COMMON FIXED-POINT THEOREMS FOR COMPATIBLE TYPE (K) MAPPINGS IN INTUITIONISTIC MENGER SPACE

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ABSTRACT
The main aim of this paper is to demonstrate common fixed point theorems in intuitionistic Mengerspace. In our main result we use the notion of CLR property for type (K) Compatible mappings. The importance of CLR property is that subspaces need not be closed for the survival of fixed points. Many authors utilize the concept of CLR property to prove theorems on fixed point in intuitionistic Mengerspaces for various type of compatible mappings.The results thus obtained, generalizes and extends some known results in intuitionistic menger space.

KEYWORDS: Common fixed point, Intuitionistic Menger Spaces, type (E) Compatible mappings, type (K) Compatible mappings ,CLRgProperty.

MSC:54H25, 47H10.

RESUMEN
El objetivo principal de este artículo es demostrar los teoremas comunes del punto fijo en el espacio Menger intuicionista. En nuestro resultado principal, usamos la noción de propiedad CLR para mapeos compatibles de tipo (K). La importancia de la propiedad CLR es que no es necesario cerrar los sub-espacios para la supervivencia de los puntos fijos. Muchos autores utilizan el concepto de propiedad CLR para demostrar teoremas sobre puntos fijos en espacios Menger intuicionistas para varios tipos de aplicaciones compatibles. Los resultados así obtenidos generalizan y amplían algunos resultados conocidos en el espacio mental intuicionista.

PALABRAS CLAVE: punto fijo común, espacios Menger intuicionistas, mapeos compatibles tipo (E), mapeos compatibles tipo (K), propiedad CLRg

1. INTRODUCTION

There have been a number of generalizations of metric spaces. One such generalization is Menger space introduced in 1942 by Menger [10] who used distribution functions instead of nonnegative real numbers as values of the metric. This space was expanded rapidly with the pioneering works of Schweizer and Sklar[15,16]. Modifying the idea of Kramosil and Michalek [7], George and Veeramani [3] introduced fuzzy metric spaces which are very similar that of Menger space. Park [13] introduced the notion of intuitionistic fuzzy metric spaces as a generalization of fuzzy metric spaces.
Kutukcu et. al [8] introduced the notion of intuitionistic Menger Spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger [10]. Further they introduced the notion of Cauchy sequences and found a necessary and sufficient condition for an intuitionistic Menger Space to be complete. Sessa [17] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [5] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [11].
In 1993, G. Jungck, P. P. Murthy and Y. J. Cho [6] gave a generalization of compatible mappings called compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions. In 1996, H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang [12] introduced the concept of compatible mappings of type (P) and compared with compatible mappings of type(A) and compatible mappings. K. B. Manandhar et.al. [9] introduced the notion of compatible mappings of type (E) and obtained a common fixed point theorem for self mappings in complete fuzzy metric space in 2014. K. Jha, V. Popa and K. B. Manandhar [4] introduced the concept of compatible mappings of type (K) in metric space. Rao R. and Reddy B. [14] have obtained fixed point theorems for compatible mappings of type (K) in complete fuzzy metric space in 2016.
Sintunayarat and Kuman [18] introduced the concept of common limit in the range property. The importance of CLR property is that we don't require the closedness of subspaces for the existence of fixed points.
AouaandAliouche [1] utilize the notion of common limit range property to prove fixed point theorems for weakly compatible mapping in intuitionistic Menger spaces.

2. PRELIMINARIES

Definition2.1. A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm if $*$ is satisfying the following conditions:

(1) $*$ is commutative and associative,
(2) $*$ is continuous,
(3) $a * 1 = a$, for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d$\in [0,1]$.

Definition2.2. A binary operation $\Diamond$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a t-conorm if $\Diamond$ is satisfying the following conditions:

(1) $\Diamond$ is commutative and associative,
(2) $\Diamond$ is continuous,
(3) $a \Diamond 0 = a$, for all $a \in [0,1]$,
(4) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d$\in [0,1]$.

