# ONE-PARAMETRIC SCHEMES FOR SOLVING MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS: THEORETICAL PROPERTIES

Gemayqzel Bouza*† Ernest Quintana** and Georg Still***
*Universidad de La Habana, Cuba.
**ernestqt@yandex.com.
**** University of Twente, The Netherlands.

**ABSTRACT**
Due to the complex disjunctive structure of mathematical programs with complementarity constraints (MPCC), parametric approaches are used to overcome this difficulty. The underlying idea is to solve a program depending on the real parameter $\tau \geq 0$, where $\tau = 0$ corresponds to the original MPCC program. The paper considers seven approaches: two based on smoothing the complementarity constraints and the other five, on their regularisation. We consider the point-to-set functions that, for each value of the parameter $\tau$, define the set of feasible solutions and the set of optimal solution of the parametric problems they define. We study the distance between the feasible sets and the set of minimisers of the parametric program for $\tau$ going to zero.

**KEYWORDS:** mathematical program with complementarity constraints, smoothing scheme, regularisation scheme, order of convergence, stationarity.

**MSC:** 90C30.

**RESUMEN**
Los problemas de de progaramación matemática con restricciones de complementariedad (MPCC, por sus siglas en inglés) tienen una estructura disjuntiva. Es por eso que se han usado enfoques paramétricos para su resolución. Para ello se considera un problema que depende de un parámetro $\tau \geq 0$, tal que para $\tau = 0$ el modelo es equivalente al MPCC original. En este artículo de consideran los siete enfoques paramétricos fundamentales: dos basados en suavizar las restricciones de complementariedad y cinco que la regularizan. Consideramos las funciones conjunto-evaluadas que para cada valor del parámetro definen el conjunto de soluciones factibles y el conjunto de soluciones óptimas del problema paramétrico. En este trabajo se estudia la distancia entre dichos conjuntos cuando $\tau \to 0$.

**PALABRAS CLAVE:** programación matemática con restricciones de complementariedad, esquema de suavización, esquema de regularización, orden de convergencia estacionariedad

†Corresponding author: gema@matcom.uh.cu

## 1. INTRODUCTION

A Mathematical Program with Complementarity Constraints (MPCC for short) can be written as:

$$(\mathcal{P}): \quad minf(x) \quad s.t. \quad x \in \mathcal{M} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_j(x) \leq 0, & j = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ r_i(x)s_i(x) = 0, & i = 1, \ldots, m, \end{array} \right\} \tag{1.1}$$

with functions $f, g_1, \ldots, g_q, r_1, \ldots, r_m, s_1, \ldots, s_m : \mathbb{R}^n \rightarrow \mathbb{R}$. The conditions $r_i(x)s_i(x) = 0$, $i = 1, \ldots, m$, are called complementarity constraints. Due to the combinatorial nature of these complementarity conditions, approaches which override this difficulty are of high interest. In the past, several solutions approaches have been investigated, see for example, [4], [10],[8],[6], [13] and [12]. The main idea of these methods is to replace the complementarity conditions by equality or inequality constraints depending on a parameter $\tau \geq 0$ such that, at $\tau = 0$, the original MPCC is obtained. Then, for a sequence $\tau^k \downarrow 0$, the solutions of the parametric program are computed. It is expected that these solutions converge to a solution of the original program $\mathcal{P}$.
The present paper considers seven of these approaches. In two of them the complementarity constraints are substituted by equality constraints, leading to the following parametric problems:

$$(\mathcal{Q}_\tau): \quad minf(x) \quad s.t. \quad x \in \mathcal{M}^{\mathcal{Q}}{}_\tau = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_j(x) \leq 0, & j = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ r_i(x)s_i(x) = \tau, & i = 1, \ldots, m, \end{array} \right\} \tag{1.2}$$

and

$$(\mathcal{P}_\tau): \quad minf(x) \quad s.t. \quad x \in \mathcal{M}^{\mathcal{P}}{}_\tau = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_j(x) \leq 0, & j = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ r^T(x)s(x) = \tau. & \end{array} \right\} \tag{1.3}$$

The other five approaches are the regularisation methods proposed in [10], [8], [6], [13] and [12]. They are given, respectively, by the parametric problems

$$\begin{array}{l} (\mathcal{R}^{\mathcal{S}}_\tau) : minf(x) \\ s.t. \quad x \in \mathcal{M}^{\mathcal{S}}_\tau \end{array} \tag{1.4}$$

$$\mathcal{M}^{\mathcal{S}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ r_i(x)s_i(x) \leq \tau, & i = 1, \ldots, m. \end{array} \right\}$$

$$\begin{array}{l} (\mathcal{R}^{\mathcal{LF}}_\tau) : minf(x) \\ s.t. \quad x \in \mathcal{M}^{\mathcal{LF}}_\tau \end{array} \tag{1.5}$$

$$\mathcal{M}^{\mathcal{LF}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \ldots, q, \\ r_i(x)s_i(x) - \tau^2 \leq 0, & i = 1, \ldots, m, \\ (r_i(x) + \tau)(s_i(x) + \tau) - \tau^2 \geq 0, & i = 1, \ldots, m. \end{array} \right\}$$

$$
\begin{aligned}
(\mathcal{R}^{\mathcal{K}}_\tau): & \; min f(x)\\
s.t. & \quad x \in \mathcal{M}^{\mathcal{K}}_\tau
\end{aligned} \tag{1.6}
$$

$$
\mathcal{M}^{\mathcal{K}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \ldots, q, \\ r_i(x), s_i(x) \geq -\tau, & i = 1, \ldots, m, \\ (r_i(x) - \tau)(s_i(x) - \tau) \leq 0, & i = 1, \ldots, m. \end{array} \right\}
$$

$$
\begin{aligned}
(\mathcal{R}^{\mathcal{SU}}_\tau): & \; min f(x)\\
s.t. & \quad x \in \mathcal{M}^{\mathcal{K}}_\tau
\end{aligned} \tag{1.7}
$$

$$
\mathcal{M}^{\mathcal{SU}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ r_i(x) + s_i(x) - \phi^{SU}(r_i(x) - s_i(x); \tau) \leq 0, & i = 1, \ldots, m. \end{array} \right\}
$$

$$
\begin{aligned}
(\mathcal{R}^{\mathcal{SK}}_\tau): & \; min f(x)\\
s.t. & \quad x \in \mathcal{M}^{\mathcal{SK}}_\tau
\end{aligned} \tag{1.8}
$$

$$
\mathcal{M}^{\mathcal{SK}}_\tau = \left\{ x \in \Re^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0, & i = 1, \ldots, q, \\ r_i(x), s_i(x) \geq 0, & i = 1, \ldots, m, \\ \phi^{SK}(x, \tau) \leq 0, & i = 1, \ldots, m. \end{array} \right\}
$$

Here

$$
\phi^{SU}(a, \tau) = \begin{cases} |a|, & \text{if } |a| \geq \tau, \\ \tau\theta(a/\tau), & \text{otherwise} \end{cases}
$$

and

$$
\phi^{SK}(x, \tau) = \begin{cases} (r_i(x) - \tau)(s_i(x) - \tau), & \text{if } r_i(x) + s_i(x) \geq 2\tau, \\ -\left[(r_i(x) - \tau)^2 + (s_i(x) - \tau)^2\right]/2, & \text{otherwise.} \end{cases},
$$

where $\theta$ is a $C^2$-function regularising function, *i.e.* $\theta(1) = \theta(-1) = 1,\ \theta'(1) = -\theta'(-1) = 1,\ \theta''(-1) = \theta(1) = 0$ and $\theta''(x) > 0,$ for all $x \in (-1, 1)$.

Problems (1.2), (1.4), (1.5), (1.6), (1.7) and (1.8) have been studied in papers as [2, 9, 5, 12, 7], respectively. With these parametrizations, the singularity at points such that $r_i(x) = s_i(x) = 0$ can be treated with known algorithms. However, all properties have not been studied for all the approaches. So, a comparison among these approaches is not complete. The goal of this paper is to start to fill these gaps. For each case, we will obtain which properties are satisfied. In this part, using tools from Parametric Optimization, we will analyse the distance between the sets of feasible solutions of the parametric problems and $\mathcal{M}$. Similar properties will be studied for the set of optimal solutions. We emphasize that most of the properties of $\mathcal{Q}_\tau$ have been studied in [2]. So, we are mainly interested in analising the other approaches. Often we will simply refer to [2] and only sketch proofs if arguments or techniques are similar to those used in this reference.

For simplicity we consider only inequality constraints, but under standard extensions of the linear independence constraint qualification (LICQ) and the Mangasarian Fromovitz constraint qualification (MFCQ), all results of this paper can be extended to the case of MPCC problems with additional equality constraints.

The paper is organized as follows. In Section 2, we review some preliminary material on MPCC programs and parametric optimisation problems needed in the subsequent sections. Section 3 studies the local stability of the sets of feasible solutions of the parametric models with respect to $\mathcal{M}^{\mathcal{P}}$. Bounds (depending on $\tau$) for the Hausdorff distance between the sets will be given. The results are compared with those obtained in [2] for model $\mathcal{Q}_\tau$. In Section 4, similar results are proven for the set of solutions. Finally, we summarize the contributions of the paper.

We end this section with some basic notation that will be used throughout the text. The canonical vectors in $\mathbb{R}^n$ will be denoted by $e_i$. The open ball centered at $\bar{x} \in \mathbb{R}^n$ with radius $\epsilon > 0$ will be $B_\epsilon(\bar{x}) = \{x \mid \|x - \bar{x}\| < \epsilon\}$, where $\|x\|$ is the Euclidean norm. The distance from a point $x$ to the set $\mathcal{A}$ is $d(x, \mathcal{A}) = \inf\{\|x - y\| \mid y \in \mathcal{A}\}$. The Hausdorff distance between the sets $A$ and $B$ is defined as

$$\tilde{d}(\mathcal{A}, \mathcal{B}) = \max\left\{\max\{d(x, \mathcal{B}) \mid x \in \mathcal{A}\}, max\{d(x, \mathcal{A}) \mid x \in \mathcal{B}\}\right\}.$$

## 2. PRELIMINARIES

We start with some basic concepts and results from MPCC theory and parametric optimisation that will be needed later on.

### 2.1. MPCC theory

When dealing with MPCC problems $\mathcal{P}$, the following active index sets play an important role:

$$I_g(x) = \{j \in \{1, \dots, q\} \mid g_j(x) = 0\},$$

$$I_r(x) = \left\{ i \in \{1, \dots, m\} \left| \begin{matrix} r_i(x) & = & 0, \\ s_i(x) & > & 0 \end{matrix} \right. \right\}, I_s(x) = \left\{ i \in \{1, \dots, m\} \left| \begin{matrix} r_i(x) & > & 0, \\ s_i(x) & = & 0 \end{matrix} \right. \right\},$$

$$I_{rs}(x) = \{i \in \{1, \dots, m\} \mid r_i(x) = 0, s_i(x) = 0\},$$

**Definition 2.1. (Strict Complementarity (SC) for $\mathcal{P}$)** *Let $\bar{x} \in \mathcal{M}$. We say that SC holds for $\mathcal{P}$ at $\bar{x}$ if $I_{rs}(\bar{x}) = \emptyset$.*

The regularity conditions for MPCC constitute adaptations from their nonlinear programming counterpart, see [1]. Here we present some that will be used later.

**Definition 2.2.** *Let $\bar{x} \in \mathcal{M}$. We say that MPCC-LICQ holds at $\bar{x}$, if the set of vectors*

$$\{\nabla g_j(\bar{x}) | j \in I_g(\bar{x})\} \cup \{\nabla r_i(\bar{x}) | i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})\} \cup \{\nabla s_i(\bar{x}) | i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})\}$$

*is linearly independent. MPCC-MFCQ is said to hold at $\bar{x}$ if the system*

$$\{\nabla r_i(\bar{x}) | i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})\} \cup \{\nabla s_i(\bar{x}) | i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})\}$$

*is linearly independent and there exists some $d \in \mathbb{R}^n$ such that*

$$\nabla g_j(\bar{x})^T d < 0, \quad \forall\, j \in I_g(\bar{x}),$$

$$\nabla r_i(\bar{x})^T d = 0, \quad \forall\, i \in I_r(\bar{x}) \cup I_{rs}(\bar{x}), \qquad \nabla s_i(\bar{x})^T d = 0, \quad \forall\, i \in I_s(\bar{x}) \cup I_{rs}(\bar{x}).$$

It is clear MPCC-LICQ and MPCC-MFCQ lead to the classical LICQ and MFCQ if $m = 0$.
As a consequence of Farkas's Lemma, an alternative characterization of MPCC-MFCQ can be obtained.

