DYNAMIC PRICING AND GOODWILL INVENTORY MODEL FOR DETERIORATING ITEMS

Nita H. Shah[1], Kavita Rabari and Ekta Patel
Department of Mathematics, Gujarat University, Ahmedabad-380009, Gujarat, India

ABSTRACT
In this model, a dynamic goodwill problem for perishable objects with advertisement effort is studied. The spoilage of perishable products that takes place over time causes not only quantitative loss but results in an economic loss that occurs due to a reduction in consumption rate. Together with deterioration, selling price and advertisement of the products also play a vital role in the profit of a firm and customer's consumptions rate. It is more difficult for a company to solve a problem that includes different dynamic strategies. The goodwill effect is positively affected by the advertisements and decays due to forgetfulness. This paper studies dynamic pricing and advertisement efforts for deteriorating objects. The demand rate is a function of the selling price, goodwill effect, and market potential. A dynamic inventory model is developed to maximize the total profit function of the retailer for the decision variable i.e. cycle time and order quantity. The dynamic selling price, advertisement efforts, and goodwill effect are calculated through Pontryagin's maximum principle. A numerical example and sensitivity analysis are done related to the different inventory parameters. The model also provides some managerial implications for different decision variables.

KEYWORDS: Dynamic Optimization problem, Pontryagin's maximum principle, Hamiltonian function.

MSC:90B05

RESUMEN
En este modelo se estudia un problema de fondo de comercio dinámico para objetos perecederos con esfuerzo publicitario. El deterioro de los productos perecederos que se produce a lo largo del tiempo provoca no solo una pérdida cuantitativa sino que se traduce en una pérdida económica que se produce debido a una reducción en la tasa de consumo. Junto con el deterioro, el precio de venta y la publicidad de los productos también juegan un papel vital en el beneficio de una empresa y la tasa de consumo de los clientes. Es más difícil para una empresa resolver un problema que incluye diferentes estrategias dinámicas. El efecto de buena voluntad se ve afectado positivamente por los anuncios y decae por el olvido. Este artículo estudia los precios dinámicos y los esfuerzos publicitarios para objetos en deterioro. La tasa de demanda es una función del precio de venta, el efecto de buena voluntad y el potencial de mercado. Se desarrolla un modelo de inventario dinámico para maximizar la función de beneficio total del minorista para la variable de decisión, es decir, el tiempo del ciclo y la cantidad del pedido. El precio de venta dinámico, los esfuerzos publicitarios y el efecto de buena voluntad se calculan a través del principio máximo de Pontryagin. Se realiza un ejemplo numérico y un análisis de sensibilidad relacionado con los diferentes parámetros del inventario. El modelo también proporciona algunas implicaciones gerenciales para diferentes variables de decisión.

PALABRAS CLAVE: problema de Optimization Dinámica, priciipo del máximo Pontryagin, función Hamiltoniana f

1. INTRODUCTION

Deterioration is the process of spoilage, damage, expiration, and decay of products that reduces the quality and value of products with time. Hence, it is important to add this parameter for the development of a precise inventory model. Moreover, by suitable investment in reducing the damage rate, one can ignore unnecessary waste and consequently progress their business competitiveness. This will suggest a major prospect for development and productivity that can be optimized by various promotional tools such as advertisement. Both from the economic and working points, selling price is one of the major efficient parameters that a firm can manipulate for effective business. The demand rate is directly proportional to the selling price of the product. More will be the price, less will be the demand. For effective business, dynamic pricing is a good strategy to match supply with demand. Not amazingly, dynamic pricing policies have been accepted by automobile companies, airlines, hotels, etc. In addition to selling price, consumers are habitually tactful to the goodwill of products, and advertisement strategies increase their acceptability and influence products goodwill between customers. Customers are sensible agents who build choices depending on advertisements and different market scenarios. Since customers have memory, products may lose their goodwill due to forgetfulness. If the product is known to the customers it will help to boost the

[1] Corresponding author: [1]Prof. Nita H. Shah
nitahshah@gmail.com

demand. This ensures that the goodwill can influence the demand rate. Along with selling price and goodwill, calculating order quantity is also a challenge. Ordering a huge volume of products increases the degradation rate along with holding cost. Thus, the dynamic effects of product goodwill are considered in this model with demand rate as price and goodwill sensitive. This model answers, how much to order for a profitable business. The model is formulated for perishable objects where it is of importance to know how to handle price with time to maximize total profit function. The selling price and advertisement investments are controlled by the company, but it is not possible to stop the rate of degradation of products. The study includes all three effects and gives analytical results. The rest of this article is planned as follows: Section 2 briefly reviews the past literature related to the topic. Notations and assumptions are defined in section 3. Section 4 represents the mathematical model. Structural results of optimal functions are defined in section 5. To modify the model, numerical example together with sensitivity analysis under managerial insights is carried out in section 6. Section 7 concludes the article.

