A NEW EXPONENTIATED DISCRETE LINDLEY DISTRIBUTION FOR ANALYZING COUNT DATA

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ABSTRACT
In this article, a new discrete distribution called the exponentiated discrete Lindley distribution is derived. The usefulness of this distribution is supported by its ability to analyze different types of count data (over-, under-, equi-dispersed, positively, and negatively skewed). Also, with two parameters, it possesses a bathtub-shaped hazard rate function, which is not the case for well-known discrete distributions. We also discuss some of its important properties with numerical illustrations. Subsequently, the point and interval estimation of the parameters are performed under the classical and Bayesian paradigms. A simulation study is carried out to showcase the numerical illustration of the discussed estimation procedures. To examine the practical applicability, four real datasets are fitted, and the results are fairly compared with other well-known existing discrete models.

KEYWORDS: Bayesian estimation, Exponentiated discrete Lindley distribution, Count data, Method of maximum likelihood, Simulation study.

MSC: 62E10, 62E20, 62F10, 62F12, 62F15

RESUMEN
En este artículo, se deriva una nueva distribución discreta llamada distribución de Lindley discreta exponencial. La utilidad de esta distribución está respaldada por su capacidad para analizar diferentes tipos de datos de recuento (sobre, bajo, equidisperso, sesgado positiva y negativamente). Además, con dos parámetros, posee una función de tasa de riesgo en forma de bañera, que no es el caso de distribuciones discretas conocidas. También discutimos algunas de sus propiedades importantes con ilustraciones numéricas. Posteriormente, la estimación puntual e intervalológica de los parámetros se realiza bajo los paradigmas clásico y Bayesiano. Se lleva a cabo un estudio de simulación para mostrar la ilustración numérica de los procedimientos de estimación discutidos. Para examinar la aplicabilidad práctica, se ajustan cuatro conjuntos de datos reales, y los resultados se comparan de manera justa con otros modelos discretos existentes bien conocidos.

PALABRAS CLAVE: Estimación Bayesiana, Distribución de Lindley discreta exponencial, Datos de conteo, Método de máxima probabilidad, Estudio de simulación.

1. INTRODUCTION

In many areas, statistical distributions are used to make decisions and characterize the probabilistic behavior of random phenomena. The distributions are proven to be helpful in all fields of science. For instance, the traditional distributions, including the Weibull, normal, gamma, Gompertz, and Maxwell distributions, have drawn a lot of interest from scholars and found usage in a variety of fields, including demography, engineering, and science. Many situations produce data that are discrete in nature, either due to measurement instrument limitations or inherent characteristics. For example, in reliability engineering, the number of successful cycles prior to the failure when the device works in cycle is the number of times a device is switched on or off; in survival analysis, the survival times for those suffering from diseases like lung cancer or the period from remission to relapse may be recorded as the number of days or weeks, the number of deaths or daily cases due to the COVID-19 pandemic observed over a specified duration, etc. Moreover, in many practical problems, the count phenomenon occurs, as, for example, in the number of occurrences of earthquakes in a calendar year, the number of absences, the number of accidents, the number of kinds of species in ecology, the number of insurance claims, and so on. Therefore, it is reasonable to model such situations with a suitable discrete distribution.
Because traditional discrete distributions such as the binomial, Poisson, geometric, and negative binomial distributions were unable to adequately characterize different types of data, academics have been more interested in the discretization of continuous distributions throughout the previous two decades. Such a discretization can be performed through various methods; one can refer to Chakraborty (2015) in this regard. These methods have been used by many authors to generate the discrete analogue of any continuous distribution. See, for instance, Roy (2004), Krishna and Pundir (2009), Chakraborty and

Chakravarty (2012), Almalki and Nadarajah (2014), Nekoukhou and Bidram (2015), Tyagi et al. (2019, 2020), Singh et al. (2022), Pandey et al. (2022), and related references cited therein.
Given the existing literature, we found that many discrete distributions have been introduced over the past few decades. Yet there is still scope to introduce new plausible discrete distributions that can adequately capture the diversity of real data. This phenomenon motivates us to provide a flexible discrete model for fitting a wide spectrum of discrete real-world datasets. Therefore, in this paper, we propose the exponentiated discrete Lindley (ExDLi) distribution and its related statistical model. Some prime objectives for introducing this distribution are as follows:

- In this existing literature, many of the developed discrete distributions claim to model over- and under-dispersed datasets. But they fail or ignore studying both types of datasets. In view of this, we showcase the applicability of the present model for both types of datasets, i.e., over- and under-dispersed datasets.
- One of the important lacunae of the recent papers on discrete models is that in the application section, they compare the applicability of the over-dispersed (or under-dispersed) model with under-dispersed (or over-dispersed) models, and this always results in the superiority of the proposed model. But theoretically, it does not sound good. To overcome this issue, we compare our proposed model with over-dispersed (or under-dispersed) models for over-dispersed (or under-dispersed) datasets.
- We aim to develop a discrete model that is capable of analyzing various types of failure data, discrete data generated from many practical studies, such as mortality experiments, industrial experiments, etc., showing constant, increasing, decreasing, or bathtub-shaped failure rates.
- Last but not least, to consistently outperform other well-known discrete models in the statistical literature.

The article is organized as follows: In Section 2, we introduce the ExDLi distribution. Different distributional measures are discussed in Section 3. The other sections are devoted to the inferential aspect and applications. In Section 4, the point estimation of the model parameters is discussed using maximum likelihood (ML) and Bayesian methods. Interval estimation of the unknown parameters is pointed out in Section 5. Numerical illustrations with empirical data are examined in Section 6. Four real datasets are analyzed to observe the flexibility of the ExDLi model in Section 7. Finally, Section 8 provides some conclusions.

2. EXPONENTIATED DISCRETE LINDLEY DISTRIBUTION

Suppose that a random variable (RV) *Y* follows the one parameter continuous Lindley distribution (Lindley, 1958) with the probability density function (PDF) defined as

$$f(y,\theta)=\frac{\theta^2}{(1+\theta)}(1+y)e^{-\theta y}; y>0, \theta>0. \tag{2.1}$$

From the Methodology III of Chakraborty (2015), the discrete analogue of *Y* has the following probability mass function (PMF):

$$P(X=x_i)=\frac{f(x_i,\theta)}{\sum_{i=0}^{\infty} f(x_i,\theta)}; i=0,1,2,... \tag{2.2}$$

Using Equation (2.2) with consecutive integer values, and putting $e^{-\theta}=\tau$, the PMF of the discrete analogue of Lindley distribution can be derived as

$$P(X=x)=(1-\tau)^2(1+x)\tau^x; x=0,1,2,...,. \tag{2.3}$$

Alternatively, we can obtain this PMF by putting $\beta=1$ in Hussain et al. (2016). The cumulative distribution function (CDF) corresponding to Equation (2.3) is indicated as

$$F(x,\tau)=1-\tau^{(x+1)}(x+2)+\tau^{(x+2)}(x+1). \tag{2.4}$$

The CDF of the ExDLi distribution is

$$\begin{aligned} G(x,\tau,\sigma)&=[F(x,\tau)]^{\sigma} \\ &=\left(1-\tau^{(x+1)}(x+2)+\tau^{(x+2)}(x+1)\right)^{\sigma} \\ &=\left(1-w(x+1,\tau)\right)^{\sigma}; x=0,1,2,..., \end{aligned} \tag{2.5}$$

where $w(x,\tau)=\tau^{x}(x+1)-\tau^{(x+1)}x$.

Therefore, the PMF of the ExDLi distribution is obtained as

$$\begin{aligned} g(x,\tau,\sigma) &= G(x,\tau,\sigma)-G(x-1,\tau,\sigma) \\ &= \left(1-w(x+1,\tau)\right)^{\sigma}-\left(1-w(x,\tau)\right)^{\sigma} \\ &= \left(1-\tau^{(x+1)}\left(x+2\right)+\tau^{(x+2)}\left(x+1\right)\right)^{\sigma}-\left(1-\tau^{x}\left(x+1\right)+\tau^{(x+1)}x\right)^{\sigma};x=0,1,2,.... \end{aligned} \tag{2.6}$$

Figure 1 shows some PMF plots for various values of the parameters. From this figure, it can be inferred that the ExDLi distribution is always unimodal, which is the case for log-concave PMFs in general.