**Lemma 2.1. (cf. [1])** *Let $\bar{x} \in \mathcal{M}$. Then MPCC-MFCQ holds at $\bar{x}$ if and only if the relation*

$$0 = \sum_{j \in I_g(\bar{x})} \mu_j \nabla g_i(\bar{x}) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i \nabla r_i(\bar{x}) - \sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i \nabla s_i(\bar{x}),$$

*for $\mu \geq 0$ and (free) real vectors $\rho, \sigma$ implies $(\mu, \rho, \sigma) = 0$.*

For $\bar{x} \in \mathcal{M}$, we introduce the Lagrangean function (near $\bar{x}$),

$$L(x, \mu, \rho, \sigma) = f(x) + \sum_{j \in I_g(\bar{x})} \mu_j g_j(x) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i r_i(x) - \sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i s_i(x). \tag{2.1}$$

Now, we present main stationarity concepts:

**Definition 2.3. (Stationarity Concepts)** *Let $\bar{x} \in \mathcal{M}$. Then, $\bar{x}$ is called weakly stationary (W-stationary) if there are multipliers $(\mu, \rho, \sigma) \in \mathbb{R}^{|I_g(\bar{x})|+|I_r(\bar{x})|+|I_s(\bar{x})|}$ with $\mu \geq 0$ such that*

$$0 = \nabla f(\bar{x}) + \sum_{j \in I_g(\bar{x})} \mu_j \nabla g_j(\bar{x}) - \sum_{i \in I_r(\bar{x}) \cup I_{rs}(\bar{x})} \rho_i \nabla r_i(\bar{x}) - \sum_{i \in I_s(\bar{x}) \cup I_{rs}(\bar{x})} \sigma_i \nabla s_i(\bar{x}).$$

*A W-stationary point $\bar{x}$ with corresponding multipliers $(\mu, \rho, \sigma)$ is:*

*(a) Clarke stationary (C-stationary) if $\rho_i \sigma_i \geq 0$, for all $i \in I_{rs}(\bar{x})$.*

*(b) Mordukhovich stationary (M-stationary) if either $\rho_i \sigma_i = 0$ or $\rho_i, \sigma_i > 0$ holds, for all $i \in I_{rs}(\bar{x})$.*

*(c) Strongly stationary (S-stationary) if $\rho_i, \sigma_i \geq 0$ for all $i \in I_{rs}(\bar{x})$.*

Clearly

$$\text{S-stationarity} \Rightarrow \text{M-stationarity} \Rightarrow \text{C-stationarity} \Rightarrow \text{W-stationarity}.$$

It is worth to point out that the S-stationarity condition is equivalent to the standard KKT condition applied directly to problem (1.1).
The following necessary condition holds:

**Theorem 2.1. (First order necessary condition, cf. [3])** *Let $\bar{x}$ be a local minimiser at which MPCC-LICQ is satisfied. Then $\bar{x}$ is an S-stationary point.*

## 3. THE SET OF FEASIBLE POINTS OF THE SCHEMES.

In this section we will analyse the properties of the sets defined by the approaches. Roughly speaking, for each approach we perform a local analysis. We will study if, locally around a feasible solution in which MPCC-LICQ holds, we can expect that LICQ holds at the parametric problem defined by the approaches. Moreover, we will find a bound of the Hausdorff distance of the parametric problem and

the set of feasible solutions. We will obtain properties Most of the latter was studied earlier for $\mathcal{Q}_\tau$ in [2] and, as mentioned before, we will often refer to arguments and techniques used in this article.
We assume from now on that $g, r, s$ are $\mathcal{C}^1$ or $C^2$ functions.
If $I_g(\bar{x})$ has $q_0$ elements, by exchanging indices we can assume that the active constraints are the first $q_0$. In the case of the complementarity constraints there are some indices such that $r_i(\bar{x}) = s_i(\bar{x}) = 0$. Suppose $|I_{00}(\bar{x})| = p$. Again, after some possible exchange of the indices of the functions we can assume that $r_i(\bar{x}) = s_i(\bar{x}) = 0$, for $i = 1, \dots, p$ and $r_i^2(\bar{x}) + s_i^2(\bar{x}) > 0$, for $i = p+1, \dots, m$. For those indices, $i,\ i > p$ such that $0 = s_i(\bar{x}) < r_i(\bar{x})$, we exchange the roles of $r_i$ and $s_i$. So, without loss of generality (w.l.o.g.), in the sequel we assume

$$I_g(\bar{x}) = \{1, \dots, q_0\}, \quad I_{rs}(\bar{x}) = \{1, \dots, p\}, \quad I_r(\bar{x}) = \{p+1, \dots, m\}, \quad I_s(\bar{x}) = \emptyset. \tag{3.1}$$

We assume $\bar{x} = 0$. As in [2], recall that the MPCC-LICQ condition is satisfied at $\bar{x}$, we suppose that

$$rank(\nabla_{x_1, \dots x_{m+p+q_0}}(r_1(\bar{x}), \dots r_m(\bar{x}), s_1(\bar{x}), \dots s_p(\bar{x}), g_1(\bar{x}), \dots g_{q_0}(\bar{x})) = m + p + q_0.$$

So, we can define a local diffeomorphism $T : B_\varepsilon(\bar{x}) \to B_\delta(0)$, as

$$T(x) = (r_1(x), \dots r_m(x), s_1(x), \dots s_p(x), g_1(x), \dots g_{q_0}(x), x_{m+p+q_0}, \dots x_n).$$

$T(x)$ canonically transforms the feasible sets of the parametric problems for small $\tau \geq 0$. Hence, locally around $\bar{x}$ we can assume that

$$\begin{array}{l} r_i(x) = x_i,\ i = 1, \dots, m, \\ s_i(x) = x_{m+i},\ i = 1, \dots, p, \\ g_j(x) = x_{m+p+j},\ j = 1, \dots, q_0. \end{array} \tag{3.2}$$

and

$$s_i(x) \geq \frac{s_i(\bar{x})}{2} > 0,\ i = p+1, \dots m, \qquad g_j(x) \leq \frac{g_j(\bar{x})}{2} < 0,\ j \notin I_g(\bar{x}). \tag{3.3}$$

We refer to [2] for details. Now, we particularise these results for the different approaches.

### 3.1. Case $\mathcal{P}_\tau$

**Theorem 3.1.** *Suppose that MPCC-LICQ is satisfied at $\bar{x}$, feasible point of $\mathcal{P}$, then there exists $\overline{\tau}$ and a neighborhood $V$ of $\bar{x}$ such that for all $\tau \in (0, \overline{\tau})$ all $x_\tau \in V \cap \mathcal{M}^{\mathcal{P}}{}_\tau$, the LICQ holds.*

**Proof:** As the MPCC-LICQ is fulfilled at $\bar{x}$, we can use (3.2). Then, locally, the set of constraints is described by

$$\left\{ \begin{array}{c} x_i \geq 0,\ i = 1, \dots, m, \quad x_{m+i} \geq 0,\ i = 1, \dots, p, \quad x_{m+p+i} \leq 0,\ i = 1, \dots, q_0, \\ \sum_{i=1}^p x_i x_{m+i} + \sum_{i=p+1}^m x_i s_i(x) = \tau. \end{array} \right. \tag{3.4}$$

By the continuity of the involved functions there exists a neighborhood $V$ such that if $x_\tau \in \mathcal{M}^{\mathcal{P}}{}_\tau$, $I_g^1(x_\tau) \subset I_g(\bar{x})$, $\{i : x_i = 0\}, \subset I_r(\bar{x}) \cup I_s(\bar{x})$. On the other hand, as $\sum_{i=1}^p x_i x_{m+i} + \sum_{i=p+1}^m x_i s_i(x) = \tau$, at least one of the terms of the previous sum is non-zero. We consider two cases. If $p + 1 < i^* \leq m$, $x_{i^*} \neq 0$, w.l.o.g. $i^* = m$, and $x_i = 0$, $i = p+1, \dots, m$, The respective gradients are

$$\begin{pmatrix} I_r^1 & 0 & 0 & 0 & \\ 0 & 0 & I_s^1 & 0 & 0 \\ 0 & 0 & 0 & I_g^1 & 0 \\ \otimes & s_m + \sum_{i=p+1}^m x_i \frac{\partial s_i}{\partial x_m} & \otimes & \otimes & \otimes \end{pmatrix}^T \quad \begin{pmatrix} 0|I_r^1 & 0 & 0 & 0 \\ 0 & I_s^1 & 0 & 0 \\ 0 & 0 & I_g^1 & 0 \\ x_{m+1}|0|0 & \otimes & \otimes & \otimes \end{pmatrix}^T$$

Noting that in the first case as $x_i \to 0$, we get $s_m + \sum_{i=p+1}^m x_i \frac{\partial s_i}{\partial x_m} \neq 0$ and that in the second case $\sum_{i=1}^p x_i x_{m+i} = \tau$. implies that at least one term is non-zero, it follows that in both cases the gradients are linearly independent, for $\tau$ small enough. □

For the analysis of $\mathcal{M}^{\mathcal{P}}{}_\tau$ apart from the SC-assumption the following condition will play a role.

**Definition 3.1.** *We say that the point $\bar{x}$ is a complementarity non-degenerate point, or that CND holds at $\bar{x}$, if $I_{rs}(\bar{x}) \neq \{1, \dots, m\}$, or equivalently $p < m$. This condition says that at least for one $i_0$ it holds $r_{i_0}(\bar{x}) + s_{i_0}(\bar{x}) > 0$.*

Now we prove the (Hölder) stability of the feasible set $\mathcal{M}^{\mathcal{P}}{}_\tau$ near $\bar{x} \in \mathcal{M}$.

**Theorem 3.2.** *Assume that MPCC-LICQ holds at $\bar{x} \in \mathcal{M}$. Then there exist $\varepsilon, \tau_0, \alpha, \beta > 0$, such that for all $\tau \in [0, \tau_0)$ the following is true: there exists $x_\tau \in \mathcal{M}^{\mathcal{P}}{}_\tau \cap B_\varepsilon(\bar{x})$ satisfying $\|x_\tau - \bar{x}\| \leq \alpha\sqrt{\tau}$, and for any $\bar{x}_\tau \in \mathcal{M}^{\mathcal{P}}{}_\tau \cap B_\varepsilon(\bar{x})$ there exists $\hat{x}_\tau \in \mathcal{M} \cap B_\varepsilon(\bar{x})$ such that $\|\bar{x}_\tau - \hat{x}_\tau\| \leq \beta\sqrt{\tau}$.*
*If in addition CND holds at $\bar{x}$, then $\alpha\sqrt{\tau}$ can be replaced by $\alpha\tau$ and if SC is satisfied then, we can take $\beta\tau$ instead of $\beta\sqrt{\tau}$.*

**Proof:** According to the arguments given above, without loss of generality, we can assume $\bar{x} = 0$ and that in $B_\varepsilon(0)$, for small $\tau \geq 0$, the feasibility condition for $\mathcal{M}^{\mathcal{P}}{}_\tau$ is given by (3.4). In case that CND does not hold, we can choose the element $x_\tau$ with components

$$[x_\tau]_i = 0, \dots, m-1, m+1, \dots, 2m-1, \;\; [x_\tau]_m = [x_\tau]_{2m} = \sqrt{\tau}, \;\; [x_\tau]_j = 0, j = 2m+1, \dots, n \, .$$

Then, obviously $(\bar{x} = 0)$ $\|x_\tau - \bar{x}\| = \sqrt{2}\sqrt{\tau}$. As usual, without loss of generality, $s_m(\bar{x}) > 0$.
If CND holds, we can choose $x_\tau$ by

$$[x_\tau]_i = 0, i = 1, \dots, m-1, m+1, \dots, 2m, \;\; [x_\tau]_m = \frac{\tau}{s_m(\bar{x})}, \;\; [x_\tau]_j = 0, j = 2m+1, \dots, n \, ,$$

which satisfies $\|x_\tau - \bar{x}\| = \frac{\tau}{s_m(\bar{x})} = O(\tau)$ .
To show the second relation, let $\bar{x}_\tau \in \mathcal{M}^{\mathcal{P}}{}_\tau \cap B_\varepsilon(0)$. The last feasibility condition in (3.4) reads $\sum_{i=1}^p x_i x_{m+i} + \sum_{i=p+1}^m x_i s_i(x) = \tau$ , which implies (see (3.3))

$$0 \leq \min\{x_i, x_{m+i}\} \leq \sqrt{\tau}, \; i = 1, \dots, p \quad \text{and} \quad x_i \leq \frac{2}{s_i(\bar{x})} \tau, \; i = p+1, \dots, m.$$

Without loss of generality, we assume that $[\bar{x}_\tau]_i \leq \sqrt{\tau}, \; i = 1, \dots, p$, and, putting $c := max\{\frac{2}{s_i(\bar{x})}, \; i = p+1, \dots, m\}$, we can choose $\hat{x}_\tau \in \mathcal{M}$ as follows:

$$[\hat{x}_\tau]_i = 0, i = 1, \dots, m, \;\; [\hat{x}_\tau]_i = [\bar{x}_\tau]_i, \;\; i = m+1, \dots, n \, .$$

This yields

$$\|\bar{x}_\tau - \hat{x}_\tau\| = \sqrt{\sum_{i=1}^{m}[\bar{x}_\tau]_i^2} \leq \sqrt{p\tau + (m-p)c^2\tau^2} = O(\sqrt{\tau}) \; .$$