Remark 2.3. The concept of triangular norms (t-norms) and triangular conforms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [1] in his study of statistical metric spaces.

Definition2.4. A distance distribution function is a function $F : R \rightarrow R^+$ which is non-decreasing, left continuous on R and inf $\{F(t) : t \in R\} = 0$ and sup $\{F(t) : t \in R\} = 1$. We will denote by D the family of all distance distribution functions while H will always denote the specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

If X is a non-empty set, $F : X \times X \rightarrow D$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y}$.

Definition2.5. A non-distance distribution function is a function $L : R \rightarrow R^+$ which is non-increasing, right continuous on R and inf $\{L(t) : t \in R\} = 1$ and sup $\{L(t) : t \in R\} = 0$. We will denote by E the family of all non-distance distribution functions while G will always denote the specific distribution function defined by

$$G(t) = \begin{cases} 1, & t \leq 0 \\ 0, & t > 0. \end{cases}$$

If X is a non-empty set, $L : X \times X \rightarrow E$ is called a probabilistic non-distance on X and $L(x, y)$ is usually denoted by $L_{x,y}$.

Definition2.6. [7] A 5-tuple $(X, F, L, *, \Diamond)$ is said to be an intuitionistic Menger space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is continuous t-conorm, F is a probabilistic distance and L is a probabilistic non-distance on X satisfying the following conditions: for all x, y,z$\in$ X and t, s ≥ 0

(1)$F_{x,y}(t) + L_{x,y}(t) \leq 1$,
(2) $F_{x,y}(0) = 0$,
(3) $F_{x,y}(t) = H(t)$ if and only if x = y,
(4) $F_{x,y}(t) = F_{y,x}(t)$,
(5) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t+s) = 1$,
(6) $F_{x,z}(t+s) \geq F_{x,y}(t) * F_{y,z}(s)$,
(7) $L_{x,y}(0) = 1$,
(8) $L_{x,y}(t) = G(t)$ if and only if x = y,
(9) $L_{x,y}(t) = L_{y,x}(t)$,
(10) if $L_{x,y}(t) = 0$ and $L_{y,z}(s) = 0$, then $L_{x,z}(t+s) = 0$,
(11) $L_{x,z}(t+s) \leq L_{x,y}(t) \Diamond L_{y,z}(s)$.

The function$F_{x,y}(t)$ and $L_{x,y}(t)$ denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Remark 2.7. Every Menger space $(X, F, *)$ is intuitionistic Menger space of the form $(X, F, 1 - F, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated, that is $x \Diamond y = 1- (1-x) * (1-y)$ for any x , y $\in$ X.

Example 2.8. Let (X, d) be a metric space. Then the metric d induces a distance distribution function F defined by $F_{x,y}(t) = H(t - d(x, y))$ and a non-distance function L defined by $L_{x,y}(t) = G(t - d(x,y))$ for all x,y$\in$X and $t \geq 0$. Then (X,F,L) is an intuitionistic probabilistic metric space. We call this instutionistic probabilistic metric space induced by a metric d the induced intuitionistic probabilistic metric space. If t-norm $*$ is $a * b = \min\{a,b\}$ and t-conorm $\Diamond$ is $a \Diamond b = \min\{1,a + b\}$ for all a, b $\in [0,1]$ then $(X,F,L, *,\Diamond)$ is an intuitionistic Menger space.

Remark 2.9. Note that the above example holds even with the t-norm a∗b = min{a,b}and t-conorm a ◊ b = max {a, b}and hence (X,F,L, ∗,◊) is an intuitionistic Menger space with respect to any t-norm and t-conorm. Also note t-norm ∗ and t-conorm ◊ are not associated.

Definition 2.10. [7] Let (X,F,L, ∗,◊) be an intuitionistic Menger space with t ∗t ≥ t and (1 – t) ◊ (1 - t) ≤ (1 – t). Then:

(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every ε> 0 and λ∈ (0,1), there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ and $L_{x_n,x}(\varepsilon) < \lambda$ whenever n ≥ N.