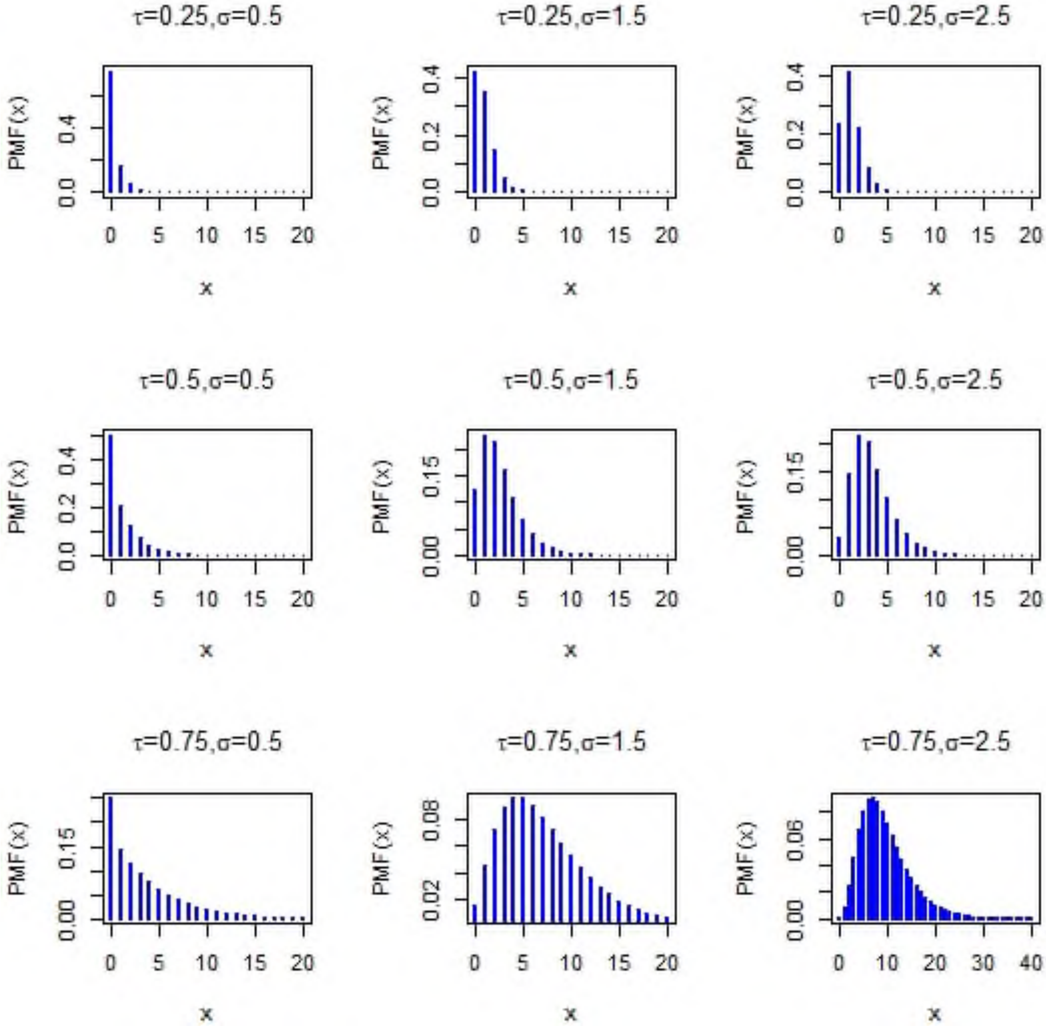

Figure 1: The PMF plots of the ExDLi distribution for different sets of parameters**.**

The survival function of the proposed distribution is

$$S\left(x,\tau,\sigma\right)=P\left(X>x\right)=1-\left(1-w\left(x+1,\tau\right)\right)^{\sigma} \quad ;x=0,1,2,..., \tag{2.7}$$

The hazard rate is a reliability characteristic that describes the system's failure behavior over time. The discrete hazard rate function (HRF) of the ExDLi distribution can be expressed as

$$h(x,\tau,\sigma)=P\left(X=x\,|\,X\geq x\right)=\frac{P\left(X=x\right)}{S\left(x-1,\tau,\sigma\right)}=\frac{\left(1-w\left(x+1,\tau\right)\right)^{\sigma}-\left(1-w\left(x,\tau\right)\right)^{\sigma}}{1-\left(1-w\left(x,\tau\right)\right)^{\sigma}};x=0,1,2,..., \tag{2.8}$$

provided that $S\left(x-1,\tau,\sigma\right)>0$. Figure 2 presents some PMF plots for various values of the parameters.

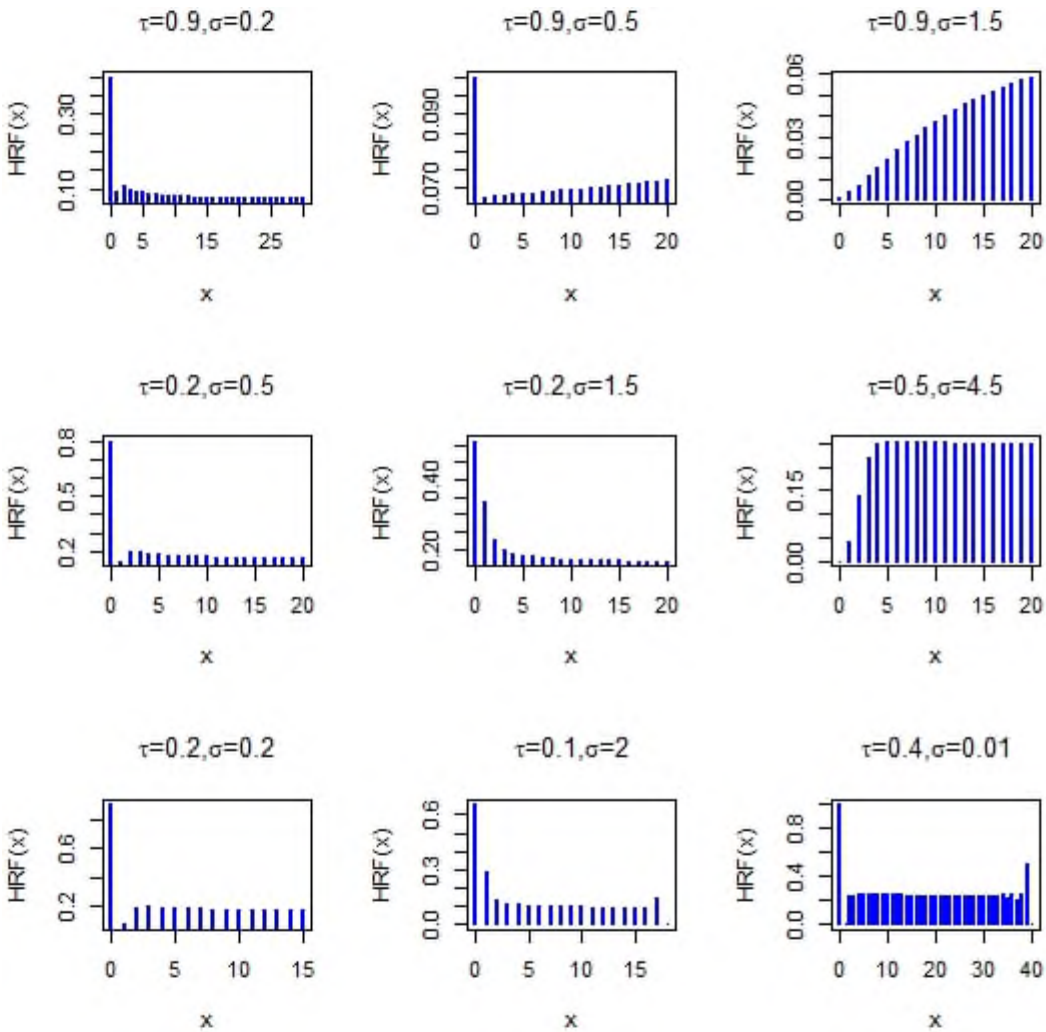

Figure 2: The HRF plots of the ExDLi distribution for different sets of parameters**.**