In case SC holds, i.e., $p = 0$, we, obviously, have $\|\bar{x}_\tau - \hat{x}_\tau\| \leq \sqrt{mc^2\tau^2} = O(\tau)$ .
From a global viewpoint we have the following result

**Theorem 3.3.** *Suppose that the sets $\mathcal{M}^{\mathcal{P}}{}_\tau$ are contained in some compact set $K$. Then, there exists $\overline{\tau}$ such that for all $\tau \in (0, \overline{\tau})$ the Hausdorff distance between $\mathcal{M}$ and $\mathcal{M}^{\mathcal{P}}{}_\tau$ is bounded by $O(\sqrt{\tau})$ and the LICQ is satisfied for all $x_\tau$, feasible point of $\mathcal{P}_\tau$.*

*Proof:* Suppose that $\mathcal{M}, \mathcal{M}^{\mathcal{P}}{}_\tau$ are subsets of a compact set $X \subset \mathbb{R}^n$. Using compactness arguments, Theorem 3.2. leads to a corresponding global (Hölder/Lipschitz) continuity result for $\mathcal{M}^{\mathcal{P}}{}_\tau$ (cf., the corresponding result for $\mathcal{M}^{\mathcal{Q}}{}_\tau$ in [2, Lemma 4.3]). That is the Hausdorff distance between $\mathcal{M}^{\mathcal{P}}{}_\tau$ and $\mathcal{M}$ is $O(\sqrt{\tau})$ or $O(\tau)$ if SC holds at all feasible point.
The same arguments will be used to prove the second part: for each $\bar{x} \in \mathcal{M}^{\mathcal{P}}{}_\tau$, take $B(\bar{x}, \delta_{\bar{x}})$, the ball centered in $\bar{x}$ and radium $\delta_{\bar{x}}$ where the transformation (3.2) is valid as a diffeomorphism and the LICQ holds for all $x \in \mathcal{M}^{\mathcal{P}}{}_\tau \cap B(\bar{x}, \delta_{\bar{x}})$, recall Theorem 3.1.. As $\mathcal{M}$ is a closed set included in a compact, it is compact. So, we can take $X^0$ a set of finitely many points $\bar{x}$ such that $\cup_{\bar{x} \in X^0} B(\bar{x}, \delta_{\bar{x}})$ covers $\mathcal{M}$. Defining $\delta = \min\{\delta_{\bar{x}}, \; \bar{x} \in X\}$, it is clear that for $\tau$ small enough

$$dist(\mathcal{M}^{\mathcal{P}}{}_\tau, \mathcal{M}) = O(\sqrt{\tau}) \leq \delta.$$

So,

$$\mathcal{M}^{\mathcal{P}}{}_\tau \subset \mathcal{M} + B(0, \delta_{\bar{x}}) \subset \cup_{\bar{x} \in X^0} B(\bar{x}, \delta_{\bar{x}})$$

and the result follows. □
Now, we prove analogous results for the other approaches. From the previous analysis, it is clear that it is enough to show a result similar to that obtained in Theorem 3.1. and bounds of the type $O(\tau^p)$, $p > 0$. This is the scheme we will follow from now on.

### 3.2. $\mathcal{Q}_\tau$

A result analogous to Theorem 3.2. has been obtained for $\mathcal{Q}_\tau$ in [2, Lemma 4.2]. We prove now the local fulfillment of the LICQ using the local diffeomorphism.

**Theorem 3.4.** *Let MPCC-LICQ holds at all feasible point of $\mathcal{P}$. Then, for all $\bar{x}$ feasible point of $\mathcal{P}$, there exists $\overline{\tau}$ and a neighborhood $V$ of $\bar{x}$ such that for all $(\tau, x)$, $\tau \in (0, \overline{\tau})$ and $x_\tau \in V \cap \mathcal{M}^{\mathcal{Q}}{}_\tau$, the LICQ holds. Moreover if the sets $\mathcal{M}^{\mathcal{Q}}{}_\tau$ are contained in some compact set $K$, there exists $\overline{\tau}$ such that for all $\tau \in (0, \overline{\tau})$ the LICQ is satisfied for all $x_\tau$, feasible point of $\mathcal{Q}_\tau$.*

*Proof:* Again we apply the local diffeomorphism defined in (3.2). Locally the set of feasible sets is

$$\begin{array}{rclrrcl} x_i x_{m+i} & = & \tau, \; i = 1, \ldots, p, & x_{p+i}s_{p+i}(x) & = & \tau, & i = 1, \ldots, m-p, \\ x_{m+p+i} & \geq & 0, \; i = 1, \ldots, q_0, & x_i, \; x_{m+i} & \geq & 0, & i = 1, \ldots, p, \\ & & \multicolumn{5}{l}{x_{p+i}, \; s_{p+i}(x) \geq 0, \; i = 1, \ldots, m-p.} \end{array} \tag{3.5}$$

Define

$$B(x) = \begin{bmatrix} X^3(x) & x_i\nabla_{1,\dots,p}s_i(x), i = p+1,\dots,m, & 0 \\ 0 & [x_i\nabla_{p+1,\dots,m}s_i(x), i = p+1,\dots,m,] + diag(s_{p+1}(x),\dots,s_m(x)) & 0 \\ X^1(x) & x_i\nabla_{m+1,\dots,m+p}s_i(x), i = p+1,\dots,m, & 0 \\ 0 & x_i\nabla_{m+p+1,\dots,m+p+q_0}s_i(x), i = p+1,\dots,m, & I_{q_0} \\ 0 & x_i\nabla_{m+p+q_0+1,\dots,n}s_i(x), i = p+1,\dots,m, & 0 \end{bmatrix}.$$

Here $X^1(x) = diag(x_1,\dots,x_p)$, $X^3(x) = diag(x_{m+1},\dots,x_{m+p})$ and $\nabla_{a,\dots,b}Y = \left(\frac{\partial Y}{\partial x_a},\dots,\frac{\partial Y}{\partial x_b}\right)$. As $s_{p+1}(0),\dots,s_m(0) > 0$, the matrix

$$B_0(x) = [x_{p+i}\nabla_{p+1\dots m}s_i(x), i = p+1,\dots,m,] + diag(s_{p+1}(x),\dots,s_m(x)) \tag{3.6}$$

is regular for $x = 0$. So, after some easy algebraic manipulations, we get that $B(x)$ has full column rank and, therefore, LICQ holds. □
As in the case of Theorem 3.3., a global result also follows.

### 3.3. Case $\mathcal{R}^S_\tau$

In this part we present analogous results for the regularisation proposed in [10]. In this case it is clear that $\mathcal{M} \subset \mathcal{M}^{\mathcal{S}}{}_\tau$. Again using the canonical transformation (3.2), it holds that $x \in B_\varepsilon(\bar{x})$ is a feasible solution of $\mathcal{R}^{\mathcal{S}}{}_\tau$ if and only if

$$\begin{cases} x_i \geq 0, & i = 1,\dots,m, \quad x_{m+i} \geq 0, \quad i = 1,\dots,p, \quad x_{m+p+i} \leq 0,\ i = 1,\dots,q_0, \\ x_ix_{m+i} \leq \tau, & i = 1,\dots,p \quad x_is_i(x) \leq \tau, \quad i = p+1,\dots,m. \end{cases} \tag{3.7}$$

Next result shows that the elements of $\mathcal{M}^{\mathcal{S}}{}_\tau$ are also not far from $\mathcal{M}$.

**Theorem 3.5.** *Assume that MPCC-LICQ holds at $\bar{x} \in \mathcal{M}$. Then there exist $\varepsilon, \tau_0, \beta > 0$, such that for all $\tau \in [0, \tau_0)$ and for any $\bar{x}_\tau \in \mathcal{M}_{\mathcal{S}}{}^\tau \cap B_\varepsilon(\bar{x})$ there exists $\hat{x}_\tau \in \mathcal{M} \cap B_\varepsilon(\bar{x})$ fulfilling $\|\bar{x}_\tau - \hat{x}_\tau\| \leq \beta\sqrt{\tau}$. If in addition SC is satisfied then, $\beta\sqrt{\tau}$ can be replaced by $\beta\tau$.*

*Proof:* Fix $\bar{x}_\tau \in \mathcal{M}_{\mathcal{S}}{}^\tau \cap B_\varepsilon(\bar{x})$, where $B_\varepsilon(\bar{x})$ is a neighborhood as in Theorem 3.2.. So, we can consider the canonical transformation and assume that for $i > p$, $s_i(x) > M > 0$, for all $x \in B_\varepsilon(\bar{x})$. In particular, for every $i = 1,\dots,p$ either $x_i$ or $x_{m+i}$ is smaller than or equal to $\sqrt{\tau}$. W.l.o.g. we assume that $x_i \leq \sqrt{\tau}$. The result follows after taking $x_i = 0,\ i = 1,\dots,m$ and $x_i = [\bar{x}_\tau]_i$ otherwise. It is clear that the bound can be sharpened to $O(\tau)$ if and only if $SC$. □.

**Remark 3.1.** *The local fulfillment of the LICQ was already proven in [11]. Actually an analogous relation was established between MPCC-MFCQ and MFCQ, see [12]. We want to point out that the local diffeomorphism* (3.2) *simplifies the proof of the first case.*

### 3.4. Case $\mathcal{R}^{\mathcal{LF}}{}_\tau$

Now we consider the regularisation proposed in [8]. Again $\mathcal{M} \subset \mathcal{M}^{\mathcal{LF}}{}_\tau$. By the canonical transformation (3.2), it holds that $x \in B_\varepsilon(\bar{x})$ is a feasible solution of $\mathcal{M}^{\mathcal{LF}}{}_\tau$ if and only if

$$\begin{cases} (x_i+\tau)(x_{m+i}+\tau) \geq \tau^2, \quad i=1,\dots,p, & (x_i+\tau)(s_i(x)+\tau) \geq \tau^2,\ i=p+1,\dots,m, \\ x_{m+p+i} \leq 0, \quad i=1,\dots,q_0, & x_i x_{m+i} \leq \tau^2,\ i=1,\dots,p, \\ x_i s_i(x) \leq \tau^2, \quad i=p+1,\dots,m. & \end{cases} \tag{3.8}$$

**Theorem 3.6.** *Assume that MPCC-LICQ holds at $\bar{x} \in \mathcal{M}$. Then there exist $\varepsilon, \tau_0, \beta > 0$, such that for all $\tau \in [0,\tau_0)$ and for any $\bar{x}_\tau \in \mathcal{M}^{\mathcal{LF}}{}_\tau \cap B_\varepsilon(\bar{x})$ there exists $\hat{x}_\tau \in \mathcal{M} \cap B_\varepsilon(\bar{x})$ satisfying $\|\bar{x}_\tau - \hat{x}_\tau\| \leq \beta\tau$.*

*Proof:* Take $\bar{x}_\tau \in \mathcal{M}^{\mathcal{LF}}{}_\tau \cap B_\varepsilon(\bar{x})$, where $B_\varepsilon(\bar{x})$ is a neighborhood as in Theorem 3.2.. This means that the canonical transformation can be used and that for all $x \in B_\varepsilon(\bar{x})$, it holds $s_i(x) > M > 0$, $i = p+1,\dots,m$.
For those indices such that $x_i s_i(x) \leq \tau^2$, as $s_i(x) > M$, $x_i \leq \tau^2/M \leq \tau/M$.
Since for every $i = 1,\dots,p$, $(x_i+\tau)(x_{m+i}+\tau) \geq \tau^2 \geq 0$, either $x_i, x_{m+i} \geq -\tau$ or $x_i, x_{m+i} \leq -\tau$. In the last case, if one of the inequalities is strict, we get the contradiction $x_i x_{m+i} > \tau^2$. Moreover, $x_i x_{m+i} + \tau(x_i + x_{m+i}) \geq 0$. So, at least one of them, $x_i$ or $x_{m+i}$ is non-negative. Then we consider two cases.
If $x_i x_{m+i} \geq 0$ since their product is smaller than or equal to $\tau^2$ at least the modulus of one of them is smaller than $\tau$. Then, we assume $|x_i| \leq \tau,\ i = 1,\dots,p$. The result follows after taking $\bar{x}_\tau \in \mathcal{M}$ given by $(\bar{x}_\tau)_i = 0,\ i = 1,\dots,m$ and $(\bar{x}_\tau)_i = x_i$ otherwise.
If $x_i < 0 \leq x_{m+i}$ since $x_{m+i} + \tau > 0$, we get that $x_i + \tau \geq 0$, so $0 \geq x_i \geq -\tau$. Taking $\bar{x}_\tau$ such that $(\bar{x}_\tau)_i = 0,\ i = 1,\dots,m$ and $(\bar{x}_\tau)_i = x_i$, otherwise; the desired bound is obtained.
Again $O(\tau)$ is the sharpest bound, $\tau^2$ can be taken if and only if $SC$ is satified. $\square$.
We want to point out that $\tau^2$ plays the role of $\tau$ in the other approaches. Indeed, the product of the complementarity functions is of order $\tau^2$. If we use $\tau^2$ instead of $\tau$ in the other approaches, we get bounds of the same order under similar conditions.
As in the case of $\mathcal{R}^{\mathcal{S}}\tau$ the local fulfillment of the LICQ at the parametric problem under MPCC-LICQ is known, see [12, Theorem 7.6]. An alternative proof based on the canonical transformation can be found at Appendix A.