(2) A sequence $\{x_n\}$ in X is called Cauchy sequence if, for every ε> 0 and λ∈ (0,1), there exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1- \lambda$ and $L_{x_n,x_m}(\varepsilon) < \lambda$ whenever n, m ≥ N.

(3) An intuitionistic Menger space (X,F,L, ∗,◊) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

The proof of the following lemmas is on the lines of Mishra [8].

Lemma 2.11. Let (X,F,L, ∗,◊) be an intuitionistic Menger space with t∗t ≥ t and (1 - t) ◊ (1 - t) ≤ (1 - t) and $\{y_n\}$ be a sequence in X. If there exists a number k ∈ (0, 1) such that:

(1) $F_{y_{n+2},y_{n+1}}(kt) \geq F_{y_{n+1},y_n}(t)$,

(2) $L_{y_{n+2},y_{n+1}}(kt) \leq L_{y_{n+1},y_n}(t)$ for all t > 0 and n =1,2,3,4,… Then$\{y_n\}$ is a Cauchy sequence inX.

Proof. By simple induction with the condition (1), we have for all t > 0 and n = 1,2,3,…,

$F_{y_{n+1},y_{n+2}}(t) \geq F_{y_1,y_2}(t/k^n)$, $L_{y_{n+1},y_{n+2}}(t) \leq L_{y_1,y_2}(t/k^n)$.

Thus by Definition 2.6 (6) and (11), for any positive integer m ≥ n and number t > 0, we have

$$F_{y_n, y_m}(t) \geq F_{y_n,y_{n+1}}\left(\frac{t}{m-n}\right) * F_{y_{n+1},y_{n+2}}\left(\frac{t}{m-n}\right) * \ldots * F_{y_{m-1},y_m}\left(\frac{t}{m-n}\right)$$

$$\geq \overbrace{(1-\lambda) * (1-\lambda) * \ldots\ldots\ldots\ldots * (1-\lambda)}^{m-n} > (1-\lambda),$$

and
$$L_{y_n, y_m}(t) \leq L_{y_n,y_{n+1}}\left(\frac{t}{m-n}\right) \Diamond L_{y_{n+1},y_{n+2}}\left(\frac{t}{m-n}\right) \Diamond \ldots \Diamond L_{y_{m-1},y_m}\left(\frac{t}{m-n}\right)$$

$$\leq \overbrace{\lambda \diamond \lambda \diamond \ldots\ldots\ldots\ldots \diamond \lambda}^{m-n} < \lambda,$$

which implies that $\{y_n\}$ is a Cauchy sequence in X. This completes the proof.

Lemma 2.12. Let (X,F,L, ∗,◊) be an intuitionistic Menger space with t ∗t ≥ t and (1 –t) ◊(1–t) ≤ (1 – t) and for all x,y∈ X , t > 0 and if for a number k ∈ (0,1)

$F_{x,y}(kt) \geq F_{x,y}(t)$ and $L_{x,y}(kt) \leq L_{x,y}(t)$ (I)

then x = y.

Proof. Since t > 0 and k∈ (0 , 1) we get t > kt. In intuitionistic Menger space (X,F,L, ∗,◊) , $F_{x,y}$ is non decreasing and $L_{x,y}$ is non-increasing for all x,y∈ X, then we have

$F_{x,y}(t) \geq F_{x,y}(kt)$ and $L_{x,y}(t) \geq L_{x,y}(kt)$.

Using (I) and the definition of intuitionistic Menger space, we have x = y.

Definition 2.13. The self-maps A and B of an intuitionistic Menger space (X,F,L, ∗,◊) are said to be compatible if for all t > 0,

$$\lim_{n\to\infty} F_{ABx_n,BAx_n}(t) = 1 \quad \text{and} \quad \lim_{n\to\infty} L_{ABx_n,BAx_n}(t) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z ∈ X.

