IMPLEMENTATION OF A FAIR HERMITE INTERPOLATION SCHEME BASED ON QUADRATIC A-SPLINE *ELASTICA*

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ABSTRACT
The minimization of an energy functional is the main ingredient of several segmentation and geometric modeling problems. When the solution of this kind of optimization problem is described by a curve, the most popular approach consists in representing the curve as a parametric curve and to compute the minimum in terms of the free parameters of the curve. In free form design tasks, the fairness (energy) functional depends of the arc length and the bending energy of the curve and the classical approach requires to compute first and second derivatives. This work presents a Hermite interpolating subdivision scheme, based on Bézier rational curves, with local tension parameters and discusses an efficient software implementation of the algorithm for energy minimization of the functional. The curve that minimizes the functional is called the fair curve, and it shows excellent properties to be used for design purposes. The novelty of the proposed method lies in the fact it is derivative free. Also we include a discussion of the implementation of our method and show some numerical results.

KEYWORDS: Fairness, Hermite interpolation, subdivision scheme, elastica, Bézier rational curves.
MSC: 65D15, 65D17, 65D07, 65K10, 90C56

RESUMEN
La minimización de un funcional de energía es el ingrediente principal de varios problemas de segmentación y modelación geométrica. Cuando la solución de este tipo de problema de optimización es descrita por una curva, el enfoque más popular consiste en representar la curva en forma paramétrica y calcular el mínimo en términos de los parámetros libres de la curva. En los problemas de diseño libre, el funcional de energía (*fairness*) depende de la longitud de arco y de la eneregía de deformación (bending) de la curva y se requiere calcular primeras y segundas derivadas. Este trabajo presenta un esquema de subdivisión interpolante de Hermite, basado en curvas racionales de Bézier, con parámetros de control local y se discute la implenetación eficiente del algoritmo de minimización de la energía. La curva que minimiza el funcional se denomina curva *fair* is muestra excelentes propiedades para el diseño. Lo novedoso de este trabajo consiste en el hecho que es libre de derivadas. Además se incluye una discusión de la implementación del método y se muestran varios resultados numéricos.

PALABRAS CLAVE: Fairness, interpolación de Hermite, esquema de subdivisión, elástica, curva racional de Bézier.

1. INTRODUCTION

A very popular technique in curve design is to construct a spline curve that interpolates a sequence of points on the plane. But the interpolation is not only limited to these points, it can also be specified that the curve is tangent to certain directions at each of the interpolated points (Hermite interpolation). There are infinitely many smooth curves that satisfy these interpolation conditions, so it is expected that the designer

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will select the one that best fits the data. In this sense, it is introduced the notion of elastic curve, which is one that minimizes a certain functional from the Theory of Elasticity, which depends on the function and its derivatives.

Our first step is to study the Hermite subdivision scheme with rational quadratic Bézier curves and prepare a computational implementation of it. A second step consists in the approximate calculation of the elastic energy of the curve and, finally, to calculate a value of the tension parameter that minimizes an elastic energy functional, this is, that achieves the fair curve.

1.1. Related work

A-spline curves, introduced in the founding article [2], are polynomials in barycentric coordinates with respect to a reference triangle, written in Berntein-Bézier form. Subsequently, in [3], A-spline interpolation schemes are presented that minimize certain energies from the Theory of Elasticity (see [15]); and for this reason they are called *elastica*. Since the calculation of the A-spline curve that minimizes these functionals results in the calculation of complicated integrals that can only be solved numerically, the authors introduce in [3] a simplified formulation of these energies and develop in detail the case of the quadratic A-splines which are (approximately) *elastica*.

The sections of the quadratic A-spline curves have different representations, both as implicit or rational parametrized (rational Bézier spline) curves. From each representation, it is possible to calculate, with more or less computational effort, the coordinates of points on the curve, as well as their tangent directions and curvature values. More recently, in [8, 12, 7] a G^1-continuous Hermite interpolation subdivision scheme is studied with very good properties for the design of curves, whose limit curve is a quadratic A-spline curve. Particularly in [12] an initial approximation of the limit curve is achieved, however, the evaluation of the bending energy functional is performed by calculating the parametric values of the points generated by the subdivision scheme using an inversion formula, and these values are substituted in the calculation of the curvature values of those points, which results in a computationally expensive method.