### 3.5. Case $\mathcal{R}^{\mathcal{K}}{}_\tau$

Now we study the set of feasible solutions for the regularisation proposed in [6]. By the canonical transformation (3.2), it holds that $x \in B_\varepsilon(\bar{x})$ is a feasible solution of $\mathcal{M}^{\mathcal{K}}{}_\tau$ if and only if

$$\begin{cases} x_i, x_{m+i} \geq -\tau,\ i=1,\dots,p\ x_i, s_i(x) \geq -\tau,\ i=p+1,\dots,m.\ x_{m+p+i} \leq 0,\ i=1,\dots,q_0, \\ (x_i-\tau)(x_{m+i}-\tau) \leq 0,\ i=1,\dots,p\ (x_i-\tau)(s_i(x)-\tau) \leq 0,\ i=p+1,\dots,m. \end{cases} \tag{3.9}$$

**Theorem 3.7.** *Assume that MPCC-LICQ holds at $\bar{x} \in \mathcal{M}$. Then there exists $\varepsilon, \tau_0, \alpha, \beta > 0$, such that for all $\tau \in [0,\tau_0)$ the following is true: there exists $x_\tau \in \mathcal{M}^{\mathcal{K}}{}_\tau \cap B_\varepsilon(\bar{x})$ satisfying $\|x_\tau - \bar{x}\| \leq \alpha\tau$, and for any $\bar{x}_\tau \in \mathcal{M}^{\mathcal{P}}{}_\tau \cap B_\varepsilon(\bar{x})$ there exists $\hat{x}_\tau \in \mathcal{M} \cap B_\varepsilon(\bar{x})$ such that $\|\bar{x}_\tau - \hat{x}_\tau\| \leq \beta\tau$.*

*Proof:* After the canonical transformation, $\bar{x} = 0$. As in Theorem 3.2., we assume that $B_\varepsilon(\bar{x})$ is a neighborhood where the canonical transformation describes the set of feasible solutions and that for $i > p$, $s_i(x) > M > 0$, for all $x \in B_\varepsilon(\bar{x})$.
We take $x$ such that $x_{m+i} = \tau \; i = 1, \ldots, p$, and $x_i = 0$, otherwise.
Since, for $\tau$ small enough $\tau < M$, the constrains $(x_i - \tau)(s_i(x) - \tau) \leq 0,\; x_i, s_i(x) \geq -\tau, i = p+1, \ldots, m$ are satisfied at this point. So, $x \in \mathcal{M}_\tau$ as desired.
Take $x \in \mathcal{M}^{\mathcal{K}}{}_\tau \cap B_\varepsilon(\bar{x})$. For those indices such that $x_i s_i(x) \leq \tau^2$, as $s_i(x) > M$, we can assume that $x_i \leq \tau^2/M \leq \tau/M$.
For the other indices, first note that for every $i = 1, \ldots, p$, either $-\tau \leq x_i \leq \tau \leq x_{m+i}$ or $x_i \geq \tau \geq x_{m+i} \geq -\tau$. W.l.o.g., we assume the first case holds. Then, taking $\bar{x}$ such that $(\bar{x})_i = 0,\; i = 1, \ldots, m$ and $(\bar{x}_\tau)_i = x_i$, otherwise, the result follows.
□
As shown in [12], LICQ is not guaranteed. Actually, using the following example, it is clear that if MPCC-LICQ holds and SC fails there exists for $\tau$, small enough, a point $x_\tau$ such that $r_i(x_\tau) = s_i(x_\tau) = \tau$ and, so, the constraint qualification is violated.

**Example 3.1.**

$$x_1, x_2 \geq -\tau,\; (x_1 - \tau)(x_2 - \tau) \leq 0$$

The point $(\tau, \tau)$ is feasible and the gradient of the active constraint, $(x_1 - \tau)(x_2 - \tau) \leq 0$ is 0. □

### 3.6. Cases $\mathcal{R}^{\mathcal{SU}}{}_\tau$ and $\mathcal{R}^{\mathcal{SK}}{}_\tau$.

For the regularisations proposed in [13, 12], we use [12, Lemma 7.15] and, applying the canonical transformation (3.2), we obtain that if $x \in B_\varepsilon(\bar{x})$ is a feasible solution of $\mathcal{R}^{\mathcal{SU}}{}_\tau$, the point is included in

$$\left\{ \begin{array}{l} x_i, x_{m+i} \geq 0,\; i = 1, \ldots, p\; x_i, s_i(x) \geq 0,\; i = p+1, \ldots, m.\; x_{m+p+i} \leq 0,\; i = 1, \ldots, q_0, \\ x_i x_{m+i} = 0 \text{ or } x_i + x_{m+i} \leq \tau,\; i = 1, \ldots, p,\; x_i s_i(x) = 0 \text{ or } x_i + s_i(x) \leq \tau,\; i = p+1, \ldots, m. \end{array} \right. \tag{3.10}$$

Similarly, for regularisation (1.7) we get

$$\left\{ \begin{array}{l} x_i, x_{m+i} \geq 0,\; i = 1, \ldots, p\; x_i, s_i(x) \geq 0,\; i = p+1, \ldots, m.\; x_{m+p+i} \leq 0,\; i = 1, \ldots, q_0, \\ \min\{x_i, x_{m+i}\} \leq \tau,\; i = 1, \ldots, p,\; \min\{x_i, s_i(x)\} \leq \tau,\; i = p+1, \ldots, m. \end{array} \right. \tag{3.11}$$

for more details, see [12]. So, we have the following results.

**Theorem 3.8.**

$$\mathcal{M} \subset \mathcal{M}^{\mathcal{SU}}{}_\tau,\; \mathcal{M} \subset \mathcal{M}^{\mathcal{SK}}{}_\tau.$$

*Assume that MPCC-LICQ holds at $\bar{x} \in \mathcal{M}$. Then there exists $\varepsilon, \tau_0, \beta > 0$, such that for any $\bar{x}_\tau \in \mathcal{M}^{\mathcal{SU}}{}_\tau \cap B_\varepsilon(\bar{x})$ there exists $\hat{x}_\tau \in \mathcal{M} \cap B_\varepsilon(\bar{x})$ satisfying $\|\bar{x}_\tau - \hat{x}_\tau\| \leq \beta\tau$. If in addition SC holds at $\bar{x}$, then, in both bounds $\tau$ can be replaced by $\tau^2$*

*Proof:* The first part is clear.
After the canonical transformation, $\bar{x} = 0$. As in Theorem 3.2., we assume that $B_\varepsilon(\bar{x})$ is a neighborhood where the canonical transformation describes the set of feasible solutions and that for $i > p$, $s_i(x) > M > 0$, for all $x \in B_\varepsilon(\bar{x})$. For regularisation (1.7), consider an element of the set described at (3.10). In particular, for those indices such that $i = p+1, \ldots, m$, as $s_i(\bar{x}_\tau) > M$, for $\tau$ small enough $(\bar{x}_\tau)_i = 0$. If $i = 1, \ldots, p$ either $(\bar{x}_\tau)_i, (\bar{x}_\tau)_{m+i}$ are non-negative and complement or $(\bar{x}_\tau)_i + (\bar{x}_\tau)_{m+i} \leq \tau$, $(\bar{x}_\tau)_i, (\bar{x}_\tau)_{m+i} \geq 0$.
W.l.o.g. we assume that $(\bar{x}_\tau)_i \leq \tau/2$ and take $x$ such that $\bar{x}_i = 0$ $i = 1, \ldots, m$ and $\bar{x}_{m+i} = (\bar{x}_\tau)_{m+i}$ otherwise. So, the result follows.
For the regularisation (1.8), the result is analogous, we only need to recall that, w.l.o.g., $(\bar{x}_\tau)_i \leq (\bar{x}_\tau)_{m+i}$.
Again it is the sharpest bound, $\tau^2$ can be taken if and only if $SC$ is satisfied.
As before, the global bound holds can be found under the compactness assumption. $\square$.
After this analysis it is clear that the Hausdorff distance of the set of feasible solutions of the parametric problem and of the $\mathcal{P}$ have the same order under similar conditions. However, LICQ is not fulfilled in the last three cases, as has been remarked in [12].

## 4. THE SET OF SOLUTIONS OF $\mathcal{P}$ AND THE PARAMETRIC APPROACHES

In this part we study the set of solutions. We consider the g.c. points as solutions concepts. By g.c. points we understand feasible solutions such that the gradient of the objective functions, the (classical) equality constraints( *i.e.* $r_i(x)s_i(x) = 0$ are not considered) and the active inequalities are linearly dependent. Non-degeneracy is related with the non-singularity of the matrix of the derivatives of the system describing the linear dependency.
At the g.c. points of regular MPCC, MPCC-LICQ, MPCC-SC and MPCC-SOC are fulfilled. In this section we will study the consequences of these properties to the parametric problems $\mathcal{P}_\tau$, $\mathcal{Q}_\tau$, $\mathcal{R}^S_\tau$, $\mathcal{R}^{LF}_\tau$, $\mathcal{R}^K_\tau$, $\mathcal{R}^{SU}_\tau$, $\mathcal{R}^{SK}_\tau$. Roughly speaking, we will analyse if the critical points of the corresponding problems are non-degenerated and we will provide bounds to the Hausdorff distance of the set of critical points of $\mathcal{P}_\tau, \mathcal{Q}_\tau, \mathcal{R}^S_\tau, \mathcal{R}^{LF}_\tau, \mathcal{R}^K_\tau, \mathcal{R}^{SU}_\tau, \mathcal{R}^{SK}_\tau$ and $\mathcal{P}$. For a unified notation we define $\Phi^A(P)$ as the set of solutions of type $A$ of problem $P$. Here $A$ may represent minimizers, local minimizers, critical points and g.c. points.
As the MPCC-LICQ holds, we apply the canonical transformation (3.2) and w.l.o.g. we assume that the solution is 0. As before, we consider the partition $x_1 = (x_1, \ldots x_p)$, $x_2 = (x_{p+1}, \ldots x_m)$, $x_3 = (x_{m+1}, \ldots x_{m+p})$, $x_4 = (x_{m+p+1}, \ldots x_{m+p+q_0})$, $x_5 = (x_{m+p+1}, \ldots x_n)$. The g.c. condition implies $\nabla f(0) = (\rho_1, \rho_2, \sigma_1, -\mu, 0)$. By the MPCC-SC, all the components of $(\rho_1, \sigma_1, \mu)$ are non-zero. In particular, close to 0, $\nabla f(x) = (\rho_1, \rho_2, \sigma_1, -\mu, 0) + q(x)$ where $q(x) = O(\|x\|)$, see [2]. Vector $q(x)$ is also divided in $(q_1(x), \ldots, q_5(x))$, where $q_i(x) = \nabla_{x_i} f(x) - \nabla_{x_i} f(0)$. As MPCC-SOC holds, as in [2, Theorem 5.1.], it is easy to prove that $\nabla_{x_5} q_5^T(0)$ is non-singular.

### 4.1. Problem $\mathcal{P}_\tau$

From Theorem 3.1., it is known that for $\tau > 0$ small enough, the solutions of problem $\mathcal{P}_\tau$ are critical points, i.e., LICQ holds. However we cannot guarantee that non-degeneracy holds. In fact, the following example shows that SC fails for $\mathcal{P}_\tau$ even if the problem is regular.

**Example 4.1.** *Using the canonical transformation suppose that locally,*

$$\mathcal{P}: \quad \min x_1 + x_2 \quad s.t. \quad 0 \leq x_1 \perp 1 - x_2 \geq 0, \; 0 \leq x_2 \perp 1 + x_1 \geq 0.$$

The point $\bar{x} = (0,0)$ is a minimiser of $\mathcal{P}$ (of order one) with corresponding multipliers $\gamma_1 = \gamma_2 = 1$. So, MPCC-SC and MPCC-SOC holds (the latter holds trivially since $T_{\bar{x}}\mathcal{M} = \{0\}$). The minimisers of $\mathcal{P}_\tau$ are $x_\tau = \alpha(0,\tau) + (1-\alpha)(\tau,0),\; \alpha \in [0,1]$. Hence, they are degenerated.
Note that $\bar{x}_\tau = (\tau,\tau)$, the minimisers of $\mathcal{Q}_\tau$ are non-degenerate critical points, for $\tau > 0$. $\square$
Nevertheless, next result shows that g.c. points of $\mathcal{P}_\tau$ are not far from critical points of $\mathcal{P}$ and vice-versa.