Definition 2.14 The self-maps A and B of an intuitionistic Menger space (X,F,L, ∗,◊) are said to be compatible of type (A) if $\lim_{n\to\infty} F_{ABx_n,BBx_n}(t) = 1$, $\lim_{n\to\infty} F_{BAx_n,AAx_n}(t) = 1$ and $\lim_{n\to\infty} L_{ABx_n,BBx_n}(t) = 0$, $\lim_{n\to\infty} L_{BAx_n,AAx_n}(t) = 0$whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X .

Definition 2.15. The self-maps A and B of an intuitionistic Menger space (X,F,L, ∗,◊) are said to be compatible of type (P) if for all t > 0,

$$\lim_{n\to\infty} F_{AAx_n,BBx_n}(t) = 1 \quad \text{and} \quad \lim_{n\to\infty} L_{AAx_n,BBx_n}(t) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}$ Ax_n= $\lim_{n\to\infty}$ Bx_n = z for some z ∈ X.

Definition 2.16 The self-maps A and B of an intuitionistic Menger space (X,F,L, ∗,◊) are said to be compatible of type (E)
iff $\lim_{n\to\infty} F_{AAx_n,ABx_n}$(t) = 1 , $\lim_{n\to\infty} F_{AAx_n,Bz}$(t) = 1 , $\lim_{n\to\infty} F_{BBx_n,BAx_n}$(t) = 1, $\lim_{n\to\infty} F_{BBx_n,Az}$(t) = 1 and $\lim_{n\to\infty} L_{AAx_n,ABx_n}$ (t) = 0, $\lim_{n\to\infty} L_{AAx_n,Bz}$ (t) = 0 , $\lim_{n\to\infty} L_{BBx_n,BAx_n}$ (t) = 0, $\lim_{n\to\infty} L_{BBx_n,Az}$ (t) = 0 whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty}$Ax_n = $\lim_{n\to\infty}$ Bx_n = z , for some z in X .
Definition 2.17. The self-maps A and B of an intuitionistic Menger space (X,F,L, ∗,◊) are said to be compatible of type (K)ifffor all t > 0,

$\lim_{n\to\infty} F_{AAx_n,Bz}$(t) = 1 , $\lim_{n\to\infty} F_{BBx_n,Az}$(t) = 1 and $\lim_{n\to\infty} L_{AAx_n,Bz}$ (t) = 0, $\lim_{n\to\infty} L_{BBx_n,Az}$ (t) = 0

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}$ Ax_n= $\lim_{n\to\infty}$ Bx_n = z for some z ∈ X.

Obviously pair of two compatible of type (E) maps is also compatible of type (K) however
the converse is not true.
Definition 2.17A pair of self mapping A and B of an intuitionistic Menger space (X,F,L, ∗,◊) is said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}$ Ax_n= $\lim_{n\to\infty}$Bx_n = Bu , for some u ∈ X .
Definition 2.18 Two pairs (A,S) and (B,T) of self mappings of an intuitionistic Menger space (X,F,L, ∗,◊) are said to share CLRg of S property if there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that$\lim_{n\to\infty}$ Ax_n = $\lim_{n\to\infty}$ Sx_n = $\lim_{n\to\infty}$ By_n =$\lim_{n\to\infty}$ Ty_n = Sz, for some z∈X.
Example 2.19 Let X = [0,∞) be the usual metric space . Define g , h : X→X by gx= x+ 3 and hx = 4x , for all x∈X . We consider the sequence $\{x_n\}$ = { 1 + 1/n }. Since, $\lim_{n\to\infty}$gx_n = $\lim_{n\to\infty}$hx_n = 4 = h(1) ∈X Therefore g and h satisfy the (CLRg) property.

