n=T=0 we get,

$$H_o(m\,,0) = \frac{1}{m!0!}\left[\frac{\partial^m}{\partial X^m} h_o(X,0)\right]_{X=0} \quad (29)$$

Applying initial condition in equation (19) and putting different values of m , i.e m=0,1,2,3,4,5,… we get

H_o (0,0) =0.8 , H_o (1,0) = -0.8 , H_o (2,0) = $\frac{0.8}{2}$, H_o (3,0) = $-\frac{0.8}{6}$, H_o (4,0) = $\frac{0.8}{24}$, H_o (5,0) = $-\frac{0.8}{120}$ (30)

Putting different values of m and n and B=0.4 in equation (27) we got different H_o (X,T) which are as follows:

Table 4: Value of H_o at different m and n's

H_o (0,0) = 0.8	H_o (1,0) = -0.8	H_o (2,0) = 0.4	H_o (3,0) = -0.133
H_o (0,1) = 2.08	H_o (1,1) = -3.36	H_o (2,1) = 2.729	H_o (3,1) = -1.474
H_o (0,2) = 6.828	H_o (1,2) = -17.678	H_o (2,2) =22.90	H_o (3,2) = -19.77
H_o (0,3) = 36.90	H_o (1,3) = -123.97	H_o (2,3) = 208.29	H_o (3,3) = -233.29

And so on.

Now by equation (13) we get,

$$h_o(x,t) = 0.8 - 0.8x + 0.4x^2 - 0.133x^3 - 0.208t + 0.336xt - 0.2729x^2t + 0.1474x^3t + 0.06828t^2 - 0.17678xt^2 + 0.2290x^2t^2 - 0.1977x^3t^2 - 0.036907t^3 + 0.12397xt^3 - 0.20829x^2t^3 + 0.23329x^3t^3 \quad (31)$$

6.2. Numerical and graphical presentation

We used MATLAB coding to generate numerical and graphical representations of Equation (31). The graph of height h_o vs. x for fixed time t=0.1 to 1.0, with the difference of 0.1 is shown in **Figure 5**. The table below shows the numerical values for height for various distances x at fixed times t=0.1 to 1.0, with the difference of 0.1.

Table 5: Tabular value of water mound when external output is applied to take out water

	h_o (x,t)									
x	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5	t=0.6	t=0.7	t=0.8	t=0.9	t=1.0
0.1	0.7067	0.6904	0.6748	0.6598	0.6453	0.6310	0.6168	0.6025	0.5881	0.5732
0.2	0.6404	0.6265	0.6132	0.6003	0.5878	0.5756	0.5635	0.5514	0.5393	0.5269
0.3	0.5802	0.5683	0.5568	0.5457	0.5349	0.5244	0.5139	0.5036	0.4932	0.4827
0.4	0.5257	0.5154	0.5056	0.4960	0.4867	0.4777	0.4687	0.4598	0.4510	0.4421
0.5	0.4761	0.4674	0.4589	0.4508	0.4428	0.4349	0.4273	0.4196	0.4120	0.4045
0.6	0.4313	0.4238	0.4166	0.4095	0.4027	0.3959	0.3891	0.3828	0.3763	0.3698
0.7	0.3907	0.3843	0.3781	0.3721	0.3662	0.3604	0.3547	0.3491	0.3435	0.3380

0.8	0.3539	0.3484	0.3431	0.3380	0.3329	0.3279	0.3230	0.3182	0.3134	0.3087
0.9	0.3208	0.3162	0.3116	0.3072	0.3028	0.2986	0.2944	0.2902	0.2861	0.2820
1.0	0.2902	0.2863	0.2824	0.2785	0.2748	0.2712	0.2676	0.2640	0.2605	0.2570