**Theorem 4.1.** *Let $\bar{x}$ be a stationary point of $P$ such that MPCC-LICQ , MPCC-SC and MPCC-SOC holds. Then*

- *If $\bar{x}$ is a C-stationary point and $I_{00}(\bar{x}) = \{1,\ldots,m\}$, then for all $\tau > 0$ (small enough), there exists $x_\tau$ a stationary point of $\mathcal{P}_\tau$ such that $\|x_\tau - \bar{x}\| = O(\sqrt{\tau})$.*
- *If $\bar{x}$ is a S-stationary point and $I_{00}(\bar{x}) \neq \{1,\ldots,m\}$, then for all $\tau > 0$ (small enough), there exists $x_\tau$ a stationary point of $\mathcal{P}_\tau$ such that $\|x_\tau - \bar{x}\| = O(\tau)$.*

*Proof:* The proof is done by construction. Again using the canonical transformation it is clear that as 0 is a C-stationary point, we get that $\rho_1\sigma_1 \geq 0, \mu \geq 0$. By the MPCC-SC, the inequality is strict. The point $x_\tau$ shall fulfill for some multipliers $(a,\rho_1,\rho_2,\sigma,\mu)_\tau$ the following system that describes the condition of been a critical point of $\mathcal{P}_\tau$:

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + q^T(x) + a_\tau \begin{pmatrix} (x_3)_\tau + \sum_{i=p+1}^m (x_2)_\tau \nabla_{x_1} s_i \\ [s_{p+1},\ldots,s_m]^T + \sum_{i=p+1}^m (x_2)_\tau \nabla_{x_2} s_i \\ (x_1)_\tau + \sum_{i=p+1}^m (x_2)_\tau \nabla_{x_3} s_i \\ \otimes \\ \otimes \end{pmatrix} - \begin{pmatrix} (\rho_1)_\tau/0 \\ (\rho_2)_\tau/0 \\ (\sigma_1)_\tau/0 \\ -\mu_\tau/0 \\ 0 \end{pmatrix} = 0 \qquad (4.1)$$

We will analyse the two cases:

**Case $p = m$ and $\bar{x}$ is a C-Stationary point**: Since $x_1^T x_3 = \tau$, there exists $i$ such that $x_{1,i}x_{3,i} \neq 0$. W.l.o.g. we assume that $i = m$. Define $(c,d)$ as $\rho_{1,m} + q_m(x) = c,\; \sigma_{1,m} + q_{2m}(x) = d$.
The MPCC-SC, MPCC-LICQ hold, and $\bar{x}$ is a C-stationary point, so, $\rho_{1,m}\sigma_{1,m} > 0$. As $q(x) = O(\|x\|)$, reducing the neighborhood such that $\|x\|$ is small enough, we can guarantee that $c$ and $d$ have the same sign.
Now we define $a_\tau = \sqrt{\frac{cd}{\tau}}$, $(x_1)_\tau = \left(0,\ldots,0,\sqrt{\frac{c\tau}{d}}\right)$, $(x_1)_\tau = \left(0,\ldots,0,\sqrt{\frac{d\tau}{c}}\right)$, and $(x_2)_\tau = 0,\; (x_4)_\tau = 0,\;\; (x_5)_\tau = 0$.

The multipliers are $(\rho_{1,1,\ldots,m-1})_\tau = \rho_{1,1,\ldots,m-1}+q_{1,1,\ldots,m-1}(x_\tau)$, $(\sigma_{1,1,\ldots,m-1})_\tau = \sigma_{1,1,\ldots,m-1}+q_{2,1,\ldots,m-1}(x_\tau)$, $\mu + q_3(x_\tau) = \mu_\tau$. Combining the definition of C-stationary points, and the fulfillment of the MPCC-LICQ and MPCC-SC, we obtain that $\rho_1, \sigma_1, \mu > 0$. As $\|x_\tau\| = O(\sqrt{\tau})$, $q(x_\tau) = O(\|x\|) = O(\sqrt{\tau})$, $(\rho_1, \sigma_1, \mu)_\tau > 0$. So, $x_\tau$ is the desired stationary point.

**Case $p < m$ and $\overline{x}$ is a M-Stationary point**: We will prove that there exists a vector $(x_{2,m}, x_5) \in \mathbb{R} \times \mathbb{R}^{n-m-p-q_0}$ such that the feasible solution $x_\tau = (x_1, x_2, x_3, x_4, x_5)_\tau = (0, (0, \ldots, 0, x_{2,m}), 0, 0, x_5)$, solves system (4.1) for suitable multipliers. Indeed, if $x_1 = x_3 = 0, x_2 = (0, \ldots, 0, x_{2,m}), x_4 = 0$, the conditions reads

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + q^T(x) + a_\tau \begin{pmatrix} x_{2,m}\nabla_{x_1} s_m \\ [s_{p+1}, \ldots, s_m]^T + x_{2,m}\nabla_{x_2} s_m \\ x_{2,m}\nabla_{x_3} s_m \\ x_{2,m}\nabla_{x_4} s_m \\ x_{2,m}\nabla_{x_5} s_m \end{pmatrix} - \begin{pmatrix} (\rho_1)_\tau \\ (\rho_2)_\tau/0 \\ (\sigma_1)_\tau \\ -\mu_\tau \\ 0 \end{pmatrix} = 0 \tag{4.2}$$

It is clear that as $s_m(x)$ is continuous near to 0, $s_m(x) > M > 0$ and $x_{2,m} = \tau/s_m(x)$ is $O(\tau)$. So, $a_\tau$ is well defined as $a_\tau = -\frac{\rho_{2,m}+q_{2,m}(x)}{s_m+x_{2,m}\frac{\partial s_m(x)}{\partial x_{2,m}}}$. Furthermore

$$\begin{array}{rcl} q_5^T(x_\tau) - \frac{\rho_{2,m}+q_{2,m}(x)}{s_m+x_{2,m}\frac{\partial s_m(x)}{\partial x_{2,m}}} x_{2,m}\nabla_{x_5} s_m(x_\tau) & = & 0 \\ x_{2,m} s_m(0, (0, \ldots, 0, x_{2,m}), 0, 0, x_5) & = & \tau \end{array} \tag{4.3}$$

Derivating with respect to $(x_{2,m}, x_5)$ and evaluating at 0 we obtain the full row rank matrix

$$\begin{array}{cc} \otimes & \nabla_{x_5} q(0) \\ s_m(0, (0, \ldots, 0, x_{2,m}), 0, 0, x_5) & 0 \end{array}$$

So, we can apply the Implicit Function Theorem and obtain, for all $\tau$ small enough, the existence of $(x_{2,m}, x_5)_\tau = O(\tau)$ solution of (4.3).

Now, given $x_\tau$, the multipliers are computed as follows

$$(\rho_1)_\tau = [\rho_1 + q_1(x)] - \frac{\rho_{2,m} + q_{2,m}(x)}{s_m + x_{2,m}\frac{\partial s_m(x)}{\partial x_{2,m}}} x_{2,m}\nabla_{x_1} s_m$$

$$(\rho_2)_\tau = [\rho_{2,p+1,\ldots,m-1} + q_{2,p+1,\ldots,m-1}(x)] - \frac{\rho_{2,m} + q_{2,m}(x)}{s_m + x_{2,m}\frac{\partial s_m(x)}{\partial x_{2,m}}} x_{2,m}\nabla_{x_{2,p+1,\ldots,m-1}} s_m$$

$$(\rho_3)_\tau = [\sigma_1 + q_3(x)] - \frac{\rho_{2,m} + q_{2,m}(x)}{s_m + x_{2,m}\frac{\partial s_m(x)}{\partial x_{2,m}}} x_{2,m}\nabla_{x_3} s_m$$

$$\mu_\tau = [\mu - q_4(x)] + \frac{\rho_{2,m} + q_{2,m}(x)}{s_m + x_{2,m}\frac{\partial s_m(x)}{\partial x_{2,m}}} x_{2,m}\nabla_{x_4} s_m$$

As the MPCC-SC holds, $q(x) = O(\tau)$, and $\overline{x}$ is a S-stationary point, it is clear that $(\rho_1, \rho_2, \sigma_1, \mu)_\tau > 0$. So, $x_\tau$ shows the existence of the desired stationary point of $\mathcal{P}$ for $\tau$ small enough. □

Now we prove an analogous results for generalized critical points.

**Theorem 4.2.** *Let $\overline{x}$ be a g.c. point of $P$ such that MPCC-LICQ , MPCC-SC and MPCC-SOC holds. Then for all $\tau > 0$ (small enough), and all $x_\tau$ g.c. point of $\mathcal{P}_\tau$ (near $\overline{x}$) it holds that $\|x_\tau - \overline{x}\| = O(\sqrt{\tau})$.*

*Proof:* It is clear, recall Theorem 3.1., that the gradients of the active constraints are l.i. So, the g.c. points of $\mathcal{P}_\tau$ are critical point of the parametric problem and satisfy the system. On the other hand, since

$$\sum_{i=1}^{p} x_{\tau,i}x_{m+i,\tau} + \sum_{i=p+1}^{m} x_{i,\tau}s_i(x_\tau) = \tau \ , \tag{4.4}$$

there exists $i$ such that either $(x_1)^i_\tau, (x_3)^i_\tau \neq 0$ or $(x_2)^\tau_i \neq 0$. Moreover, as $(x_1)^i_\tau (x_3)^i_\tau \leq \tau$, either $(x_1)^i_\tau$ or $(x_3)^i_\tau = O(\sqrt{\tau})$. Similarly as $s_i(x) > m > 0$ in a neighborhood of 0, $(x_2)^i_\tau = O(\tau)$. From (4.4) it also follows that there are two cases: either $(x_2)_\tau \neq 0$ or $(x_2)_\tau = 0$ and there exists $i^*$: $(x_{1,i^*})_\tau$, $(x_{3,i^*})_\tau \neq 0$. Now, suppose the first case is true, w.l.o.g. we assume that $(x_{2,m})_\tau \neq 0$. Hence,

$$\rho_{2,m} + O(\|x\|) + a_\tau[s_m + \sum_{i=p+1}^{m} (x_2)_\tau \nabla_{x_{2,m}} s_i] = 0$$

In particular $a_\tau$ is bounded. By the MPCC-SC $\mu, \rho_1, \sigma_1 \neq 0$. After a suitable shrink of the neighborhood of $\bar{x}$, we can assume that

$$-|(\mu/2)_i| < q_4(x) + [a_\tau[(\sum_{j=p+1}^{m} (x_2)_\tau \nabla_{x_4} s_j]_i < |(\mu/2)_i|.$$

So, $\mu + q_4(x) + a_\tau[+\sum_{j=p+1}^{m}(x_2)_\tau \nabla_{x_4} s_j]]_i \neq 0$. In particular, $\mu + q_4(x) + a_\tau[+\sum_{j=p+1}^{m}(x_2)_\tau \nabla_{x_4} s_j]]_i \neq 0$. Hence, $-\mu_\tau \neq 0$ and $(x_4)_\tau = 0$, for $\tau$ small enough.
Analogously, we prove that $-|(\rho_1/2)_i| < q_1(x) + [a_\tau[(x_3)_\tau + \sum_{j=p+1}^{m}(x_2)_\tau \nabla_{x_1} s_j]_i < |(\rho_1/2)_i|$ and $-|(\sigma_1/2)_i| < q_3(x) + [a_\tau[(x_1)_\tau + \sum_{j=p+1}^{m}(x_2)_\tau \nabla_{x_3} s_j]_i < |(\sigma_1/2)_i|$. So, $(\rho_1)_\tau, (\sigma_1)_\tau \neq 0$, component-wise. Then, $(x_1)_\tau = (x_3)_\tau = 0$. Therefore, $\rho_2 + a(s_{p+1}, \ldots, s_m) + (\rho_{2,\tau}/0) = 0$. So, $x_\tau$ fulfills that $(x_1, x_3, x_4)_\tau = 0$, $x_2 = O(\tau)$ and

$$q_5(x_\tau) + a_\tau(\sum_{i=p+l+1}^{m} (x_2)_\tau \nabla_{x_5} s_i((0, x_2, 0, 0, x_5)_\tau) = 0 \tag{4.5}$$

At $(x_5, \tau) = 0$, the system has a solution. Taking derivatives with respect to $x_5$ we get, $\nabla_{x_5} q_5(0)$. By the MPCC-SOC, this matrix is non-singular. Using the Implicit Function Theorem, for all $(a_\tau, x_2, \tau)$ close to $(a, 0, 0)$ the solutions of (4.5) are $O(\tau)$, recall $a_\tau \to a$, $\tau \to 0$.
So, the solutions of the system (4.1) with $(x_2)_\tau \neq 0$ satisfies $x_\tau = (0, O(\tau), 0, 0, O(\tau))$.
Now, we consider the case $(x_2)_\tau = 0$ and $(x_{1,p})_\tau, (x_{3,p})_\tau \neq 0$. System (4.1) reads