In other nearby works, the problem is also attacked from similar points of view. In [14] it is considered energy minimization for quadratic Bézier polynomial curves, but not for rational curves. In [5], the minimization of the bending energy for rational Bézier curves is modified, imposing the calculation of the values of the weights w_i such that the sum of the values of the local bending energy at the ends of the spline sections is minimized, however this approach does not take into account the behavior in the interior of the curve sections. In [6] cubic curves are used, but the functional only uses the second derivative, the curvatures are not calculated. In [16], another simplifications of energies are discussed.

The present work may be included in a recent trend for curve and surface subdivision schemes, that consists in taking advantage of the hierarchical nature of subdivision schemes to provide algorithms for the efficient computation of energy functionals, see for instance [1],[9] and [10].

1.2. Our contribution

The main objective of this research is to propose and implement an efficient algorithm to compute a fair Hermite interpolation scheme based on quadratic A-spline *elastica*. Exploiting the self-similarity of the geometric information encapsulated in the subdivision scheme of [7, 13], which allows generating increasingly dense samples of points on the limit spline curve and based on the fact that for uniform sampling of regular smooth curves, Richardson extrapolation can be applied repeatedly giving a sequence of derivative free arc length estimates of arbitrarily high orders of accuracy [11], we show that it is possible to compute efficiently quadratic A-spline *elastica*, that interpolate given assigned points and tangent directions.

2. SUBDIVISION SCHEME

Subdivision schemes have become an efficient method for generating curves and surfaces in the Computer Aided Geometric Design (CAGD) environment. An univariate subdivision process defines a curve as the limit of a sequence of polygons that are refinements of an initial polygon. This schemes are important because they allow control of the shape of the limit curve and reproduce a wide family of curves used in Computer Graphics, such as conic sections and polynomial curves. In addition, they provide valuable information about the limit curve, without the need to use its analytical expression.

In [8], a Hermite subdivision interpolation scheme for rational quadratic Bézier curves with local tension parameter ω is proposed. The subdivision rule consists, essentially, in inserting between each pair of consecutive points of the scheme the shoulder point corresponding to that subsection. Choosing this point avoids the need to assign it a tangent direction and, moreover, it is easy to calculate with little numerical error. When performing the first refinement, the control triangle associated with each conic segment is split into two new control triangles and the conic section is split into two consecutive conic sections, and although these two new conic sections are arcs of the previous conic, it is necessary to find the new tension parameters that identify them in their Bernstein-Bézier form with respect to the new triangles. In [7], it is shown that the parameter ω_i^j associated with the conic that interpolates the i-th edge of the A-spline in the j-th step can be calculted with the recursion formula:

$$\omega_{2i-1}^{j+1} = \omega_{2i}^{j+1} = \sqrt{\frac{1+\omega_i^j}{2}}$$

In this way, giving an initial points and tangent directions, we can define a recursion, adding in each step the shoulder point corresponding to each new section as follows:

$$\begin{aligned}
P_{2i-1}^{j+1} &= P_i^j \\
P_{2i}^{j+1} &= \frac{P_i^j + \omega_i^j P_{i+1}^j}{1+\omega_i^j} \\
P_{2i+1}^{j+1} &= \frac{P_i^j + 2\omega_i^j P_{i+1}^j + P_{i+2}^j}{2(1+\omega_i^j)} \\
P_{2i+2}^{j+1} &= \frac{P_{i+2}^j + \omega_i^j P_{i+1}^j}{1+\omega_i^j} \\
P_{2i+3}^{j+1} &= P_{i+2}^j.
\end{aligned}$$

where P_{2i-1}^{j+1} and P_{2i+3}^{j+1} are the extreme points of the section, and P_{2i+1}^{j+1} is the corresponding shoulder point, for a better idea of the scheme see **Fig 1**.

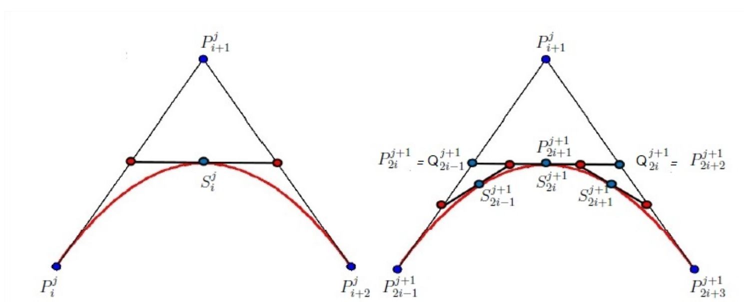

Figure 1: Refinement of an edge. Taken from [12]

This subdivision scheme manages to reproduce conic sections with any non-uniform distribution of the data. In addition, it satisfies many of the main current requirements of CAGD applications, since it interpolates not only points, but also the tangents associated with them, and the limit curve to which it converges is smooth. The existence of tension parameters allows local control of the geometry of the curve. The scheme also provides the ability to accurately reproduce arcs of conics, is invariant under affine transformations and it is relatively simple to implement computationally.