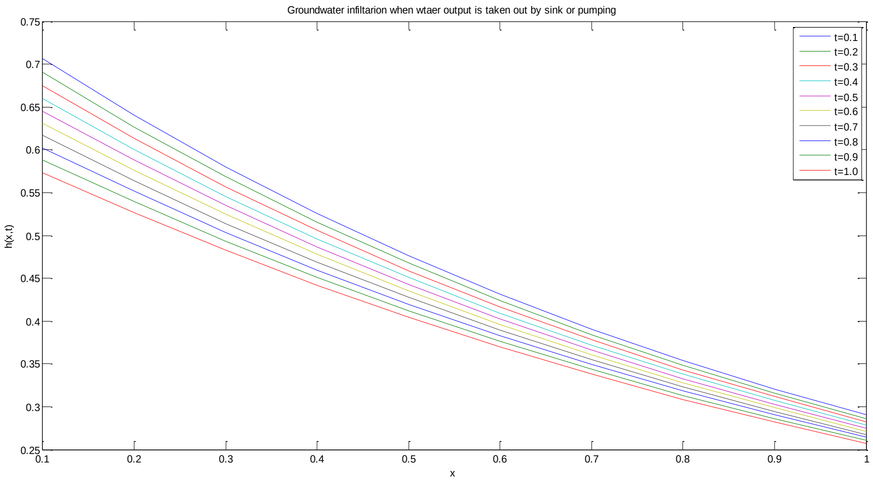

Figure 5: Groundwater infiltration phenomenon when water output is taken out by sink or pumping

7. DIFFERENCE CREATED BY EXTERNAL EFFECT OF INPUT AND OUTPUT ON GROUNDWATER INFILTRATION PHENOMENON IN HORIZONTAL DIRECTION

The Comparative Study of Groundwater infiltration phenomena when the external effect of Input and output on the groundwater infiltration phenomenon has been done numerically and graphically. The graphs have been done by MATLAB coding.

Table 6: Numerical difference by applying external input and output

	t=0.1		t=0.5		t=0.9	
x	h_i	h_o	h_i	h_o	h_i	h_o
0.1	0.7165	0.7067	0.6903	0.6453	0.6652	0.5881
0.2	0.6492	0.6404	0.6282	0.5878	0.6085	0.5393
0.3	0.5880	0.5802	0.5710	0.5349	0.5551	0.4932
0.4	0.5326	0.5257	0.5189	0.4867	0.5062	0.4510
0.5	0.4823	0.4761	0.4713	0.4428	0.4612	0.4120
0.6	0.4366	0.4313	0.4278	0.4027	0.4197	0.3763
0.7	0.3954	0.3907	0.3884	0.3662	0.3818	0.3435
0.8	0.3580	0.3539	0.3524	0.3329	0.3472	0.3134
0.9	0.3244	0.3208	0.3199	0.3028	0.3156	0.2861
1.0	0.2934	0.2902	0.2898	0.2748	0.2865	0.2605

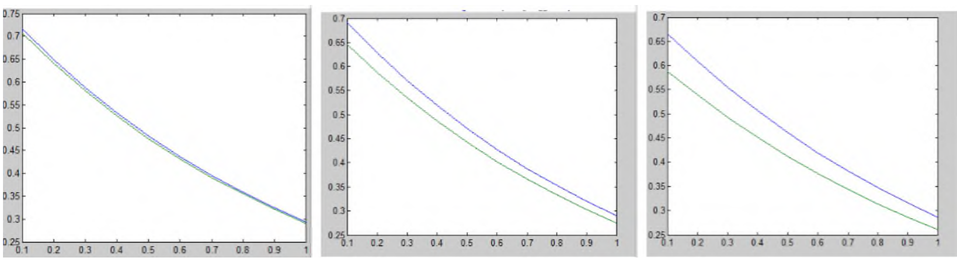

Figure 6: Comparative Study of Groundwater infiltration phenomena when the external effect of Input and output on the groundwater infiltration phenomenon when t = 0.1, 0.5, 0.9.