3. MAIN RESULT

Now we prove our main result
Theorem 3.1. Let (X, F, L, ∗, ◊) be a complete intuitionistic Menger space with t∗t≥t and (1-t) ◊ (1-t) ≤ (1-t) and let A, B, S and T be selfmappings of X such that the following conditions are satisfied :

(i) A(X) ⊆ T(X) , B(X) ⊆ S(X) ,
(ii) (B,T) is compatible of type (K),

(iii) There exists k ∈ (0,1) such that for every x, y∈ X and t > 0,

$$F_{Ax,By}(kt) \geq \{F_{Sx,Ty}(t) * F_{Ax,Sx}(t) * F_{By,Ty}(t) * F_{Ax,Ty}(t)\} \tag{1}$$

and $$L_{Ax,By}(kt) \leq \{L_{Sx,Ty}(t) \lozenge L_{Ax,Sx}(t) \lozenge L_{By,Ty}(t) \lozenge L_{Ax,Ty}(t)\} \tag{2}$$

If the pair (A,S) and (B, T) share the common limit in the range of S property , then A, B, S and T have a unique common fixed point.
Proof Let $x_0$be an arbitrary point in X. Since A(X) ⊂ T(X) and B(X) ⊂ S(X), there exist x_1, x_2 ∈ X such that Ax_0 = Tx_1 and Bx_1 = Sx_2. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$$

for n = 0,1,2,….Now putting in (1) and (2) x = x_{2n}, y = x_{2n+1}, we obtain

$$F_{Ax_{2n},Bx_{2n+1}}(kt) \geq \{F_{Sx_{2n},Tx_{2n+1}}(t) * F_{Ax_{2n},Sx_{2n}}(t) * F_{Bx_{2n+1},Tx_{2n+1}}(t) * F_{Ax_{2n},Tx_{2n+1}}(t)\}$$

that is

$$F_{y_{2n+1},y_{2n+2}}(kt) \geq \{F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+2},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n+1}}(t)\}$$
$$F_{y_{2n+1},y_{2n+2}}(kt) \geq \{F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n+2}}(t)\}$$
$$F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}(t)$$

and

$$L_{Ax_{2n},Bx_{2n+1}}(kt) \leq \{L_{Sx_{2n},Tx_{2n+1}}(t) \lozenge L_{Ax_{2n},Sx_{2n}}(t) \lozenge L_{Bx_{2n+1},Tx_{2n+1}}(t) \lozenge L_{Ax_{2n},Tx_{2n+1}}(t)\}$$

that is

$$L_{y_{2n+1},y_{2n+2}}(kt) \leq \{L_{y_{2n},y_{2n+1}}(t) \lozenge L_{y_{2n+1},y_{2n}}(t) \lozenge L_{y_{2n+2},y_{2n+1}}(t) \lozenge L_{y_{2n+1},y_{2n+1}}(t)\}$$
$$L_{y_{2n+1},y_{2n+2}}(kt) \leq \{L_{y_{2n},y_{2n+1}}(t) \lozenge L_{y_{2n+1},y_{2n+2}}(t)\}$$

$$L_{y_{2n+1},y_{2n+2}}(kt) \leq L_{y_{2n},y_{2n+1}}(t)$$

Similarly,

$$F_{y_{2n+2},y_{2n+3}}(kt) \geq F_{y_{2n+1},y_{2n+2}}(t) \text{ and } L_{y_{2n+2},y_{2n+3}}(kt) \leq L_{y_{2n+1},y_{2n+2}}(t).$$

Thus, we have

$$F_{y_{n+1},y_{n+2}}(kt) \geq F_{y_n,y_{n+1}}(t) \text{ and } L_{y_{n+1},y_{n+2}}(kt) \leq L_{y_n,y_{n+1}}(t) \quad \text{for n = 1,2,3, ….}$$

Therefore, we have

$$F_{y_n,y_{n+1}}(t) \geq F_{y_n,y_{n+1}}\left(\frac{t}{q}\right) \geq F_{y_{n-1},y_n}\left(\frac{t}{q^2}\right) \geq \ldots \geq F_{y_1,y_2}\left(\frac{t}{q^n}\right) \to 1$$

$$\text{and} L_{y_n,y_{n+1}}(t) \leq L_{y_n,y_{n+1}}\left(\frac{t}{q}\right) \leq L_{y_{n-1},y_n}\left(\frac{t}{q^2}\right) \leq \ldots \leq L_{y_1,y_2}\left(\frac{t}{q^n}\right) \to 0 \text{ when } n \to \infty$$