3. FAIRNESS FUNCTIONAL

There are different functions that provide ways to quantify the fairness of a curve. One of the most commonly used in literature is a linear combination of the bending energy E and stretching energy S, (see [3], [13]):

$$\begin{aligned} F_\lambda &= E + \lambda S \\ &= \int_0^1 \left(\frac{x'y'' - y'x''}{((x')^2 + (y')^2)^{3/2}} \right)^2 \frac{ds}{dt} dt + \lambda \int_0^1 \sqrt{(x')^2 + (y')^2} dt. \end{aligned}$$

where $(x(t), y(t))$ is a parametrization of the curve, with $t \in [0,1]$, $ds = \sqrt{(x')^2 + (y')^2} dt$ and $'$ is the derivative with respect to t.
Due to the complexity of the calculations of these integrals, and the fact that the previously explained subdivision rule offers a sample of points on the curve as dense as desired, then we can approximate these integrals using numerical methods, such as the method of the trapezoids.

Given $j > 0$, the 2^j chords $\overline{P^j_{2i} P^j_{2i+2}}$, with $i = 0, 1, ..., 2^j - 1$, have their ends on the conic section. Hence, the corresponding chord length of the conic section is:

$$\Delta^j_i s := \left|\left| P^j_{2i} - P^j_{2i+2} \right|\right|, i = 0, 1, ..., 2^j - 1$$

Then, the length of the curve section can be approximated by the sum:

$$S_j := \sum_{i=0}^{2^j - 1} \Delta^j_i s$$

Let $k(s)$ be the curvature of a rational quadratic Bézier spline section, parameterized by arc length, then the bending energy of this section is defined as

$$\int_0^l k^2(s) ds$$

where s is the arc length and l is the arc length of the section. Let k^j_i be the curvature of $c(t)$ at P^j_{2i}, with $i = 0, 1, ..., 2^j - 1$. Then, using the trapezoid rule for non-uniform spacing, the bending energy can be approximated by the sum:

$$E_j := \frac{1}{2} \sum_{i=0}^{2^j - 1} \left((k^j_i)^2 + (k^j_{i+1})^2 \right) \Delta^j_i s$$

In [13] is introduced a expression to calculate curvature corresponding to shoulder points in rational quadratic Bézier curves

$$k_{sp} = \frac{8 \omega A_T}{||P_2 - P_1||^3}$$

where A_T is the area of the control triangle.

4. ALGORITHM AND NUMERICAL EXAMPLES

The subdivision rule and the algorithms to calculate the energies and to minimize the fairness functional is implemented using Flutter framework, creating an iterative algorithm that receives the points, the tangent vectors and the initial values of the tension parameters ω corresponding to each section of the spline curve, and returns a list of points on the fair subdivision curve, as well as the value of the elastic energy and the value of the tension parameter ω that minimizes the functional.
The algorithms used in the program are shown in pseudocode below. Due the local nature of the problem, the numerical examples are shown for only one section of the spline curve, but in the software they are implemented for splines with several sections.

4.1. Subdivision algorithm

The following is a pseudocode for the implementation of the subdivision scheme. The initial conditions ($j = 0$) are:

- $P^0 = \{P_i^0, i = 0, 1, 2\}$ starting points,
- w^0 tension parameter,
- A^0 area of the triangle formed by the P^0 points.

Procedure AlgSub(j, P^j, ω^j, A_0^j)
$\omega^{j+1} = \sqrt{\frac{1+\omega^j}{2}}$, $A^{j+1} = \frac{1}{2}\frac{\omega^j A^j}{(1+\omega^j)^2}$, $P_{2^{j+2}}^{j+1} = P_{2^{j+1}}^j$
for $i = 1$ to $2^j - 1$ **do**
    $P_{4i+1}^{j+1} = P_{2i}^j$
    $P_{4i}^{j+1} = \frac{P_{2i}^j + \omega^j P_{2i+1}^j}{1+\omega^j}$
    $P_{4i+2}^{j+1} = \frac{P_{2i+2}^j + \omega^j P_{2i+1}^j}{1+\omega^j}$
    $P_{4i+3}^{j+1} = \frac{P_{4i+1}^{j+1} + P_{4i+3}^{j+1}}{2}$
end for
output:$P^{j+1}, \omega^{j+1}, A^{j+1}$