As we see from this figure, a characteristic of the ExDLi distribution is that its HRF can be increasing, decreasing, decreasing–increasing–decreasing, increasing–decreasing–increasing, unimodal, bathtub, and J-shaped, which makes the proposed distribution able to fit suitably different datasets. As a matter of fact, the ExDLi distribution is clearly more flexible than other discrete distributions. Also, the reversed hazard rate function (RHRF) of the ExDLi distribution can be expressed as follows:

$$h^{*}(x,\tau,\sigma)=P(X=x\,|\,X\le x)=\frac{P(X=x)}{G(x,\tau,\sigma)}=\frac{\left(1-w(x+1,\tau)\right)^{\sigma}-\left(1-w(x,\tau)\right)^{\sigma}}{\left(1-w(x+1,\tau)\right)^{\sigma}};x=0,1,2,...,$$

3. DISTRIBUTIONAL PROPERTIES

3.1 Moments, Skewness and Kurtosis

Moments of a probability distribution are important tools for measuring its different properties such as mean, variance, skewness, kurtosis, etc. If $G(x)$ is the CDF of a discrete RV *X*, then the r^{th} raw moments of this RV can be obtained by using the following formula:

$$E(X^{r})=\sum_{x=0}^{\infty}\left\{\left((x+1)^{r}-x^{r}\right)\left(1-G(x)\right)\right\}.$$

Using the above expression, the r^{th} raw moment of a RV *X* with the ExDLi distribution denoted by $\mu_r^{'}$ can be written as

$$\mu_r' = E(X^r) = \sum_{x=0}^{\infty}\left\{\left((x+1)^r - x^r\right)\left(1-\left(1-w(x+1,\tau)\right)^{\sigma}\right)\right\}. \quad (3.1)$$

Using the ratio test, we can easily observe that the expression in Equation (3.1) is convergent. It implies the existence of the r^{th} moment of the proposed distribution.

Now, using Equation (3.1), the first four raw moments of the ExDLi distribution are

$$\mu_1' = E(X) = \sum_{x=0}^{\infty}\left\{1-\left(1-w(x+1,\tau)\right)^{\sigma}\right\}, \quad (3.2)$$

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty}\left\{(2x+1)\left(1-\left(1-w(x+1,\tau)\right)^{\sigma}\right)\right\}, \quad (3.3)$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{\infty}\left\{\left(3x^2+3x+1\right)\left(1-\left(1-w(x+1,\tau)\right)^{\sigma}\right)\right\}, \quad (3.4)$$

and

$$\mu_4' = E(X^4) = \sum_{x=0}^{\infty}\left\{\left(4x^3+6x^2+4x+1\right)\left(1-\left(1-w(x+1,\tau)\right)^{\sigma}\right)\right\}. \quad (3.5)$$

The variance of the ExDLi distribution is obtained as

$$Var(X) = E\left(X^2\right) - E\left(X\right)^2$$

$$= \sum_{x=0}^{\infty}\left\{(2x+1)\left(1-\left(1-w(x+1,\tau)\right)^{\sigma}\right)\right\} - \left[\sum_{x=0}^{\infty}\left\{1-\left(1-w(x+1,\tau)\right)^{\sigma}\right\}\right]^2.$$

Using the raw moments in Equations (3.2) -(3.5), we can easily find the skewness and kurtosis from the following relations:

$$S = \frac{E(X^4) - 3E(X^2)E(X) + 2\left(E(X)\right)^3}{\left(Var(X)\right)^{3/2}},$$

and

$$K = \frac{E(X^4) - 4E(X^2)E(X) + 6E(X^2)\left(E(X)\right)^2 - 3\left(E(X)\right)^4}{\left(Var(X)\right)^2}, \text{ respectively.}$$

Tables 1-4 present some numerical results of the mean, variance, skewness and kurtosis for the ExDLi distribution by considering diverse combinations of τ and σ.

Table 1. Some values of the mean of the ExDLi distribution**.**

$\sigma \downarrow \tau \rightarrow$	0.025	0.05	0.1	0.2	0.4	0.6	0.8
0.25	0.0131	0.0273	0.0594	0.1417	0.4178	1.0293	2.9994
0.5	0.0260	0.0539	0.1162	0.2715	0.7714	1.8317	5.1288
1	0.0513	0.1053	0.2222	0.5000	1.3333	3.0000	8.0000
2	0.1001	0.2010	0.4075	0.8588	2.0914	4.4355	11.3471
5	0.2332	0.4396	0.8047	1.4875	3.2279	6.4957	16.0843

Table 2. Some values of the variance of the ExDLi distribution**.**

$\sigma \downarrow \tau \rightarrow$	0.025	0.05	0.1	0.2	0.4	0.6	0.8
0.25	0.0139	0.0307	0.0746	0.2179	0.9592	3.7811	22.5825
0.5	0.0273	0.0593	0.1399	0.3886	1.5737	5.7787	32.4436
1	0.0526	0.1108	0.2469	0.6250	2.2222	7.5000	40.0000
2	0.0980	0.1937	0.3880	0.8464	2.6111	8.3799	43.9230
5	0.1985	0.3284	0.5117	0.9232	2.7277	8.6759	45.2818

Table 3. Some values of the skewness of the ExDLi distribution**.**

$\sigma \downarrow \tau \rightarrow$	0.025	0.05	0.1	0.2	0.4	0.6	0.8
0.25	90.8306	50.2438	29.2631	17.8122	10.9628	8.1387	6.5501

0.5	44.2721	24.0723	13.7378	8.2443	5.1368	3.9685	3.4003
1	21.0125	11.0250	6.0500	3.6000	2.4500	2.1333	2.0250
2	9.4228	4.5799	2.3547	1.5352	1.4297	1.4530	1.4576
5	2.6037	0.9686	0.5674	0.9057	1.1141	1.1616	1.1793

Table 4. Some values of the kurtosis of the ExDLi distribution**.**

$\sigma \downarrow \tau \rightarrow$	0.025	0.05	0.1	0.2	0.4	0.6	0.8
0.25	102.6584	61.1713	38.8921	25.8762	17.4862	13.9119	11.9258
0.5	50.8751	30.3618	19.5630	13.4970	9.8360	8.4303	7.7678
1	25.0125	15.0250	10.0500	7.6000	6.4500	6.1333	6.0250
2	12.1416	7.5012	5.6175	5.2139	5.3142	5.3439	5.3463
5	4.6313	3.5210	4.0692	4.8308	4.9769	5.0323	5.0518

From these tables, we can infer that:

1. For a fixed value of σ and increasing value of τ as well as for a fixed value of τ and increasing value of σ, the mean and variance of the ExDLi distribution increase.
2. The skewness and kurtosis of the ExDLi distribution decrease for a fixed value of σ and increasing value of τ as well as for a fixed value of τ and increasing value of σ, implying that the proposed model is appropriate for modelling positively skewed and leptokurtic data.