For each ε > 0 and t > 0, we can choose $n_0 \in$N

such that$F_{y_n,y_{n+1}}(t) > 1 - \varepsilon$ and $L_{y_n,y_{n+1}}(t) < \varepsilon$ for each $n \geq n_0$

For m, $n \in$ N, we suppose $m \geq n$. Then, we have

$F_{y_n,y_m}(t) \geq F_{y_n,y_{n+1}}\left(\frac{t}{m-n}\right) * F_{y_{n+1},y_{n+2}}\left(\frac{t}{m-n}\right) * \ldots * F_{y_{m-1},y_m}\left(\frac{t}{m-n}\right)$

> ((1−ε) ∗ (1−ε) ∗...(*m*−*n*)*times*... ∗ (1−ε))

≥ (1−ε),

and $L_{y_n,y_m}(t) \leq L_{y_n,y_{n+1}}\left(\frac{t}{m-n}\right) \lozenge L_{y_{n+1},y_{n+2}}\left(\frac{t}{m-n}\right) \lozenge \ldots \lozenge L_{y_{m-1},y_m}\left(\frac{t}{m-n}\right)$

< ((ε)◊(ε)◊...(*m*−*n*)*times*...◊(ε))

≤(ε).

$F_{y_n,y_m}(t) >$ (1−ε),$L_{y_n,y_m}(t) <$.ε .

Hence $\{y_n\}$ is a Cauchy sequence in *X*. As *X* is complete, $\{y_n\}$ converges to some point $z \in X$. Also, its subsequences converges to this point $z \in X$, i.e.$\{Bx_{2n+1}\} \to z$, $\{Sx_{2n}\} \to z$,$\{Ax_{2n}\} \to z$,$\{Tx_{2n+1}\} \to z$.

Since the pair (A, S) and (B,T) share the common limit in the range of S property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n$ = Sz for some z∈X.

First we prove that Az = Sz

By (1), putting x = z and y = y_n , we get

$F_{Az,By_n}(kt) \geq \{F_{Sz,Ty_n}(t) * F_{Az,Sz}(t) * F_{By_n,Ty_n}(t) * F_{Az,Ty_n}(t)\}$

Taking limit$n \to \infty$, we get

$F_{Az,Sz}(kt) \geq \{F_{Sz,Sz}(t) * F_{Az,Sz}(t) * F_{Sz,Sz}(t) * F_{Az,Sz}(t)\}$

$F_{Az,Sz}(kt) \geq F_{Az,Sz}(t)$ (3)

By (2), putting x = z and y = y_n , we get

$L_{Az,By_n}(kt) \leq \{L_{Sz,Ty_n}(t) \lozenge L_{Az,Sz}(t) \lozenge L_{By_n,Ty_n}(t) \lozenge L_{Az,Ty_n}(t)\}$

Taking limit$n \to \infty$, we get

$L_{Az,Sz}(kt) \leq \{L_{Sz,Sz}(t) \lozenge L_{Az,Sz}(t) \lozenge L_{Sz,Sz}(t) \lozenge L_{Az,Sz}(t)\}$

$L_{Az,Sz}(kt) \leq L_{Az,Sz}(t)$ (4)

By lemma 2.12 , Az = Sz (5)

Since, A(X) ⊆ T(X), therefore there exist u∈ X, such that Az = Tu (6)

Again by inequality (1), putting x = z and y = u, we get

$$F_{Az,Bu}(kt) \geq \{F_{Sz,Tu}(t) * F_{Az,Sz}(t) * F_{Bu,Tu}(t) * F_{Az,Tu}(t)\}$$