4.2. Energy approximation algorithm

Algorithm to approximate the fairness functional. The initial conditions are:

- $P^0 = \{P_i^0, i = 0, 1, 2\}$, starting points,
- w^0, tension parameter,
- A^0, area of the triangle formed by the P^0 points,
- $K^0 = \{k_0^0, k_1^0\}$
- $\Delta^0 = \{\Delta_0^0 s\}$
- $j*$

Procedure AlgEn(j, $P^j, \omega^j, A^j, K^j, \Delta^j, \lambda$)
$\{P^{j+1}, \omega^{j+1}, A^{j+1}\} = AlgSub(P^j, \omega^j, A^j)$
for $i = 1$ to $j * -1$ **do**
    $\Delta_i^{j+1} s = \left\|P_{2i+3}^{j+1} - P_{2i+1}^{j+1}\right\|$
    $k_{2i}^{j+1} = k_i^j$
    $k_{2i+1}^{j+1} = \frac{8\omega^j A_0^j}{(\Delta_i^j s)^3}$

$S^{j+1} = \sum_{i=0}^{2^j-1} \Delta_i^{j+1} s$

$E^{j+1} = \dfrac{1}{2} \sum_{i=0}^{2^j-1} \left((k_i^{j+1})^2 + (k_{i+1}^{j+1})^2\right) \sum_{r=i2^{j*-i}}^{(i+1)2^{j*-i}-1} \Delta_r^j s$

end for

$S = RichardsonExtrapolation\{S^1, S^2, ..., S^{j*}\}$

$E = RichardsonExtrapolation\{E^1, E^2, ..., E^{j*}\}$

output:$\{S, E\}$

4.3. Numerical examples

With the previous algorithms, for a given tension parameter ω we can efficiently approximate both energies from the fairness functional. Using these energy approximations, we can compute the tension parameter ω that minimizes the fairness functional with a derivative free a numerical method, such as Golden Section.
In **Fig 2** we can observe an example developed in [12], and in **Fig 3** we can observe the numerical results of the example for the approximation of the energies S, E and F.
It becomes apparent that our results improve the previous related works. Furthermore, we see that with few iterations (5), very good approximations are obtained with a computational cost in time that is acceptable for free design CAGD applications.

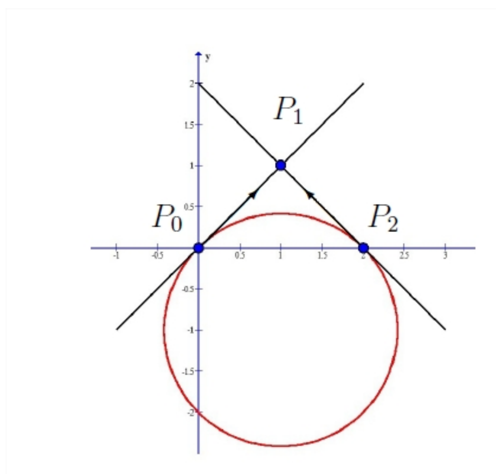

Figure 2: Numerical example. Taken from [4]

S(w)			
Iterations	Approximation	Absolute Error	Relative Error
3	2.2214446440	3.17E-06	7.15E-07
5	2.2214414064	6.27E-08	1.41E-08
7	2.2214414133	5.57E-08	1.25E-08
9	2.2214414184	5.06E-08	1.14E-08

E(w)			
Iterations	Approximation	Absolute Error	Relative Error
3	1.1107191216	1.61E-06	1.45E-06
5	1.1107207658	3.13E-08	2.82E-08
7	1.1107207623	2.78E-08	2.50E-08
9	1.1107207583	2.38E-08	2.15E-08

Iterations	Optimal value	Absolute error	Average time (s)
3	0.70711808232	1.13E-05	0.629
5	0.70710655986	2.21E-07	3.399
7	0.70710664875	2.15E-07	6.407
9	0.70710667798	1.03E-07	23.7858

Figure 3: Numerical results [4]