3.3 Index of dispersion and coefficient of variation

The index of dispersion (IOD) is a metric used to determine whether data is over- or under-dispersed. An IOD greater than one indicates over-dispersion, whereas an IOD lower than one indicates under-dispersion. Equi-dispersion is indicated when the IOD is equal to one. The expression for the IOD of a RV *X* with the ExDLi distribution is

$$IOD(X)=\frac{Var(X)}{E(X)}=\frac{\sum_{x=0}^{\infty}\left\{(2x+1)\left(1-\left(1-w(x+1,\tau)\right)^{\sigma}\right)\right\}-\left[\sum_{x=0}^{\infty}\left\{1-\left(1-w(x+1,\tau)\right)^{\sigma}\right\}\right]^{2}}{\sum_{x=0}^{\infty}\left\{1-\left(1-w(x+1,\tau)\right)^{\sigma}\right\}}. \tag{3.6}$$

Furthermore, the coefficient of variation (COV) is a measure of data variability. It is commonly used to compare the variability of independent samples. The larger the coefficient of variation (COV), the more erratic the data. The COV of a RV *X* with the ExDLi distribution may be represented as

$$COV(X)=\frac{(Var(X))^{1/2}}{E(X)}=\frac{\left[\sum_{x=0}^{\infty}\left\{(2x+1)\left(1-\left(1-w(x+1,\tau)\right)^{\sigma}\right)\right\}-\left[\sum_{x=0}^{\infty}\left\{1-\left(1-w(x+1,\tau)\right)^{\sigma}\right\}\right]^{2}\right]^{1/2}}{\sum_{x=0}^{\infty}\left\{1-\left(1-w(x+1,\tau)\right)^{\sigma}\right\}}. \tag{3.7}$$

Some numerical values of the IOD and COV are shown in Tables 5 and 6, respectively, for a variety of parametric values.

Table 5. Some values of the IOD of the ExDLi distribution**.**

$\sigma \downarrow \tau \rightarrow$	0.025	0.05	0.1	0.2	0.4	0.6	0.8
0.25	1.0625	1.1255	1.2547	1.5372	2.2956	3.6736	7.5289
0.5	1.0501	1.1006	1.2043	1.4310	2.0401	3.1548	6.3258
1	1.0256	1.0526	1.1111	1.2500	1.6667	2.5000	5.0000
2	0.9786	0.9639	0.9520	0.9855	1.2485	1.8893	3.8709
5	0.8511	0.7471	0.6360	0.6207	0.8450	1.3356	2.8153

Table 6. Some values of the COV of the ExDLi distribution**.**

$\sigma \downarrow \tau \rightarrow$	0.025	0.05	0.1	0.2	0.4	0.6	0.8
0.25	9.0211	6.4248	4.5940	3.2933	2.3439	1.8892	1.5843
0.5	6.3608	4.5192	3.2191	2.2957	1.6263	1.3124	1.1106
1	4.4721	3.1623	2.2361	1.5811	1.1180	0.9129	0.7906
2	3.1264	2.1901	1.5284	1.0713	0.7727	0.6526	0.5841

5	1.9106	1.3037	0.8890	0.6460	0.5117	0.4535	0.4184

From these tables, it is observable that, for a fixed value of σ and increasing value of τ, the IOD increases and the COV decreases, whereas, for a fixed value of τ and increasing value of σ, the IOD and COV both decrease. Rather than this, we can clearly see that the value of IOD is greater than or less than 1, indicating that the proposed distribution is appropriate for modelling over-dispersed and under-dispersed data.

3.4 Order statistics

Order statistics have several applications in reliability engineering and life testing. Let $Y_1, Y_2, ..., Y_n$ be a random sample drawn from ExDLi distribution. Also, let $Y_{(1)} \le Y_{(2)} \le ... \le Y_{(n)}$, denote the corresponding order statistics. Then, the CDF of the r^{th} order statistic, say, $Z = Y_{(r)}$, is given by

$$\begin{aligned} G_r(z,\tau,\sigma) &= \sum_{i=r}^{n} \binom{n}{i} G^i(z,\tau,\sigma)\left[1-G(z,\tau,\sigma)\right]^{n-i} \\ &= \sum_{i=r}^{n}\sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k}\left[1-w(z+1,\tau)\right]^{\sigma(i+k)}; z = 0,1,2,..., \end{aligned} \tag{3.8}$$

The corresponding PMF of r^{th} order statistic is

$$\begin{aligned} g_r(z,\tau,\sigma) &= G_r(z,\tau,\sigma) - G_r(z-1,\tau,\sigma) \\ &= \sum_{i=r}^{n}\sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k}\left[\left(1-w(z+1,\tau)\right)^{\sigma(i+k)} - \left(1-w(z,\tau)\right)^{\sigma(i+k)}\right]; z = 0,1,2,.... \end{aligned} \tag{3.9}$$

Particularly, by putting $r = 1$ and $r = n$ in Equation (3.9), we can obtain the PMF of the minimum order statistic, and the PMF of the maximum order statistic, respectively.

4. ESTIMATION OF PARAMETERS

In this section, two estimation methods are utilized to estimate the ExDLi distribution's unknown parameters, namely the ML estimation (MLE) and Bayesian methods.

4.1 Maximum Likelihood Estimation

In this section, we estimate the unknown parameter of the ExDLi distribution using the MLE method. Let $\underset{\sim}{X} = (x_1, x_2, ..., x_n)$ be a random sample of size *n* drawn from the ExDLi distribution. Then, the likelihood function is given by

$$L(\underset{\sim}{X} \mid \tau,\sigma) = \prod_{i=1}^{n}\left[\left(1-\tau^{(x_i+1)}(x_i+2)+\tau^{(x_i+2)}(x_i+1)\right)^{\sigma} - \left(1-\tau^{x_i}(x_i+1)+\tau^{(x_i+1)}x_i\right)^{\sigma}\right]. \tag{4.1}$$

The corresponding log-likelihood (LL) function is given by

$$\log L(\underset{\sim}{X} \mid \tau,\sigma) = \sum_{i=1}^{n} \log\left\{\left(1-\tau^{(x_i+1)}(x_i+2)+\tau^{(x_i+2)}(x_i+1)\right)^{\sigma} - \left(1-\tau^{x_i}(x_i+1)+\tau^{(x_i+1)}x_i\right)^{\sigma}\right\}. \tag{4.2}$$

By maximizing this function with respect to of τ and σ, we get the corresponding ML estimates. These estimates are also the solutions of the following non-linear equations:
Let us set

$$\frac{\partial \log L(\underset{\sim}{X} \mid \tau,\sigma)}{\partial \tau} = 0 \tag{4.3}$$

$$\frac{\partial \log L(\underset{\sim}{X} \mid \tau,\sigma)}{\partial \sigma} = 0 \tag{4.4}$$

Unfortunately, these equations cannot yield analytical solutions. Therefore, we use an iterative approach such as Newton-Raphson (NR) to calculate the estimates computationally.

4.2 Bayesian Estimation

In this section, we compute estimates of the parameters of the ExDLi distribution in a Bayesian setup. This method enables the investigator to combine prior beliefs (knowledge) about variations in parameters available in the form of prior densities with the sample information at hand. Choosing priors for the unknown model parameters is an important and difficult problem. There is no clear methodology to choose the best priors in such a setting. Here, we assume the independent prior densities as $\tau \sim Beta_I\left(a_1, b_1\right)$ and $\sigma \sim Gamma\left(a_2, b_2\right)$ with the following respective PDFs:

$$f_1\left(\tau\right) = \frac{1}{B_I\left(a_1, b_1\right)} \tau^{a_1 - 1}\left(1-\tau\right)^{b_1 - 1}; 0 < \tau < 1, \left(a_1, b_1\right) > 0, \tag{4.5}$$

$$f_2\left(\sigma\right) = \frac{a_2^{b_2}}{\Gamma\left(b_2\right)} \sigma^{b_2 - 1} e^{-a_2 \sigma}; \sigma > 0, \left(a_2, b_2\right) > 0, \tag{4.6}$$

where $B_I\left(a_1, b_1\right)$ in Equation (4.5) denotes the standard beta integral function and $\Gamma\left(b_2\right)$ in Equation (4.6) represents the standard gamma function.