Using (5) and (6)

$$F_{Tu,Bu}(kt) \geq \{F_{Tu,Tu}(t) * F_{Tu,Tu}(t) * F_{Bu,Tu}(t) * F_{Tu,Tu}(t)\}$$

$$F_{Tu,Bu}(kt) \geq F_{Tu,Bu}(t)$$

By (2), putting x = z and y = u , we get

$$L_{Az,Bu}(kt) \leq \{L_{Sz,Tu}(t) \lozenge L_{Az,Sz}(t) \lozenge L_{Bu,Tu}(t) \lozenge L_{Az,Tu}(t)\}$$

Using (5) and (6)

$$L_{Tu,Bu}(kt) \leq \{L_{Tu,Tu}(t) \lozenge L_{Tu,Tu}(t) \lozenge L_{Bu,Tu}(t) \lozenge L_{Tu,Tu}(t)\}$$

$$L_{Tu,Bu}(kt) \leq L_{Tu,Bu}(t)$$

By lemma 2.12 ,

Tu = Bu (7)

Thus from (5), (6) ,(7) , we get

Az = Sz = Tu = Bu (8)

Now we will prove that Az = z

By inequality (1), putting x = z and y = x_{2n+1},

$$F_{Az,Bx_{2n+1}}(kt) \geq \{F_{Sz,Tx_{2n+1}}(t) * F_{Az,Sz}(t) * F_{Bx_{2n+1},Tx_{2n+1}}(t) * F_{Az,Tx_{2n+1}}(t)\}$$

Taking $\lim_{n\to\infty}$, we get

$$F_{Az,z}(kt) \geq \{F_{Sz,z}(t) * F_{Az,Sz}(t) * F_{z,z}(t) * F_{Az,z}(t)\}$$

$$F_{Az,z}(kt) \geq F_{Az,z}(t)$$

By (2), putting x = z and y = x_{2n+1},

$$L_{Az,Bx_{2n+1}}(kt) \leq \{L_{Sz,Tx_{2n+1}}(t) \Diamond L_{Az,Sz}(t) \Diamond L_{Bx_{2n+1},Tx_{2n+1}}(t) \Diamond L_{Az,Tx_{2n+1}}(t)\}$$

Taking limit$n \to \infty$, we get

$$L_{Az,z}(kt) \leq \{L_{Sz,z}(t) \Diamond L_{Az,Sz}(t) \Diamond L_{z,z}(t) \Diamond L_{Az,z}(t)\}$$

$$L_{Az,z}(kt) \leq L_{Az,z}(t)$$

By lemma 2.12, Az = z.. Thus from (8), we get z = Tu = Bu

Now (B, T) is compatible of type (K) then

$$\lim_{n\to\infty} BBy_{2n+1} = \text{Tz} \quad \text{and} \quad \lim_{n\to\infty} TTy_{2n+1} = \text{Bz}$$

that is Bz = Tz

Now putting x = z and y = z in inequality (1), we get

$$F_{Az,Bz}(kt) \geq \{F_{Sz,Tz}(t) * F_{Az,Sz}(t) * F_{Bz,Tz}(t) * F_{Az,Tz}(t)\}$$

$$F_{Az,Bz}(kt) \geq F_{Az,Bz}(t)$$

By (2), we get

$$L_{Az,Bz}(kt) \leq \{L_{Sz,Tz}(t) \Diamond L_{Az,Sz}(t) \Diamond L_{Bz,Tz}(t) \Diamond L_{Az,Tz}(t)\}$$

$$L_{Az,Bz}(kt) \leq L_{Az,Bz}(t)$$

By lemma 2.12, Az = Bz and hence Az = Bz = z

Combining all results , we get z = Az = Bz = Sz = Tz.

From this we conclude that z is a common fixed point of A, B, S and T.