8. OBSERVATIONS AND CONCLUSION:

Equation (23) shows height of infiltrated water when external input is applied. **Table 4** and **figure 4** shows numerical and graphical presentation of equation (23). Equation (31) shows height of infiltrated water when external output is applied. **Table 5** and **figure 5** shows numerical and graphical presentation of equation (31). In groundwater infiltration phenomena when we apply external force there is difference of height of the water table. When external output is applied height is decreased as compared to external input is applied. **Table 6** and **figure 6** shows the comparison. As the water mound changes, the result will change. The physical phenomena depend on the equation.

As there is lot of scarcity of water, it is important to know how much water is pumped out and how much water is going underground. This method easy to compute.

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REFERENCES

[1] AYAZ, F., (2004): Solutions of the systems of differential equations by differential transform method, **Appl. Math. Comput.**, 147, 547-567.

[2] BORANA, R.N, PRADHAN V.H. andMEHTA M.N., (2013): Numerical solution of Boussinesq equation arising in one dimensional infiltration phenomenon by using finite difference method, **International Journal of Research in Engineering and Technology**, 2, 202-209.

[3] BOUSSINESQ, J., (1903): **Comptes Rendus Acad. Science Journal, Math. Pures Appl.**, 10, 5-78.

[4] CHAVAN, S.S. and PANCHAL, M.M., (2014): Solution of porous medium equation arising in fluid flow through porous media by homotopy perturbation method using Elzaki transform, **International Journal of Futuristic Trends in Engineering and Technology**, 1, 32-35.

[5] CHEN, C.K., HO, S.H., (1999): Solving partial differential equations by two-dimensional differential transform method, **Appl. Math. Comput.**, 106 ,171-179.

[6] DARCY, H., (1999): **Les Fontaines publiques de loville de Dijon,** ***Dalmont***, *Paris*, 1856, 305-310.

[7] DESAI, N.B., (2017): Similarity solution of nonlinear Boussinesq's equation arising in infiltration of incompressible fluid flow, **International Journals of Advanced Research in Computer Science and Software Engineering**, 7, 59-67.

[8] PARIKH, A.K., (2015) Numerical solution of Boussinesq's equation in groundwater infiltration phenomenon by differential quadrature method, **Paripex-Indian Journal Of Research**, 4, 165-167.

[9] PATEL, K.K., MEHTA, M.N. and SINGH, T.R, (2014): A solution of Boussinesq's equation for infiltration phenomenon in unsaturated porous media by homotopy analysis method, **IOSR Journal of Engineering**, 41-8.

[10] PATEL, M.A. and DESAI, N.B., (2018): An Approximate Analytical Solution of Boussinesq's Equation for Infiltration Phenomenon in Unsaturated Porous Medium, International Journal of Mathematics and its Applications: **Int. J. Math. And Appl.**, 6, C463–470 ISSN: 2347-1557.

[11] PHILIP, J.R., (1957): The theory of infiltration: 1. **The infiltration equation and its solution, Soil science**, 83, 345-358.

[12] SHAH, K., PARIKH A.K.,(2021) Solution of Boussinesq's Equation for Infiltration Phenomenon in Horizontal Direction by Differential Transform Method, **International Journal of Innovative Science, Engineering and Technology**, V8_I05_61 ISSN:2348-7968 , 630-638.

[13] SRIVASTAVA, R. and YEH, T.C.J., (1991): Analytical solutions for one-dimensional, transient infiltration toward the water table in homogeneous and layered soils, **Water Resources Research**, 27, 753-762.

[14] WITELSKI, T.P., (1998): Horizontal infiltration into wet soil, Water Resources Research, 34(71859-1863.

[15] WOJNAR, R., (2010): Boussinesq equation for flow in an aquifer with time dependent porosity, **Bulletin of Polish Academy of Sciences: Technical Sciences**, 58, 1165-170.

[16] ZHOU, J.K., (1986): **Differential Transformation and Its Applications for Electrical Circuits**, Huazhong University Press, Wuhan, China.