These priors become non-informative if we set $a_1 = b_1 = a_2 = b_2 = 0$. Given the likelihood function in Equation (4.1) and the prior distributions in Equations (4.5) and (4.6), the unnormalized joint posterior distribution of $\left(\tau, \sigma\right)$ given data is

$$\pi\left(\tau, \sigma \mid \underset{\sim}{X}\right) \propto f_1\left(\tau\right).f_2\left(\sigma\right).L(\underset{\sim}{X} \mid \tau, \sigma). \tag{4.7}$$

To draw Bayesian inferences on the parameters, one needs the marginal posterior distribution of each parameter, which cannot be developed in this case. To overcome this difficulty, we use the well-known Markov Chain Monte Carlo (MCMC) technique and the Metropolis-Hastings (M-H) algorithm (Metropolis and Ulam, 1949; Hasting, 1970) within a Gibbs sampler, which generates parametric draws from the unnormalized posterior distribution of each of the parameters using the current value of the given parameters. After removing the sufficient burn-in-sample and checking the draws' convergence to their target distributions, we use these parametric draws to find posterior sample-based Bayes estimates of unknown parameters under different loss functions. The unnormalized marginal posterior distributions of τ and σ are as follows:

$$\pi_1\left(\tau \mid \underset{\sim}{X}, \sigma\right) \propto L(\underset{\sim}{X} \mid \tau, \sigma).\tau^{a_1 - 1}\left(1-\tau\right)^{b_1 - 1}, \tag{4.8}$$

$$\pi_2\left(\sigma \mid \underset{\sim}{X}, \tau\right) \propto L(\underset{\sim}{X} \mid \tau, \sigma).\sigma^{b_2 - 1} e^{-a_2 \sigma}. \tag{4.9}$$

5. INTERVAL ESTIMATION

In this section, two types of confidence intervals for parameters, namely asymptotic confidence intervals, and highest posterior density (HPD) intervals, are constructed.

5.1 Asymptotic Confidence Intervals

Here, we develop asymptotic confidence intervals (ACI) for the model parameters $\Theta \equiv \left(\tau, \sigma\right)$ based on large sample theory as the exact sampling distribution of the MLEs cannot be obtained explicitly. By using the general theory of MLEs, the asymptotic distribution of $\left(\Theta - \hat{\Theta}\right)$ is $N_2\left(0, I^{-1}(\Theta)\right)$, where $I\left(\Theta\right)$ is the Fisher's information matrix which can be approximated as

$$I(\hat{\Theta}) \approx -\begin{pmatrix} \dfrac{\partial^2 \log L}{\partial \tau^2} & \dfrac{\partial^2 \log L}{\partial \tau \partial \sigma} \\ \dfrac{\partial^2 \log L}{\partial \sigma \partial \tau} & \dfrac{\partial^2 \log L}{\partial \sigma^2} \end{pmatrix}_{\Theta = \hat{\Theta}}$$

Thus, for a large sample, the $100\times(1-\gamma)\%$ ACI for $\Theta_i; i=1,2$, where $\Theta_1=\tau$ and $\Theta_2=\sigma$ is given by

$$\left(\hat{\Theta}_i - Z_{\gamma/2}\sqrt{Var(\hat{\Theta}_i)}, \hat{\Theta}_i + Z_{\gamma/2}\sqrt{Var(\hat{\Theta}_i)}\right),$$

where $Var(\hat{\Theta}_i)$ is the $(i,i)^{th}$ diagonal element of $I^{-1}(\hat{\Theta})$ and $Z_{\gamma/2}$ is the $100(1-\gamma/2)$ % percentile of a standard normal variate.

5.2 Highest Posterior Density Intervals

Let $\Theta_{N+1} < \Theta_{N+2} < \Theta_{N+3} ... \Theta_M$ be the ordered values of a generated sequence $\Theta_{N+1}, \Theta_{N+2}, \Theta_{N+3} ... \Theta_M$ of Θ from the M-H within the Gibbs algorithm. Then, using the method proposed by Chen and Shao (1999), the $(1-\gamma)\times 100$ % HPD intervals for Θ is given by $\left(\Theta_{N+i^*}, \Theta_{N+i^*+[(1-\gamma)(M-N)]}\right)$, where i^* is chosen so that

$$\left(\Theta_{N+i^*}, \Theta_{N+i^*+[(1-\gamma)(M-N)]}\right) - \Theta_{N+i^*} = \underset{N\le i\le (M-N)-[(1-\gamma)(M-N)]}{min}\left(\left(\Theta_{N+i} + \Theta_{N+i+[(1-\gamma)(M-N)]}\right) - \Theta_{N+i}\right).$$

6. NUMERICAL ILLUSTRATION USING SIMULATED DATA

This section deals with an extensive simulation study to investigate the performance of the classical and Bayesian estimation methods. For this purpose, we generate samples of different sizes, i.e. *n*=25, 50, 100, and 200 from the ExDLi distribution with four sets of parametric values as (0.4, 1.5), (0.4, 3), (0.8, 1.5), and (0.8, 3). All these combinations of true parameters and sample size are used to obtain classical as well as Bayesian estimates of the unknown model parameters. In frequentist point estimation, we obtain the MLEs along with their standard errors (SEs) through the NR method, and these values are reported in Table 7. Under classical interval estimation, we find the 95% ACIs for the unknown model parameters and the same is summarized in Table 7.

In a Bayesian paradigm, under the squared error loss function (SELF), we compute the Bayes estimates with non-informative priors (NIPs) as well as informative priors (IP). For the Bayesian estimation with IPs, the prior PDFs for the parameters τ and σ are taken to be as $Beta_I(a_1,b_1)$ and $Gamma(a_2,b_2)$, respectively. The hyper-parameters in these prior PDFs are chosen in such a way that the mean of the priors' PDF is approximately equal to the corresponding assumed value of the unknown parameter. On the contrary, all these hyper-parameters are equated to zero to obtain Bayes estimates with NIPs. To compute these aforesaid Bayesian estimates under such a setup, we generate 11,000 realizations of the MCMC of τ and σ from their conditional posterior distributions given in Equations (4.8) and (4.9), respectively, using the M-H algorithm within the Gibbs sampler. The initial 1000 burn-in values for each of the chains are discarded to remove the effects of the starting values of the parameters. Also, we store every 10th observation so that the autocorrelation of successive draws would diminish. The convergence diagnostics are performed through Geweke's (Geweke, 1991) criterion, by selecting a 95% credibility level.

After the successful convergence of each of the generated chains, we use these simulated posterior samples to obtain Bayes estimates with their associated posterior standard error (PSE) under the SELF, and they are listed in Tables 8 and 9. Using these generated MCMC, we also compute the HPD credible intervals for the unknown model parameters of the proposed distribution, and the same is presented in Tables 8 and 9. All the above numerical computations are performed using the open-source R programming language. The following fruitful conclusions can be drawn from the above simulation study:

- We notice that as the sample size increases, the SE as well as the PSE of all the estimates tend to decrease. The same trend is observed in the case of the lengths of various confidence intervals.
- Overall, Bayes estimates with IP outperform Bayes estimates with NIP and MLEs in terms of estimation errors.
- The MLEs outperform Bayes estimates under NIPs because they have smaller estimation errors in the comparison of Bayes estimates with NIPs.

- In the case of interval estimation of unknown model parameters, the HPD credible interval with IP comes out to be a better interval estimate than the HPD credible interval under NIP and ACI in terms of the length of the CIs.
- For both estimation procedures, the estimation of σ is more sensitive than the estimation of τ, because it produces a larger estimation error as compared to the other parameter.

Table 7. Simulation results for the MLEs for varying values of τ and σ.