4.4. Some images

We show some images obtained by our software implementation

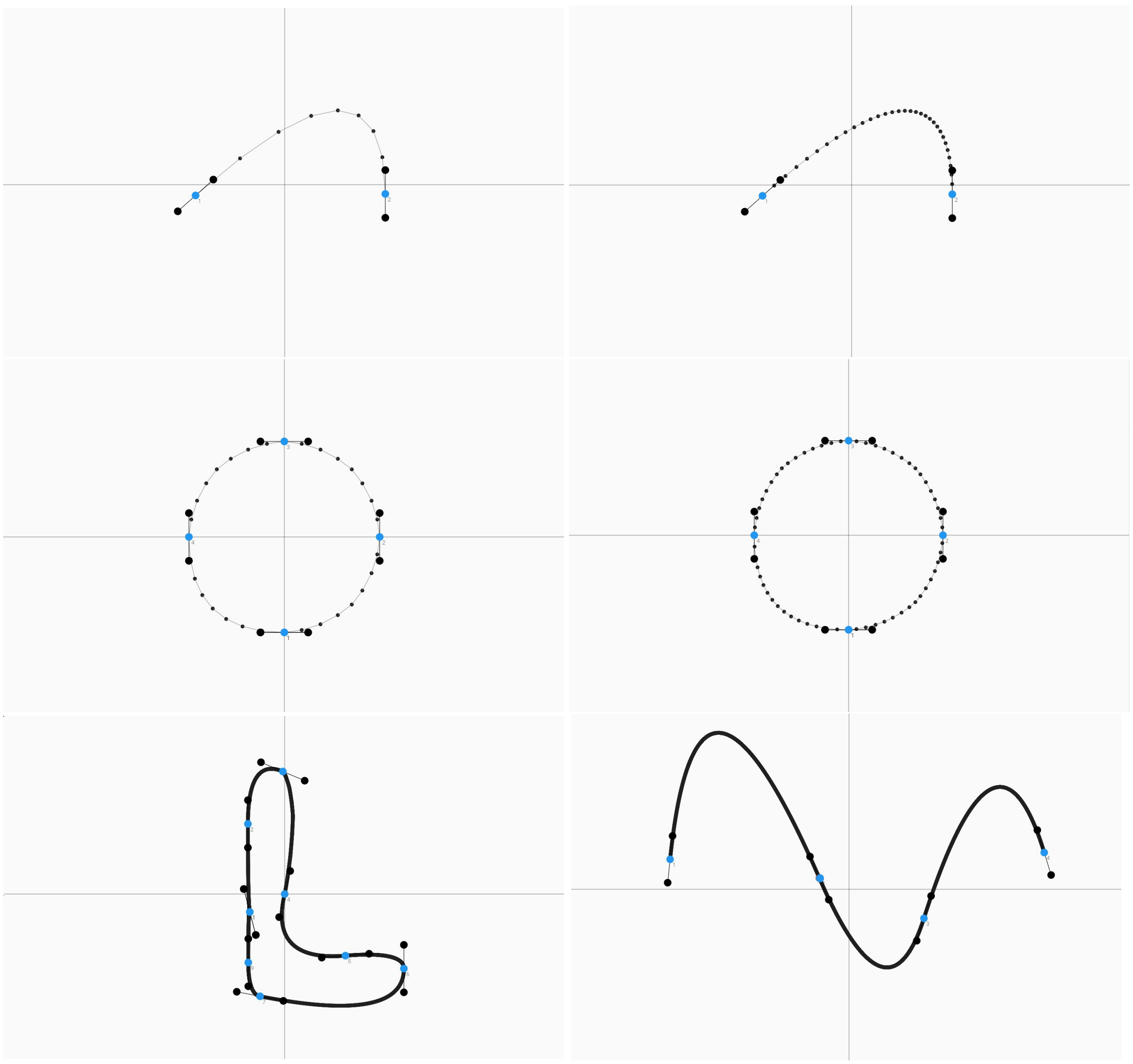

Figure 4: Examples of the software. Upper row: a single one-section subdivision example with 3 and 5 steps. Middle row: subdivision of a circle with 3 and 5 steps. Bottom row: another two simple outputs from the software with 11 subdivision steps.

5. CONCLUSIONS

A subdivision scheme to compute a dense sample of points of rational quadratic Bézier splines (i.e., quadratic A-splines) with local tension parameters is presented. Exploiting the self-similarity of the geometric information encapsulated in the subdivision scheme, it is proposed an efficient algorithm to determine the arc lengths of the spline sections, as well as their elastic energy, without using evaluations of the functions or

their first and second derivatives, obtaining as a result, a computationally cheap and derivative free method to compute the tension parameters that minimize the fairness functional.
The proposed algorithms and implementation of our method may be extended to those curve subdivision schemes with known formulae for the position of the limit points, their tangent vectors and curvatures, enabling the implementation of a multiscale optimization strategy, where the energy (fairness) functional is optimized in a coarse-to-fine fashion.

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