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + q^T(x) + a_\tau \begin{pmatrix} (x_3)_\tau \\ [s_{p+1}, \ldots, s_m]^T \\ (x_1)_\tau \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} (\rho_1)_\tau/0 \\ 0 \\ (\sigma_1)_\tau/0 \\ -\mu_\tau/0 \\ 0 \end{pmatrix} = 0 \tag{4.6}$$

As $q(x) = O(\|x\|)$, and $x_i \neq 0$ it is clear that $a_\tau$ is bounded. Moreover, we can assume that the neighborhood of $x_\tau$ is such that

$$-|\rho_{1,i}/2| < q_{1,i}(x) < |\rho_{1,i}/2|,\ -|\sigma_{1,i}/2| < q_{3,i}(x) < |\sigma_{1,i}/2|,\ -|\mu_i/2| < q_{4,i}(x) < |\mu_i/2|.$$

By the MPCC-SC $\rho_1, \sigma_1, \mu \neq 0$ component-wise. Using the previous inequalities all the components of $(\rho_1, \sigma_1, \mu) + (q_1(x), q_3(x), q_4(x))$ are non-zero. In particular, $\mu_\tau \neq 0$ component-wise. Hence, $x_4 = 0$. On the other hand

$$(\rho_{1,p})_\tau + q_1(x_\tau) + a_\tau (x_{3,p})_\tau = 0.$$

$$(\sigma_{1,p})_\tau + q_3(x_\tau) + a_\tau (x_{1,p})_\tau = 0.$$

As already noticed, either $(x_{1,p})_\tau = O(\sqrt{\tau})$ or $(x_{3,p})_\tau = O(\sqrt{\tau})$. W.l.o.g. we assume that the first case holds. Then, as $(\rho_{1,p}, \sigma_{1,p}) \neq 0$, $a_\tau O(\sqrt{\tau}) = -\rho_{1,p} - q_{1,p}(x_\tau)$ is bounded. So,

$$a_\tau = 1/O(\sqrt{\tau}).$$

Using this fact, we obtain that also $(x_{3,p})_\tau = O(\sqrt{\tau})$. Furthermore, noting that $\rho_2 + q_2(x)$ and $s_i(x)$, $i = p+1, \ldots, m$ are bounded, it is clear that the growth of $a_\tau$ implies that $p = m$. For the other indices we have the following options:

Case 1 $x_{1,i} = 0$, $x_{3,i} = 0$.

Case 2 $x_{1,i} = 0$, $x_{3,i} > 0$ or $x_{3,i} = 0$, $x_{1,i} > 0$: Then at system (4.6) $\sigma_{1,i} + q(x) = 0$. This is impossible at the considered neighborhood.

Case 3 $x_{1,i} > 0$, $x_{3,i} > 0$: Analogous to the case $i = p$, $x_{1,i} = O(\sqrt{\tau})$, $x_{3,i} = O(\sqrt{\tau})$.

So, either $x_{1,i} = 0 = x_{3,i} = 0$ or $x_{1,i}, x_{3,i} = O(\sqrt{\tau})$. As the critical point condition reads $q_5(x_\tau) = 0$, using again that MPCC-SOC holds and the Implicit Function Theorem, the only possible solutions fulfill that $(x_5)_\tau = O(\|(x_1, x_2, x_3, x_4)_\tau\|) = O(\sqrt{\tau})$. So, $\|x_\tau\| \leq O(\sqrt{\tau})$ at all point that solves system (4.1).□

### 4.2. Problem $\mathcal{Q}_\tau$

Analogously, for this approach, the following result can be proven

**Theorem 4.3.** *Let $x$ be a g.c. point of $P$ such that MCPC-LICQ , MPCC-SC and MPCC-SOC holds. Then for all $\tau > 0$ (small enough) the g.c. points of $x_\tau$ of $\mathcal{Q}_\tau$ (near $x$) are non-degenerated critical points uniquely determined and satisfy $\|x_\tau - x\| = O(\sqrt{\tau})$.*

*Proof:* For the uniqueness and the rate of convergence, see [2]. We only need to prove the non-degeneracy of $x_\tau$ critical point of the non-linear program $\mathcal{Q}_\tau$. Here we only sketch the proof. The involved algebraic work can be found in Appendix B. Consider the canonical representation of the system (3.5). The g.c. point condition at $\mathcal{P}_\tau$ is

$$\nabla f(x^k) + \begin{bmatrix} X^3(x^k) \\ 0 \\ X^1(x^k) \\ 0 \\ 0 \end{bmatrix} \mu_1^k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ eye(q_0, I_{g(x^k k)}) \\ 0 \end{bmatrix} \lambda^k +$$

$$\begin{bmatrix} x_i^k \nabla_{1,\dots,p} s_i(x^k), i = p+1,\dots,m \\ B_0(x^k) \\ x_i^k \nabla_{m+1,\dots,m+p} s_i(x^k), i = p+1,\dots,m, \\ x_i^k \nabla_{m+p+1,\dots,m+p+q_0} s_i(x^k), i = p+1,\dots,m \\ x_{p+i}^k \nabla_{m+p+q_0+1,\dots,n} s_i(x^k), i = p+1,\dots,m \end{bmatrix} \mu_2^k = 0.$$

Here $X^3(x^k) = diag(x_{m+1}^k, \dots, x_{m+p})$, $X^1(x^k) = diag(x_1^k, \dots, x_p)$, $B_0(x_k)$ is defined as in (3.6) and $eye(q_0, I_{g(x_k)})$ denotes the columns of the $q^0$-dimensional unit matrix whose indices belong to $I_g(x^k)$. Note that, due to the non-singularity of $B_0(x^k)$ and continuity arguments, it follows that $\overline{x}$ is a MPCC-critical point, $I_g(\bar{x}) = I_g(x_k)$ and SC holds at $x^k$ for $k$ large. The SOC follows from assuming the contrary and obtaining a contradiction with the MPCC-SOC.

For the global bound, consider the open covering given by the diffeomorphism. Taking, by the compactness a finite sub-covering, the result follows. □

### 4.3. Problem $\mathcal{R}_\tau^{\mathcal{S}}$

For the non-degenerancy, the following example shows that the parametric problem does not inherit this property from MPCC as in the previous case.

**Example 4.2.**

$$\min x_1 + x_2 + x_3^2$$

$$s.t.\ 0 \le x_1 \perp x_2 \ge 0,\ 0 \le x_3 \perp (1 + x_2) \ge 0.$$

*0 is a non degenerated critical point in the MPCC-sense. It is also a critical point of $\mathcal{R}_\tau^{\mathcal{S}}$ for all $\tau > 0$ such that the multiplier of the inequality $x_3 \ge 0$ is 0. So, the SC for nonlinear program is violated.*

With respect to the distance between of the g.c. points of the respective problems let us consider the g.c. points condition for $\mathcal{R}_\tau^{\mathcal{S}}$. As before, we consider the canonical representation given in (3.7). Assuming, again w.l.o.g., that the the index are regrouped in such a way that $(x_{1,i})_\tau (x_{3,i})_\tau = \tau, i = 1,\dots,p_1$ and $(x_{2,i})_\tau s(x_\tau) = \tau, i = p+1,\dots,p+p_2$

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + q^T(x^\tau) + \begin{bmatrix} (X^3(x^\tau)/0) \\ 0 \\ (X^1(x^\tau)/0/0) \\ 0 \\ 0 \end{bmatrix} \mu_1^\tau - \begin{bmatrix} (0/\rho_1^\tau) \\ 0 \\ (0/\sigma_1^\tau) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ eye(q_0, I_{g(x_\tau)}) \\ 0 \end{bmatrix} \lambda_\tau +$$

$$\begin{bmatrix} (x_i)_\tau \nabla_{1,\ldots,p} s_i(x_\tau), i = p+1,\ldots,p_2 \\ B_0(x_\tau) \\ (x_i)_\tau \nabla_{m+1,\ldots,m+p} s_i(x_\tau), i = p+1,\ldots,p_2 \\ (x_i)_\tau \nabla_{m+p+1,\ldots,m+p+q_0} s_i(x_\tau), i = p+1,\ldots,p_2 \\ (x_{p+i})_\tau \nabla_{m+p+q_0+1,\ldots,n} s_i(x_\tau), i = p+1,\ldots,p_2 \end{bmatrix} (\mu_2)_\tau = 0.$$

$$(x_1)_\tau(\rho_1)_\tau = 0,\ (x_3)_\tau(\sigma_1)_\tau = 0,\ (x_2)_\tau(\rho_2)_\tau = 0.$$

Following the ideas of Theorem 4.1., we can assume that $x_2 = O(\tau)$ and the previous condition reads

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + O(\tau) + \begin{bmatrix} (X^3(x^\tau)/0) \\ 0 \\ (X^1(x^\tau)/0) \\ 0 \\ 0 \end{bmatrix} \mu_1^\tau - \begin{bmatrix} (0/\rho_1^\tau) \\ 0 \\ (0/\sigma_1^\tau) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ eye(q_0, I_{g(x_\tau)}) \\ 0 \end{bmatrix} \lambda_\tau + \begin{bmatrix} O(\tau) \\ B_0(x_\tau)(\mu_2)_\tau \\ O(\tau) \\ O(\tau) \\ O(\tau) \end{bmatrix} = 0.$$

Furthermore by the MPCC-SC, $\mu \neq 0$ and $\rho_{1,i}, \sigma_{1,i} \neq 0$. Hence, $(x_4)_\tau = 0$ and either $(x_{1,i})_\tau, (x_{3,i})_\tau = \tau$ or $(x_{1,i})_\tau = (x_{3,i})_\tau = 0$. For the first case $\sqrt{\rho_{1,i}\sigma_{1,i}/\tau} = \mu_{1,i}$. In particular it is only possible if $\rho_{1,i}$ and $\sigma_{1,i}$ have the same sign. So, $x_1, x_3 = O(\sqrt{\tau})$. By the MPCC-SOC, $\nabla_{x_5} q(0)$ is regular. Using the same arguments of Theorem 4.1. based on the Implicit Function Theorem at the g.c. points also, $x_5 = O(\sqrt{\tau})$. We have proven the following result

**Theorem 4.4.** *Let $x$ be a g.c. point of $P$ such that MCPC-LICQ , MPCC-SC and MPCC-SOC holds. Then for all $\tau > 0$ (small enough) the g.c. points of $x_\tau$ of $\mathcal{R}_\tau^{\mathcal{S}}$ (near $x$) satisfy $\|x_\tau - x\| = O(\sqrt{\tau})$.*

### 4.4. Problem $\mathcal{R}_\tau^{\mathcal{LF}}$

For this problem we have the following result

**Theorem 4.5.** *Let $x$ be a g.c. point of $P$ such that MPCC-LICQ , MPCC-SC and MPCC-SOC holds. Then for all $\tau > 0$ (small enough) the g.c. points of $x_\tau$ of $\mathcal{R}_\tau^{\mathcal{S}}$ (near $x$) satisfy $\|x_\tau - x\| = O(\tau)$.*

*Proof:* Using the ideas of the proof of the previous theorem we get that the g.c. point condition is

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + O(\tau) + \begin{pmatrix} X_1^3(x) + \tau I & \otimes & 0 & 0 & \otimes \\ 0 & \otimes & 0 & X_2^3(x) & \otimes \\ 0 & S_1 + \tau I + \otimes & 0 & 0 & \otimes \\ 0 & \otimes & 0 & 0 & S_2 + \otimes \\ X_1^1(x) + \tau I & \otimes & 0 & 0 & \otimes \\ 0 & \otimes & 0 & X_2^1(x) & \otimes \\ 0 & \otimes & I_{q_0} & 0 & \otimes \\ 0 & \otimes & 0 & 0 & \otimes \end{pmatrix} \begin{pmatrix} \rho_\tau^1 \\ \rho_\tau^2 \\ \sigma_\tau^1 \\ \sigma_\tau^2 \\ \mu_\tau \end{pmatrix} = 0.$$

and that the expressions $\otimes = x_i \nabla s_i$ are $O(\tau)\nabla s_i$. As in the previous cases, by the MPCC-SC, $x_4 = 0$ and either $(x_i + \tau)(x_{m+i} + \tau) = \tau^2$ or $x_i x_{m+i} = \tau^2, i = 1,\ldots,p$. As both terms, $(x_i + \tau)$ and

$(x_{m+i} + \tau)$ in the first case and $x_i$ and $x_{m+i}$ in the second shall have the same order, it follows that $(x_i + \tau), (x_{m+i} + \tau), x_i, x_{m+i} = O(\tau)$. This implies that $x_1, x_2, x_3, x_4 = O(\tau)$. The relation $x_5 = O(\tau)$ is obtained after the MPCC-SOC. □

As in the previous case the non-degeneracy of the MPCC-model does not imply the degeneracy of the solutions of $\mathbb{R}_\tau^{LF}$ as it is shown in the next example

**Example 4.3.** *Let us consider the problem that locally corresponds with*

$$\min x_1 + x_3 \quad s.t.\ 0 \le x_1 \perp x_3 \ge 0\ 0 \le x_2 \perp (x_1^2 + 1) \ge 0.$$

$x = 0$ *is its unique solution and it is, evidently, non degenerated.*
*The parametric problem is*

$$\begin{array}{rrclcrcl} & & & \min x_1 + x_3 & & & & \\ s.t. & x_1 x_3 & \le & \tau^2, & & (x_1+\tau)(x_3+\tau) & \ge & \tau^2, \\ & x_2(x_1^2+1) & \le & \tau^2, & & (x_2+\tau)(x_1^2+1+\tau) & \ge & \tau^2. \end{array}$$

$x = (0, \tau^2, 0)$ *is a critical point and SC fails* □.