(τ,σ)	n	Parameter	Estimate	SE	95% ACIs		
					Lower	Upper	Width
(0.4, 1.5)	25	τ	0.4007	0.077	0.275	0.5839	0.3089
		σ	1.6575	0.6515	0.7672	3.581	2.8138
	50	τ	0.4332	0.0551	0.3376	0.5558	0.2182
		σ	1.4472	0.3946	0.8481	2.4695	1.6214
	100	τ	0.4325	0.0404	0.3602	0.5193	0.1591
		σ	1.3189	0.2596	0.8967	1.9398	1.0431
	200	τ	0.4052	0.0284	0.3531	0.4649	0.1118
		σ	1.4611	0.2085	1.1047	1.9325	0.8278
(0.4, 3)	25	τ	0.3819	0.065	0.2736	0.533	0.2594
		σ	4.1974	1.8385	1.7789	9.9039	8.125
	50	τ	0.429	0.0464	0.347	0.5303	0.1833
		σ	2.8846	0.8077	1.6663	4.9936	3.3273
	100	τ	0.4218	0.0331	0.3616	0.492	0.1304
		σ	2.8677	0.5713	1.9407	4.2375	2.2968
	200	τ	0.4067	0.0235	0.3632	0.4554	0.0922
		σ	2.9135	0.4132	2.2064	3.8471	1.6407
(0.8,1.5)	25	τ	0.7839	0.035	0.7183	0.8555	0.1372
		σ	2.2867	0.8211	1.1313	4.6221	3.4908
	50	τ	0.8231	0.022	0.7812	0.8673	0.0861
		σ	1.4108	0.3187	0.906	2.1967	1.2907
	100	τ	0.817	0.0161	0.7861	0.8492	0.0631
		σ	1.4028	0.2252	1.0242	1.9214	0.8972
	200	τ	0.8071	0.0117	0.7846	0.8303	0.0457
		σ	1.4943	0.1688	1.1974	1.8647	0.6673
(0.8, 3)	25	τ	0.7837	0.0335	0.7206	0.8523	0.1317
		σ	4.7433	2.1371	1.9615	11.4705	9.509
	50	τ	0.8214	0.0198	0.7835	0.8611	0.0776
		σ	2.6406	0.672	1.6035	4.3485	2.745
	100	τ	0.8165	0.0141	0.7893	0.8447	0.0554
		σ	2.6686	0.4712	1.888	3.772	1.884
	200	τ	0.8083	0.0105	0.788	0.8292	0.0412
		σ	2.8643	0.376	2.2145	3.7048	1.4903

Table 8. Simulation results for the Bayes estimates using non-informative priors for varying values of τ and σ.

(τ,σ)	n	Parameter	Estimate	SE	95% HPD Intervals		
					Lower	Upper	Width
(0.4, 1.5)	25	τ	0.4091	0.0780	0.2691	0.5725	0.3033
		σ	1.7074	0.6963	0.5942	3.0725	2.4783
	50	τ	0.4365	0.0557	0.3294	0.5447	0.2152
		σ	1.4726	0.4106	0.7379	2.2586	1.5207
	100	τ	0.4337	0.0406	0.3580	0.5150	0.1570
		σ	1.3313	0.2656	0.8683	1.8799	1.0116
	200	τ	0.4062	0.0286	0.3514	0.4618	0.1104

		σ	1.4667	0.2128	1.0747	1.8862	0.8115
(0.4, 3)	25	τ	0.3917	0.0636	0.2681	0.5128	0.2447
		σ	4.2856	1.9237	1.3238	8.0384	6.7146
	50	τ	0.4329	0.0462	0.3451	0.5245	0.1794
		σ	2.9169	0.8286	1.5145	4.5797	3.0652
	100	τ	0.4234	0.0330	0.3585	0.4861	0.1276
		σ	2.8882	0.5828	1.8270	4.0737	2.2467
	200	τ	0.4076	0.0236	0.3623	0.4533	0.0910
		σ	2.9207	0.4188	2.1784	3.7859	1.6075
(0.8, 1.5)	25	τ	0.7850	0.0350	0.7179	0.8535	0.1357
		σ	2.3680	0.8772	0.8626	4.0314	3.1688
	50	τ	0.8240	0.0219	0.7839	0.8704	0.0865
		σ	1.4150	0.3254	0.8083	2.0384	1.2300
	100	τ	0.8171	0.0160	0.7853	0.8478	0.0625
		σ	1.4092	0.2248	0.9974	1.8527	0.8553
	200	τ	0.8073	0.0116	0.7856	0.8310	0.0454
		σ	1.4940	0.1702	1.1803	1.8396	0.6593
(0.8, 3)	25	τ	0.7878	0.0315	0.7237	0.8471	0.1235
		σ	4.7824	2.0602	1.5832	8.8996	7.3164
	50	τ	0.8230	0.0198	0.7826	0.8604	0.0778
		σ	2.6280	0.6836	1.4390	4.0055	2.5665
	100	τ	0.8168	0.0143	0.7884	0.8440	0.0556
		σ	2.6800	0.4872	1.8005	3.6720	1.8715
	200	τ	0.8086	0.0103	0.7881	0.8282	0.0401
		σ	2.8628	0.3687	2.1497	3.5650	1.4153

Table 9. Simulations results for the Bayes estimates using informative priors for varying values of τ and σ .

(τ,σ)	n	Parameter	Estimate	SE	95% HPD Intervals		
					Lower	Upper	Width
(0.4, 1.5)	25	τ	0.4097	0.0414	0.3347	0.4945	0.1597
		σ	1.5366	0.2404	1.0957	2.0278	0.9321
	50	τ	0.4229	0.0347	0.3612	0.4979	0.1367
		σ	1.5074	0.2192	1.0793	1.9226	0.8433
	100	τ	0.4180	0.0292	0.3621	0.4750	0.1129
		σ	1.4280	0.1879	1.0626	1.7773	0.7147
	200	τ	0.4036	0.0229	0.3592	0.4481	0.0889
		σ	1.4801	0.1628	1.1752	1.8026	0.6274
(0.4, 3)	25	τ	0.4180	0.0334	0.3575	0.4869	0.1294
		σ	3.0960	0.3701	2.3717	3.8311	1.4593
	50	τ	0.4211	0.0270	0.3699	0.4750	0.1051
		σ	2.9979	0.3449	2.3564	3.6724	1.3160
	100	τ	0.4159	0.0220	0.3734	0.4594	0.0860
		σ	2.9767	0.3198	2.3664	3.5994	1.2329

	200	τ	0.4043	0.0176	0.3702	0.4390	0.0687
		σ	2.9681	0.2768	2.4432	3.5160	1.0728
(0.8, 1.5)	25	τ	0.8096	0.0214	0.7686	0.8519	0.0833
		σ	1.6282	0.2527	1.1423	2.1166	0.9742
	50	τ	0.8190	0.0168	0.7859	0.8519	0.0660
		σ	1.4710	0.2071	1.0691	1.8602	0.7912
	100	τ	0.8142	0.0134	0.7883	0.8405	0.0522
		σ	1.4489	0.1725	1.1152	1.7882	0.6729
	200	τ	0.8069	0.0104	0.7876	0.8284	0.0407
		σ	1.4977	0.1431	1.2307	1.7801	0.5494
(0.8, 3)	25	τ	0.8103	0.0164	0.7789	0.8430	0.6410
		σ	3.1126	0.3731	2.3773	3.8293	1.4520
	50	τ	0.8142	0.0130	0.7910	0.8415	0.0506
		σ	2.9266	0.3367	2.2871	3.5869	1.2999
	100	τ	0.8111	0.0105	0.7897	0.8309	0.0412
		σ	2.8842	0.3034	2.3123	3.4820	1.1697
	200	τ	0.8068	0.0084	0.7907	0.8234	0.0328
		σ	2.9318	0.2699	2.3983	3.4533	1.0551

7. REAL DATA ANALYSIS

7.1 Statistical analysis of the real datasets

In this section, we analyze four real-world datasets to demonstrate the applicability of the ExDLi model. Several standard criteria are used to compare fitted models, including the negative log-likelihood (-LL), the Akaike information criterion (AIC), the corrected AIC (CAIC), the Bayesian information criterion (BIC), the Hannan Quinn information criterion (HQIC), and the Chi-square ($\chi 2$) statistic with its associated P-value. The descriptive summaries of the datasets are shown in Table 3. From this table, we can see that the IOD of datasets I-II is greater than 1 whereas the IOD of datasets III-IV is less than 1, indicating that the considered datasets I-II can only be modelled by discrete distributions with over-dispersion phenomena, and datasets III-IV can only be modelled by discrete distributions with under-dispersion phenomena. Table 11 lists the competitive models for the ExDLi model.