Uniqueness : Let z_1 be another common fixed point of A, B, S and T . Then

$$z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1 \text{and} \qquad z = Az = Bz = Sz = Tz$$

then by inequality (1), putting x = z and y = z_1 , we get

$$F_{Az,Bz_1}(kt) \geq \{F_{Sz,Tz_1}(t) * F_{Az,Sz}(t) * F_{Bz_1,Tz_1}(t) * F_{Az,Tz_1}(t)\}$$

$$F_{z,z_1}(kt) \geq F_{z,z_1}(t)$$

By (2) , we get

$$L_{Az,Bz_1}(kt) \leq \{L_{Sz,Tz_1}(t) \Diamond L_{Az,Sz}(t) \Diamond L_{Bz_1,Tz_1}(t) \Diamond L_{Az,Tz_1}(t)\}$$

$$L_{z,z_1}(kt) \leq L_{z,z_1}(t)$$

By lemma 2.12, we get z = z_1.

Thus z is the unique common fixed point of A, B, S and T.

If we increase the number of self maps from four to six then we have the following .

Corollary 3.2 Let (X, F, L, ∗, ◊) be a complete intuitionistic Menger space with t∗t≥t and (1-t) ◊ (1-t) ≤ (1-t) . Let A, B, S, T, I and J be selfmappings of X such that the following conditions are satisfied :

(i) AB(X) ⊆ J(X) , ST(X) ⊆ I(X) ,

(ii) (ST,J) is compatible of type (K)

(iii) There exists k ∈ (0,1) such that for every x,y∈ X and t > 0,

$$F_{ABx,STy}(kt) \geq \{F_{Ix,Jy}(t) * F_{ABx,Ix}(t) * F_{STy,Jy}(t) * F_{ABx,Jy}(t)\} \tag{9}$$

and $L_{ABx,STy}(kt) \leq \{L_{Ix,Jy}(t) \Diamond L_{ABx,Ix}(t) \Diamond L_{STy,Jy}(t) \Diamond L_{ABx,Jy}(t)\}$ (10)

If the pair (AB,I) and (ST,J) share the common limit in the range of I property , then AB, ST, I and J have a unique common fixed point. Furthermore, if the pairs (A,B), (A,I),(B,I), (S,T),(S,J) and (T,J) are commuting mapping then A, B, S, T, I and J have a unique common fixed point.

Proof. From theorem 3.1,z is the unique common fixed point of AB, ST, I and J.

Finally, we need to show that z is also a common fixed point of A, B, S, T, I, and J. For this, let z be the unique common fixed point of both the pairs (AB,I) and (ST,J). Then, by using commutativity of the pair (A,B) , (A, I) , and (B, I) , we obtain

Az = A(ABz) = A(BAz) = AB(Az) , Az = A(Iz) =I (Az) ,(11)

Bz = B(ABz) =B(A(Bz)) = BA(Bz) = AB(Bz) ,Bz = B(Iz) = I(Bz) , which shows that Az and Bz are common fixed point of(AB, I) , yielding thereby

Az = z = Bz = Iz = ABz (12)

In the view of uniqueness of the common fixed point of the pair (AB,I). Similarly, using the commutativity of (S,T) ,(S, J) , (T, J) , it can be shown that

Sz =Tz = Jz = STz = z. (13)

Now, we need to show that Az = Sz (Bz = Tz) also remains a common fixed point of both the pairs (AB, I)and(ST, J) . For this, put x = z and y = z in (9) and using (12) and (13) , we get

$$F_{ABz,STz}(kt) \geq \{F_{Iz,Jz}(t) * F_{ABz,Iz}(t) * F_{STz,Jz}(t) * F_{ABz,Jz}(t)\}$$

$$F_{Az,Sz}(kt) \geq F_{Az,Sz}(t)$$

and by (10)

$$L_{ABz,STz}(kt) \leq \{L_{Iz,Jz}(t) \lozenge L_{ABz,Iz}(t) \lozenge L_{STz,Jz}(t) \lozenge L_{ABz,Jz}(t)\}$$

$$L_{Az,Sz}(kt) \leq L_{Az,Sz}(t)$$

By lemma 2.12, we getAz = Sz. Similarly, it can be shown that Bz = Tz. Thus, z is the unique common fixed point of A, B, S, T, I, and J.

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