### 4.5. Problem $\mathcal{R}_\tau^{\mathcal{K}}$

We have already remarked that the LICQ does not hold in this case. So, we can not expect non-degenerancy. We consider the concept of g.c. points as a condition the minimal solutions satisfy because at minimizers in which the LICQ fails are critical points.

We suppose that $x_\tau$ is close to a critical point point $\bar{x}$.

In the first case, w.l.o.g., we assume that $\bar{x} = 0$ and, after rearranging conveniently the indices, we take the following partition $(x_{1,i})_\tau = -\tau, (x_{3,i})_\tau = \tau,\ i = 1, \dots, p_1,\ (x_{1,p_1+i})_\tau = -\tau, (x_{3,p_1+i})_\tau \ne \tau\ i = 1, \dots, p_2,\ (x_{1,p_1+p_2+i})_\tau = \tau,\ (x_{1,p_1+p_2+i})_\tau \notin \{\tau, -\tau\},\ i = 1, \dots, p_3,\ (x_{3,p_1+p_2+p_3+i})_\tau = \tau\ i = 1, \dots, p_4,$ $(x_{2,p+i})_\tau = -\tau,\ i = 1, \dots, p_5$ and $(x_{2,p+p_5+i})_\tau = \tau,\ i = 1, \dots, p_6$. The critical point condition reads

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + O(\tau) + \begin{pmatrix} (-I_{p_1+p_2}|0|0)^T & (0|X_3-\tau|0)^T & 0 & 0 & 0 \\ 0 & 0 & (-I_{p_5}|0)^T & (0|s_2-\tau)^T & 0 \\ 0 & \frac{X_1^1-\tau}{0}|0|\frac{0}{X_1^2-\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{I}{0} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_\tau^1 \\ \rho_\tau^2 \\ \sigma_\tau^1 \\ \sigma_\tau^2 \\ \mu_\tau \end{pmatrix} = 0.$$

Here $X_1^1 - \tau, X_1^2 - \tau,\ X_3 - \tau$ represent the diagonal matrices defined by the vectors $x_{1,i} - \tau, i = 1, \dots p_1,$ $x_{1,p_1+p_2+p_3+i} - \tau, i = 1, \dots, p_4\ x_{3,i} - \tau, i = p_1 + p_2 + 1, \dots, p_1 + p_2 + p_3 + p_4$ and $s_2 - \tau$, is similarly constructed using the vector $s_{2,i}(x) - \tau,\ i = p + p_5 + 1, \dots, p + p_5 + p_6$. By the feasibility $x_2 \in [\tau, \tau/C]$ where $C > 0$ is a lower bound of $s_i(x)$ for $i = p + 1, \dots, m$. The fulfillment of the MPCC-SC implies that $p_1 + p_2 = p,\ p_2 = p_3 = 0$ and $x_4 = 0$. In particular, $x_1, x_3 \in \{\tau, -\tau\}^p$. By the MPCC-SOC $x_5 = O(\tau)$. So,

$$\|x_\tau - \bar{x}\| \le O(\tau).$$

If the LICQ fails there exists and index $i$ such that $x_{1.i} = x_{3,i} = \tau$. In particular the direction $d = -e_{3,i}$ is feasible and as $\frac{\partial f(x_\tau)}{\partial x_{3,i}} = \sigma_{3,i} + O(\tau) > 0$, for $\tau$ small enough, $d$ is a descent direction. So, $x_\tau$ can not be a stationary point or a local minimizer.
We have the following result

**Theorem 4.6.** *Let $\bar{x}$ be a g.c. point of $P$ such that MPCC-LICQ , MPCC-SC and MPCC-SOC holds. Then for all $\tau > 0$ (small enough) there are no local minimizers of $P_\tau$. Moreover the set of g.c. points of $\mathcal{R}_\tau^{\mathcal{K}}$ $x_\tau$ (near $\bar{x}$) is not empty and for all g.c. point $x_\tau$ it holds $\|x_\tau - \bar{x}\| = O(\tau)$.*

### 4.6. Problems $\mathcal{R}_\tau^{\mathcal{SU}}$ and $\mathcal{R}_\tau^{\mathcal{SK}}$

Although LICQ fails at the parametric problems, for the critical points we have the following result

**Theorem 4.7.** *Given $\bar{x}$ a non-degenerated critical point of MPCC. Then locally around $\bar{x}$, the Hausdorff distance between the sets of critical points of $\mathcal{R}_\tau^{\mathcal{SU}}$ (resp. $\mathcal{R}_\tau^{\mathcal{SK}}$ ) and $\mathcal{P}$ is bounded by $O(\tau)$*

*Proof:* It is clear that critical points of $\mathcal{P}$ are also critical solutions of $\mathcal{R}_\tau^{\mathcal{SU}}$ and $\mathcal{R}_\tau^{\mathcal{SK}}$.
For the first approach, as already noticed in Section 3., $r_i(x) + s_i(x) \leq \phi^{SU}(r_i(x) - s_i(x); \tau)$ implies $r_i(x) + s_i(x) \leq \tau$ or $r_i(x)s_i(x) = 0$. $s_i(x) > 0, i = p+1, \ldots, m$ and locally $s_i(x) > M > \tau$, for $\tau$ small enough. We use the local dipheomorphism, see (3.2). As $s_i(x_\tau) > M > \tau$, for $\tau$ large enough, $x_{i,2}(x_\tau) + s_i(x_\tau) > \tau$ and, therefore, $x_{i,2}(x_\tau)s_i(x_\tau) = 0$ As a consequence $x_2 = 0$ and $x_{i,2}(x_\tau) + s_i(x_\tau) - \phi(s_{i,2} - x_{i,2}) = 0$.
Similarly $((x_{1,i})_\tau > \tau)$ if and only if $(x_{i,3})_\tau = 0$. Analogously $(x_{3,i})_\tau > \tau$ implies $(x_{1,i})_\tau = 0$. For simplicity we suppose $x_{1,i} \leq x_{3,i}$ Suppose we have the following partition $x_{3,i} \geq \tau,\ i = 1, \ldots p_1$, $\tau > x_{3,i} \geq x_{1,i} > 0,\ i = p_1 + 1, \ldots p_1 + p_2$, $\tau > x_{3,i} > x_{1,i} = 0,\ i = p_1 + p_2 + 1, \ldots p_1 + p_2 + p_3$, $x_{3,i} = x_{1,i} = 0,\ i = p_1 + p_2 + p_3 + 1, \ldots p$. Then the critical point condition is

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + O(\tau) - \begin{pmatrix} I_{p_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{p_3} & 0 \\ 0 & 0 & I_{p-p_1-p_2-p_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rho_\tau^1 - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \rho_\tau^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_\tau^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mu_\tau \\ 0 \end{pmatrix} + \begin{pmatrix} 2I_{p_1} & 0 & 0 \\ 0 & \otimes & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2I_{m-p} \\ 0 & 0 & 0 \\ 0 & \otimes & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_\tau \\ b_\tau \\ c_\tau \end{pmatrix} = 0$$

As, by the MPCC-SC, all the components of $\sigma_1 \neq 0$, $p_1 = p_3 = 0$ and $x_4 = 0$. So, for each $i$ $\tau > x_{3,i} \geq x_{1,i} > 0$ or $x_{3,i} = x_{1,i} = 0$. As $x_2, x_4 = 0$, we have that $(x_1, \ldots, x_4) = O(\tau)$. By the MPCC-SOC, also $x_5 = O(\tau)$.
For the second approach we have, for each $i = 1, \ldots, p$, the following possible six sets of active indices related to the complementarity constraints:
$x_{1,i} = x_{3,i} = 0$, $\phi^{SK}(x_{1,i}, x_{3,i}, \tau) < 0$,
$x_{3,i} > x_{1,i} = 0$, $\phi^{SK}(x_{1,i}, x_{3,i}, \tau) < 0$,
$x_{3,i} \geq x_{1,i} = \tau, \phi^{SK}(x_{1,i}, x_{3,i}, \tau) = 0$.
$s_i > M > 2\tau$, for $\tau$ small enough $x_{2,i} = 0$, $\phi^{SK}(x_{1,i}, x_{3,i}, \tau) < 0$.
$s_i > M > 2\tau$, for $\tau$ small enough $x_{2,i} = \tau$, $\phi^{SK}(x_{1,i}, x_{3,i}, \tau) = 0$.
$s_i > M > 2\tau$, for $\tau$ small enough $x_{2,i} \in (0, \tau)$, $\phi^{SK}(x_{1,i}, x_{3,i}, \tau) < 0$.

Without loss of generality we assume that $x_{1,i} = x_{3,i} = 0,\ i = 1, \ldots, p_1,\ x_{1,i} = 0, x_{3,i} > 0,\ i = p_1, \ldots, p_1 + p_2,\ x_{1,i} = \tau, x_{3,i} > \tau,\ i = p_1 + p_2, \ldots, p_1 + p_2 + p_3 \leq p,\ x_{2,i} = 0,\ i = p + 1, \ldots, p + p_4,$ $x_{2,i} = \tau,\ i = p + p_4, \ldots, p + p_4 + p_5$, and $x_{4,i} = 0,\ i = m + p + 1, \ldots, m + p + p_5$.
Considering these sets, the critical points satisfy the system:

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ -\mu \\ 0 \end{pmatrix} + O(\tau) - \begin{pmatrix} I_{p_1+p_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{p_4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{p_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{p_5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho^1_\tau \\ \rho^2_\tau \\ \sigma^1_\tau \\ \mu^1_\tau \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (x_{3,i})_\tau - \tau & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & s_i(x_\tau) - \tau \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_\tau \\ b_\tau \end{pmatrix} = 0$$

Again, by yhe MPCC-SC, $p_1 + p_2 + p_3 = p_1 = p$, $p_4 = q_0$. So, $x_4 = 0$ and, for each $i = 1, \ldots p$, $x_{1,i} = x_{3,i} = 0$. As $x_{2,i} = O(\tau)$, by the MPCC-SOC and the Implicit Function Theorem $x_5 = O(\tau)$. So, all critical point of the parametric problem close to $\bar{x}$ are in $B(\bar{x}, O(\tau)$ as desired. □

## 5. CONCLUSIONS

A first result shows that the (Hausdorff) distance between the feasible sets of $\mathcal{P}_\tau, \mathcal{Q}_\tau, \mathcal{R}^S_\tau$ and $\mathcal{P}$ is bounded by $O(\sqrt{\tau})$ for $\tau \to 0$. Error bounds of the same order are also obtained for the set of the local minimizers. In the case of $\mathcal{R}^{\mathcal{LF}}{}_\tau$, $\mathcal{R}^K_\tau$, $\mathcal{R}^{\mathcal{SU}}{}_\tau$ and $\mathcal{R}^{\mathcal{SK}}{}_\tau$, the order is $O(\tau)$. It is important to realize that in the these regularisations, $\tau^2$ is used to bound the complementarity constraints. So, a similar rate of convergence was obtained in all cases.
For the smoothing approach defined by $\mathcal{P}_\tau$ the g.c. points can be degenerated, even if at the original problem MPCC-LICQ, MPCC-SC and MPCC-SOC hold . In the considered regularisation approaches either LICQ or SC fails. From a numerical viewpoint non-degenarancy is an important advantage because matrices of the system defined by Newton type algorithems will be non-singular. So, from this point of view $\mathcal{Q}_\tau$ has an important advantage.
In the second part of this paper we will complete this study. We will obtain the types of points that can be obtained as limits of the critical points of the parametric problems defined by the corresponding schemes near $\tau = 0$. From a global viewpoint we will obtain which kind of solutions may appear in the generic case. In particular if they are critical points of $\mathcal{P}$ and $\mathcal{M}$ and the set of feasible solutions of the parametric problems are compact, we can extend the results shown in Section 4. and provide global bounds for the Hausdorff distance of the set of g.c. points.