Table 10: Descriptive statistics of the Datasets.

Data	*n*	Mean	Variance	Skewness	Kurtosis	IOD	COV
Dataset I	100	0.6700	1.1526	1.5715	4.5320	1.7203	1.6024
Dataset II	110	1.3909	6.1118	2.2612	7.8066	4.3941	1.7774
Dataset III	156	0.9936	0.7419	0.8023	3.5038	0.7467	0.8669
Dataset IV	40	6.3250	3.6096	-0.5893	2.6273	0.5707	0.3004

Table 11: The competitive models of the ExDLi model.

Distribution	**Abbreviation**	**Author(s)**
Geometric	Geo	-
Discrete Lindley	DLi	Gómez-Déniz and Calderín-Ojeda (2011)
Discrete Lindley-Two Parameter	DLi-II	Hussain et al. (2016)
Discrete Pareto	DPa	Krishna and Pundir (2009)
Discrete Linear Failure Rate	DLFR	Kumar et al. (2017)

Discrete Inverse Weibull	DIW	Jazi et al. (2010)
Discrete Log-Logistic	DLog-L	Para and Jan (2016b)
Discrete Burr-XII	DB-XII	Para and Jan (2016a)
Discrete Lomax	DLo	Para and Jan (2016a)
Discrete Burr-Hatke	DBH	El-Morshedy et al. (2020)
Generalized Geometric	GGeo	Gómez-Déniz (2010)
Binomial	Binomial	-
Discrete Generalized Exponentiated Type II	DGE2	Nekoukhou et al. (2013)
Type I Discrete Quasi Xgamma	DQX1	Mazucheli et al. (2020)
Type II Discrete Quasi Xgamma	DQX2	Mazucheli et al. (2020)
Discrete Weibull	DW	Khan et al. (1989)
Discrete Generalized Odd Lindley–Weibull	DGOL-W	Aryuyuen et al. (2020)
Discrete Generalized Rayleigh	DGR	Alamatsaz et al. (2016)
Discrete Burr	DB	Krishna and Pundir (2009)
Poisson Xgamma	PX	Para and Jan (2020)
The New Discrete Lindley	TNDL	Al-Babtain et al. (2020)
Natural Discrete Lindley	NDL	Al-Babtain et al. (2020)
Discrete Poisson-Lindley	DPL	Sankaran (1970)

Dataset I: The first dataset consists of the recordings of the total number of carious teeth among the four deciduous molars in a sample of 100 children between 10 and 11 years old (Krishna and Pundir, 2009). Under these data, the fitting summary based on the MLEs for proposed and rival models is provided in Table 12.

Table 12: The MLEs and goodness of fit statistics for different models under dataset I.

X	O.Fr.	E.Fr.							
		ExDLi	Geo	DLi	DLi-II	DPa	DLFR	DIW	DLogL
0	64	63.57	59.88	57.13	59.88	69.04	59.9	63.3	62.73
1	17	19.73	24.02	26.88	24.02	15.37	24.01	22.48	22.42
2	10	9.13	9.64	10.45	9.64	6.01	9.63	6.44	7.01
3	6	4.2	3.87	3.71	3.87	3.01	3.86	2.76	2.98
>=4	3	3.36	2.59	1.83	2.59	6.57	2.6	5.02	4.86
Total	100	100	100	100	100	100	100	100	100
MLE		0.373	0.401	0.274	0.401	0.184	0.401	0.633	0.745
		0.485	-	-	0.001	-	1	1.576	1.768
− LL		111.425	112.474	113.68	112.475	116.83	112.47	116.275	115.47
AIC		226.85	226.947	229.36	228.95	235.66	228.94	236.55	234.94
CAIC		226.974	226.988	229.39	229.073	235.7	229.063	236.673	235.063
BIC		232.061	229.552	232.96	234.16	238.27	234.15	241.76	240.15
HQIC		228.959	228.001	230.41	231.058	236.72	231.048	238.658	237.048
$\chi 2$		0.737	3.347	6.638	3.347	3.225	3.34	3.503	2.783
DF		1	2	2	1	2	1	1	1
P-value		0.692	0.188	0.036	0.067	0.199	0.068	0.061	0.095

***O.Fr.: Observed frequency and E.Fr.:Expected frequency

Dataset II: The second dataset is the number of deaths due to horse kicks in the Prussian army between 1875 and 1894 (Klugman et al., 2012). The fitting summary for these data using MLEs is recorded in Table 13.

Table 13: The MLEs and goodness of fit statistics for different models under dataset II**.**

X	O.Fr.	E.Fr.									
		ExDLi	DLi	DLi-II	Geo	DW	DB-XII	DLo	DBH	GGeo	DLogL
0	65	64.96	40.25	46.03	45.98	63.64	63.32	61.89	61.94	62.79	63.19
1	14	14.45	29.83	26.77	26.76	17.45	18.19	21.01	20.06	19.66	20.1
2	10	9.02	18.36	15.57	15.57	9.3	9.29	9.65	9.65	9.43	8.64
3	6	6.13	10.35	9.05	9.06	5.68	5.49	5.24	5.52	5.43	4.66
4	4	4.31	5.53	5.27	5.28	3.73	3.52	3.17	3.49	3.46	2.86
5	2	3.08	2.86	3.06	3.07	2.56	2.39	2.06	2.34	2.35	1.92
6	2	2.23	1.44	1.78	1.79	1.82	1.69	1.42	1.65	1.66	1.37
7	2	1.61	0.71	1.04	1.04	1.32	1.23	1.02	1.19	1.21	1.02
8	1	1.17	0.35	0.6	0.61	0.98	0.92	0.76	0.89	0.9	0.79
9	1	0.85	0.17	0.35	0.35	0.74	0.7	0.58	0.67	0.68	0.62
10	1	0.62	0.08	0.2	0.21	0.57	0.55	0.46	0.52	0.52	0.5
11	2	1.57	0.07	0.28	0.28	2.21	2.71	2.74	2.08	1.91	4.33
Total	110	110	110	110	110	110	110	110	110	110	110
-LL		166.95	189.10	178.80	178.800	167.90	168.80	170.50	168.80	168.56	171.72
MLE		0.673	0.436	0.581	0.582	0.421	0.003	0.150	0.874	0.80	0.780
		0.236	-	0.001	-	0.629	12.75	1.390	-	0.188	1.208
		-	-	-	-	-	0.720	-	-	-	-
AIC		337.90	380.20	361.50	359.50	339.90	343.50	344.90	339.79	341.11	347.43
CAIC		338.01	380.30	361.60	359.60	340.10	343.80	345.10	339.83	341.23	347.55
BIC		343.30	382.90	366.90	362.20	345.40	351.60	350.40	342.49	346.51	352.835
HQIC		340.09	381.30	363.70	360.60	342.20	346.80	347.20	340.89	343.30	349.63
$\chi 2$		0.146	43.480	22.890	22.840	1.040	2.469	3.316	2.613	2.461	2.830
DF		3	4	3	4	3	3	3	4	3	3
P-value		0.986	<0.01	<0.01	<0.01	0.792	0.48	0.345	0.625	0.482	0.419

Dataset III: The third dataset (Ridout and Besbeas, 2004) contains the number of strikes in the UK coal mining industries (156 observations) over four consecutive week periods from 1948 to 1959. The values of fitting measures with MLEs for the ExDLi model and other rival models are given in Table 14.

Table 14: The MLEs and goodness of fit statistics for different models under dataset III**.**

X	O.Fr.	E.Fr.								
		ExDLi	Binomial	DGE2	DQX1	DQX2	DW	DGOL-W	DGR	DB
0	46	46.19	46.78	46.08	46.0551	46.028	48.50	46.7	47.50	47.89
1	76	74.82	65.81	75.75	74.0515	74.9294	68.70	73.27	69.32	80.87
2	24	26.90	32.62	26.11	28.2635	26.9609	31.10	28.06	31.82	18.28
3	9	6.42	7.19	6.50	6.3074	6.5022	6.84	6.65	6.67	6.07
4	1	1.67	3.60	1.56	1.3225	1.5795	0.86	1.32	0.69	2.89
Total	156	156	156	156	156	156	156	156	156	156
-LL		187.4863	188.9570	187.5340	187.56	187.395	188.1832	187.3904	188.3290	192.2100
MLE		0.16447	4.0000	4.7994	0.1758	-0.1097	1.9017	55.3489	0.9415	4.6543

	3.38659	0.2484	0.2247	2.3016	2.1335	0.3109	5.5369	0.7172	0.5941
	-	-	-	-	-	-	3.0603	-	-
	-	-	-	-	-	-	0.2562	-	-
AIC	378.9727	381.9140	379.0680	379.12	378.79	380.3664	382.7808	380.6580	388.4200
CAIC	379.0511	381.9924	379.1464	379.1984	378.8684	380.4448	394.9802	380.7364	388.4984
BIC	385.0724	388.0137	385.1677	385.22	384.89	386.4661	383.0457	386.7577	394.5197
HQIC	381.4501	384.3914	381.5454	381.5974	381.2674	382.8438	387.7357	383.1354	390.8974
$\chi 2$	0.7832	6.2021	1.3347	1.92	1.51	3.2495	1.2167	3.5623	4.8160
DF	1	2	1	2	2	3	1	1	1
P-value	0.3762	0.1022	0.8555	0.383	0.47	0.3547	0.2700	0.4685	0.3067

Dataset IV: This dataset represents 40 observations of time-to-failure (103h) of the turbocharger of one type of engine (Xu et al., 2003). Table 15 shows the values of fitting measures using MLEs for the ExDLi and other competing models.

Table 15: The MLEs and goodness of fit statistics for different models under dataset IV**.**

X	O.Fr.	E.Fr.						
		ExDLi	Binomial	DGE2	PX	TNDL	NDL	DPL
0	0	0.00	0.01	0.00	3.88	2.31	3.26	3.39
1	0	0.02	0.08	0.01	3.71	3.51	3.80	3.82
2	2	0.49	0.46	0.46	3.67	4.00	3.93	3.89
3	2	2.52	1.68	2.62	3.59	4.05	3.82	3.76
4	2	5.54	4.08	5.72	3.43	3.85	3.56	3.49
5	7	7.35	7.07	7.37	3.20	3.51	3.23	3.17
6	6	7.16	8.94	7.00	2.91	3.11	2.86	2.81
7	8	5.76	8.30	5.57	2.59	2.70	2.50	2.47
8	9	4.12	5.61	3.99	2.26	2.31	2.16	2.14
9	4	7.04	3.77	7.26	10.76	10.67	10.89	11.07
Total	40	40.00	40.00	40.00	40.00	40.00	40.00	40.00
-LL		87.172	-81.9223	87.890	107.762	105.574	108.369	108.939
MLE		0.54280	12.1662	0.608	0.3962	2.402e-01	0.22319	0.27906
		9.15391	0.5199	17.317	-	4.358e+05	-	-
AIC		178.344	167.8447	179.780	217.524	215.149	218.738	219.879
CAIC		178.668	171.2225	180.110	217.630	215.473	218.843	219.984
BIC		181.722	168.1690	183.160	219.213	218.527	220.427	221.568
HQIC		179.565	169.0660	181.010	218.135	216.370	219.349	220.490
$\chi 2$		2.149	2.3859	2.380	53.055	19.780	23.485	24.173
DF		2	2	2	3	2	3	3
P-value		0.341	0.3033	0.304	0.000	0.000	0.000	0.000

It is clear from these examples of real datasets that, in contrast to the distributions shown in Table 11, the suggested model not only provides high P-values but also the lowest AIC, CAIC, BIC, HQIC, and $\chi 2$ values. As a result, it demonstrates that, compared to the considered rival distributions, the suggested model suffers from the least amount of information loss.

7.2 Bayesian analysis

In this sub-section, we perform a Bayesian analysis for the considered datasets. In this estimation method, as we have no prior information regarding the unknown parameters, we use non-informative priors for the unknown parameters. Using the same procedure as we did in the simulations (see Section 6), we calculate the Bayes estimates with PSE, along with their standard errors (SEs). These results are summarized in Table 16. We also report the 95% ACI and HPD intervals for the unknown parameters in Table 17. From these tables (Tables 16 and 17), we obtain similar conclusions as we have observed in Section 6.

Table 16: The classical and Bayes estimates with their SEs for all the datasets under ExDLi distribution.

Dataset No.	MLE and SE				Bayes Estimate and PSE			
	τ		σ		τ		σ	
	MLE	SE	MLE	SE	Bayes	PSE	Bayes	PSE
Dataset I	0.3730	0.0650	0.4850	0.1420	0.3764	0.06561	0.4994	0.1496
Dataset II	0.6732	0.0498	0.2355	0.0499	0.6752	0.0488	0.2362	0.0510
Dataset III	0.1645	0.0263	3.3866	0.7506	0.1666	0.0259	3.4297	0.7522
Dataset IV	0.5428	0.0376	9.1539	3.2568	0.5534	0.0369	8.6245	2.9981

Table 17: The classical and HPD confidence intervals with their width for all the datasets under the ExDLi distribution.

Dataset No.	ACI and Width				HPD interval and Width			
	τ		σ		τ		σ	
	ACI	Width	ACI	Width	HPD Interval	Width	HPD Interval	Width
Dataset I	[0.2648, 0.526]	0.2612	[0.2735, 0.8594]	0.5859	[0.2630, 0.5130]	0.2499	[0.2360, 0.7933]	0.5572
Dataset II	[0.5836, 0.7766]	0.1930	[0.1555, 0.3567]	0.2012	[0.5763, 0.7689]	0.1925	[0.1438, 0.3381]	0.1943
Dataset III	[0.1205, 0.2245]	0.1040	[2.2013, 5.2101]	3.0088	[0.1169, 0.2171]	0.1001	[2.0056, 4.8629]	2.8572
Dataset IV	[0.4739, 0.6217]	0.1478	[4.5624, 18.365]	13.8029	[0.4792, 0.6233]	0.1440	[3.6360, 14.5558]	10.9198

8. CONCLUSIONS

In this article, a new discrete distribution named the exponentiated discrete Lindley distribution was elaborated and studied in depth. We have observed that the proposed distribution has great flexibility in terms of fitting, as it is capable of modelling equi-, over-, under-dispersed, positively skewed, and leptokurtic data. In addition, the ExDLi distribution can be used quite effectively for modelling a wide variety of failure data because its hazard rate function can take diverse shapes such as increasing, decreasing, increasing–decreasing–increasing, unimodal, bathtub, and J-shaped. In this study, several important statistical properties of the ExDLi distribution were discussed and numerically examined. In both classical and non-classical setups, the method of maximum likelihood and the Bayesian approach were utilized to estimate the unknown parameters of the proposed model. An extensive Monte Carlo simulation analysis was conducted to evaluate the behavior of the above-stated estimation methods. The results of simulation studies show that these two estimation procedures perform quite satisfactorily in estimating unknown parameters of the model. The flexibility of the ExDLi distribution was also exemplified by using four distinctive and well-referenced real datasets, showing that it outperforms valuable competitors. Hence, it is reasonable to apply it for the modelling count or failure data in various fields, including reliability, insurance, medicine, economics, demography, etc. Possible extensions of this work include bivariate count modelling, mixture distributions involving the ExDLi distribution, and other count regression models. These perspectives need more investigation, which we will consider in future work.

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