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## A ALTERNATIVE PROOF OF THEOREM 7.6, [12]

**Theorem A1.** *Let MPCC-LICQ holds at all feasible point of $\mathcal{P}$. Then, for all $\bar{x}$ feasible point of $\mathcal{P}$, there exists $\overline{\tau}$ and a neighborhood $V$ of $\bar{x}$ such that for all $\tau \in (0, \overline{\tau})$ and $x_\tau \in V \cap \mathcal{R}_\tau^{\mathcal{LF}}$, the LICQ holds. Moreover if the sets $\mathcal{R}_\tau^{\mathcal{LF}}$ are contained in some compact set $K$, there exists $\overline{\tau}$ such that for all $\tau \in (0, \overline{\tau})$ the LICQ is satisfied for all $x_\tau$, feasible point of $\mathcal{R}_\tau^{\mathcal{LF}}$ .*

*Proof:* First we shall note that for $\tau > 0$ at most one of the following pair of constraints can be active $(x_i + \tau)(x_{m+i} + \tau) = \tau^2$ or $x_i x_{m+i} = \tau^2$. Analogously, $(x_i + \tau)(s_i(x) + \tau) \geq \tau^2$ and $x_i s_i(x) \leq \tau^2$ cannot be active simultaneously. Rearranging the index we suppose that $(x_i + \tau)(x_{m+i} + \tau) = \tau^2, i = 1, \ldots, p_1$, $x_i x_{m+i} = \tau^2, i = p_1 + 1, \ldots, p_2 \leq p$, $(x_i + \tau)(s_i(x) + \tau) \geq \tau^2$, $i = p + 1, \ldots, p_3$ and $x_i s_i(x) \leq \tau^2$, $i = p_3 + 1, \ldots, p_4 \leq m$. We use the following notation $X_1^1(x) = diag(x_1, \ldots, x_{p_1})$, $X_2^1(x) = diag(x_{p_1+1}, \ldots, x_{p_2})$, $X_1^3(x) = diag(x_{m+1}, \ldots, x_{m+p_1})$, $X_2^3(x) = diag(x_{m+p_1+1}, \ldots, x_{m+p_2})$, $S_1(x) = diag(s_{p+1}(x), \ldots, s_{m+p_3}(x))$, and $S_2(x) = diag(s_{m+p_1+1}(x), \ldots, x_{m+p_4}(x))$. Here $\otimes$ is a matrix whose columns are $x_i \nabla_{x_I} s_i$, $i = p + 1, \ldots, m$, for a suitable indices set $I$.
The gradients are

$$B = \begin{matrix} X_1^3(x) + \tau I & \otimes & 0 & 0 & \otimes \\ 0 & \otimes & 0 & X_2^3(x) & \otimes \\ 0 & S_1 + \tau I + \otimes & 0 & 0 & \otimes \\ 0 & \otimes & 0 & 0 & S_2 + \otimes \\ X_1^1(x) + \tau I & \otimes & 0 & 0 & \otimes \\ 0 & \otimes & 0 & X_2^1(x) & \otimes \\ 0 & \otimes & I_{q_0} & 0 & \otimes \\ 0 & \otimes & 0 & 0 & \otimes \end{matrix}$$

As in the proof of Theorem 3.6., $s_i(x) > M > 0$. So, $S_1, S_2$ are diagonal matrices whose elements are larger than $M$. As $\otimes = O(\tau)$, it follows that the matrix $\begin{pmatrix} S_1 + \tau I + \otimes & \otimes \\ \otimes & S_2 + \otimes \end{pmatrix}$ is non singular.
From this fact, the result easily follows.
The second part is a consequence of the first part and Theorem 3.6.$\square$.

## B PROOF OF THEOREM 4.3.

Consider the canonical representation of the system (3.5). The g.c. point condition at $\mathcal{P}_\tau$ is

$$\nabla f(x^k) + \begin{bmatrix} X^3(x^k) \\ 0 \\ X^1(x^k) \\ 0 \\ 0 \end{bmatrix} \mu_1^k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ eye(q_0, I_{g(x^k k)}) \\ 0 \end{bmatrix} \lambda^k +$$

$$\begin{bmatrix} x_i^k \nabla_{1,\dots,p} s_i(x^k), i = p+1,\dots,m \\ B_0(x^k) \\ x_i^k \nabla_{m+1,\dots,m+p} s_i(x^k), i = p+1,\dots,m, \\ x_i^k \nabla_{m+p+1,\dots,m+p+q_0} s_i(x^k), i = p+1,\dots,m \\ x_{p+i}^k \nabla_{m+p+q_0+1,\dots,n} s_i(x^k), i = p+1,\dots,m \end{bmatrix} \mu_2^k = 0.$$

Here $X^3(x^k) = diag(x_{m+1}^k, \dots, x_{m+p})$, $X^1(x^k) = diag(x_1^k, \dots, x_p)$, $B_0(x_k)$ is defined as in (3.6) and $eye(q_0, I_{g(x_k)})$ denotes the columns of the $q^0$-dimensional unit matrix whose indices belong to $I_g(x^k)$. Note that, due to the non-singularity of $B_0(x^k)$, the multiplier $\mu_2^k$ is well-defined. Moreover, by the continuity of the involved functions, $\mu_2^k$ converges and we can define

$$\mu = \lim_{k\to\infty} diag(s_{p+1}(x^k), \dots, s_m(x^k))\mu_2^k.$$

Analogously, it can be seen that

$$\begin{aligned} X^3(x^k)\mu_1^k &\to \gamma, \\ X^1(x^k)\mu_1^k &\to \nu, \end{aligned} \tag{B1}$$

and $\lambda^k \to \lambda$. As $x_i^k \nabla s_i(x^k) \to 0$ for $i = p+1, \dots, m$, $\overline{x}$ is a MPCC-critical point and

$$\nabla f(\overline{x}) + \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_{m-p} & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_{q_0} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \gamma \\ \mu \\ \nu \\ \lambda \end{pmatrix} = 0.$$

By the regularity of $\mathcal{P}$ the point $\bar{x}$ is a non-degenerate critical point. So, $\lambda \neq 0$ and $\lambda_j^k \neq 0$ for $k$ large enough and $j \in I_g(\overline{x})$. As a consequence, for $k$ large enough $I_g(\bar{x}) \subset I_g(x_k)$. Hence $I_g(\bar{x}) = I_g(x_k) = \{1, \dots q_0\}$, for $k$ large and SC holds at $x^k$.

To analyse SOC we need to prove that the following matrix is non singular over $T_{x^k}\mathcal{M}^{\mathcal{Q}}{}_{\tau^k}$,

$$Y^k = \nabla_x^2 L(x^k, \lambda^k, \mu_1^k, \mu_2^k) = \left( \nabla_x^2 f(x^k) + \begin{pmatrix} 0 & 0 & diag(\mu_1^k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ diag(\mu_1^k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \bigotimes \right)$$

Recall that $T_{x^k}\mathcal{M}^{\mathcal{Q}}{}_{\tau^k}$ is the space orthogonal to $B(x^k) = \begin{pmatrix} X^3(x^k) & \otimes_1^k & 0 \\ 0 & B_0(x^k) & 0 \\ X^1(x^k) & \otimes_3^k & 0 \\ 0 & \otimes_k^k & I_{q_0} \\ 0 & \otimes_5^k & 0 \end{pmatrix}$.

Here $\bigotimes = \sum_{i=p+1}^m (\mu_2^k)_i \nabla_x^2 [x_i s_i(x)](x^k)$, $\otimes_1^k = [x_i^k \nabla_{1,\dots,p} s_i(x^k),\ i = p{+}1\dots, m]$, $\otimes_3^k = [x_i^k \nabla_{m+1,\dots,m+p} s_i(x^k),\ i = p+1, \dots, m]$, $\otimes_4^k = [x_i^k \nabla_{m+1,\dots,m+p} s_i(x^k),\ i = p+1, \dots, m]$, $\otimes_5^k = [x_i^k \nabla_{m+p+1,\dots,n} s_i(x^k), i = p+1, \dots, m]$ and $B_0(x^k)$, defined in (3.6), is non-singular for $k$ large enough. So, a base of $T_{x^k}\mathcal{M}^{\mathcal{Q}}{}_{\tau}$

is given by the columns of

$$\hat{B}(x^k) = \begin{pmatrix} X^1(x^k) & 0 \\ \oplus_1^k & \oplus_2^k \\ -X^3(x^k) & 0 \\ 0 & 0 \\ 0 & I_{n-m-p-q_0} \end{pmatrix},$$

where $\oplus_1^k = -B_0(x^k)^{-1}[X^1(x^k) \otimes_1^k - X^3(x^k) \otimes_3^k]$ and $\oplus_2^k = -B_0(x^k)^{-1} \otimes_5^k$,

Now consider $\mathcal{C}(x^k) = \hat{B}(x^k)^T Y^k \hat{B}(x^k)$. Then, after some algebraic manipulation we can prove that

$$\begin{aligned} \mathcal{C}(x^k) &= \begin{pmatrix} O(\tau) & O(\sqrt{\tau}) \\ O(\sqrt{\tau}) & O(\tau) + \nabla^2_{x_{m+p+q_1+1},\ldots,x_n} f(x^k) \end{pmatrix} + \begin{pmatrix} -2X^3(x^k) diag(\mu_1^k) X^1(x^k) & 0 \\ 0 & 0 \end{pmatrix} + O(\tau) \\ &= \begin{pmatrix} O(\tau) - 2X^3(x^k) diag(\mu_1^k) X^1(x^k) & O(\sqrt{\tau}) \\ O(\sqrt{\tau}) & O(\tau) + \nabla^2_{x_{m+p+q_1+1},\ldots,x_n} f(x^k) \end{pmatrix}. \end{aligned}$$

As for SOC, we need to prove that $\mathcal{C}(x^k)$ is non-singular. Assume to the contrary, that $c^k \neq 0$ is a solution of $\mathcal{C}(x^k)c^k = 0$. Without loss of generality we assume that $\|c^k\| = 1$.

Taking $k \to \infty$ and hence, $\tau^k \to 0$, without loss of generality, we can assume $c^k \to c \neq 0$. Thus, considering $c^k = (c_1^k, c_2^k) \to (c_1, c_2) = c \neq 0$, we have

$$\begin{aligned} \lim_{k\to\infty} \mathcal{C}(x^k)c^k &= \lim_{k\to\infty} \begin{pmatrix} O(\tau^k)c_1^k - 2X^3(x^k) diag(\mu_1^k) X^1(x^k) c_1^k + O(\sqrt{\tau^k}) c_2^k \\ O(\sqrt{\tau^k}) c_1^k + O(\tau^k) c_2^k + \nabla^2_{x_{m+p+q_1+1},\ldots,x_n} f(x^k) c_2^k \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \nabla^2_{x_{l+p+q_1+1},\ldots,x_n} f c_2 \end{pmatrix} = 0. \end{aligned}$$

But as MPCC-SOC holds, $\nabla^2_{x_{m+p+q_1+1},\ldots,x_n} f(x^k)$ is non-singular. Hence $c_2 = 0$.

Now, if $O(\tau^k)c_1^k - 2X^3(x^k)diag(\mu_1^k)X^1(x^k)c_1^k + O(\sqrt{\tau_k})c_2^k = 0$, dividing by $\sqrt{\tau_k}$, we obtain

$$\frac{O(\tau_k)c_1^k - 2X^3(x^k)diag(\mu_1^k)X^1(x^k)c_1^k + O(\sqrt{\tau^k})c_2^k}{\sqrt{\tau_k}} = O(\sqrt{\tau^k})c_1^k + O(1)c_2^k - \frac{2X^3(x^k)diag(\mu_1^k)X^1(x^k)c_1^k}{\sqrt{\tau^k}} = 0.$$

Taking limits for $\tau \to 0$ yields

$$\lim_{\tau\to 0} O(\sqrt{\tau_k}) - O(1)c_2^k - \frac{2X^3(x^k)diag(\lambda_1^k)X^1(x^k)c_1^k}{\sqrt{\tau_k}} = 0.$$

As $c_2^k \to 0$, it holds that $\lim_{\tau\to 0} \frac{2X^3(x^k)diag(\lambda_1^k)X^1(x^k)c_1^k}{\sqrt{\tau_k}} = 0$.

However, for $i = 1, \ldots, p$, either $x_i^k$ or $x_{m+i}^k \geq \sqrt{\tau_k}$, because $x_i^k x_{m+i}^k = \tau_k$. Without loss of generality, we assume that $x_i^k \geq \sqrt{\tau_k}$. As already obtained in (B1) $-2X^3(x^k)diag(\mu_1^k) \to \gamma \neq 0$. Then

$$\lim_{k\to\infty} \frac{X^3(x^k)diag(\mu_1^k)X^1(x^k)}{\sqrt{\tau_k}} = diag(\gamma) \lim_{k\to\infty} \frac{X^1(x^k)}{\sqrt{\tau_k}}.$$

But, as seen in [2], $\lim_{\tau_k\to 0} \frac{X^1(x^k)}{\sqrt{\tau_k}} = diag(y)$ exists and it is non zero. So,

$$\lim_{\tau_k\to 0} \frac{-2X^3(x^k)diag(\mu_1^k)X^1(x^k)c_1^k}{\sqrt{\tau}} = diag(\gamma)diag(y)c_1 = 0.$$

As $y, \gamma \neq 0$, it follows that $c_1 = 0$ and $c_2 = 0$, contradicting the assumption that $c \neq 0$.