2. **LITERATURE REVIEW**

A different model for inventory management and pricing of deteriorating objects has been designed in topical years. The inventory management system for deteriorating objects has been introduced by Goyal and Giri in 2001. Chande et al.2005 developed an inventory model for deteriorating objects and pricing in a discrete period under RFID technology. Duong et al.2015 proposed suitable performance matrices to calculate the complete inventory system for perishable products. The study deals with inventory management of objects when they have a multi-period lifetime, positive lead time, and required customer service level. Rahimi et al.2017 developed a multi-objective mathematical model for inventory routing problems assuming financial criteria, consumer satisfaction level, and natural aspects for deteriorating objects with expiration details. Also, calculates the probability of using electrical vehicles and fuels in urban areas. One of the major factors that affect inventory is the product goodwill effect, positively influenced by advertisement. Aksen 2007 developed an inventory model for the incapacitated lot-sizing problem due to loss of customer goodwill. The purpose is to maximize the total profit where the demand, cost, and price change with time. Liu et al.2015 calculated the influence of administrated and negotiated transfer pricing on the retailer's profit function, based on a differential game considering marketing and operations with advertising-dependent goodwill. Nair and Narasimhan 2006 developed a dynamic inventory model for advertising- and quality-based goodwill. This paper includes the study of both dynamic advertising and pricing. Feng et al.2015 developed an inventory model for deteriorating objects under dynamic advertising and pricing problem. An inventory optimization problem is solved to maximize the total profit function using Pontryagin's maximum principle. Liu and Shankar 2015 developed an inventory model to represent the changes in brand preference due to the negative impact of product-harm crises and dynamics in advertising effectiveness with customers' response to objects. Erickson 2009 developed an oligopoly model. Also, represent the positive impact of advertisement on sale under the negative effect of discount rate and decay rate. Bass et al.2007 introduced a dynamic Bayesian model for advertisement-sales connection. Schlosser 2015 examined a particular case of dynamic advertising and pricing model for deteriorating objects with inventory holding costs. Lu et al.2016 developed a model with reference price effect on marketing strategies. The model investigated feasible selling prices and advertising efforts to maximize the total profit function. Dye and Yang 2016 introduced a preservation technology investment model for perishable objects and dynamic pricing with price and time-sensitive demand. A sensitivity analysis is performed to modify the model. Crettez et al.2018 suggested an advertising dynamic-optimization problem for goodwill dynamic. Yang et al.2010 developed a two-stage dynamic-pricing model with myopic and strategic consumers. Keskin and Zeevi 2014 developed an inventory model for unknown demand under a dynamic pricing strategy. Herbon et al.2014 introduced a model for deteriorating objects that uses RFID-supported time-temperature indicators to analyze the quality and age of deteriorating objects. Pal et al.2015 presented two layers retailer-manufacturer supply chain inventory model where the demand rate is sensitive to quality, price, and promotional effort. Avinadav et al.2017 developed a model, that calculates dynamic promotion expenditure and selling price related to deteriorating objects. Xie and Wei 2009 developed and compare different models: a cooperative game and non-cooperative with the optimal solution under the bargaining problem. Zhang et al. 2008 introduced a stochastic demand model with joint optimization on promotion like advertisements, inventory policy, and dynamic pricing. Modeling in this manner, reflect the effect of goodwill on consumer demand.

3. **NOTATIONS AND ASSUMPTIONS**

To formulate the model following notations and assumptions are being used:

3.1 Notations

C	Purchase cost per unit (dollars/unit)
A	Ordering cost per order (dollars/order)
h	Holding cost per unit (dollars/unit)
θ	Constant deterioration rate, $0 \le \theta \le 1$
α	Market potential
μ	Constant leading decay rate of goodwill
γ	Constant governing demand sensitivity to product goodwill
r	Constant governing advertising effort
$E(t)$	Advertising investment at a time t
$p(t)$	Selling price per unit at a time t
T	Cycle time (in years)
Q	Replenishment order quantity per cycle
$G(t)$	Goodwill effect at a time t
$R(p(t),G(t))$	Demand rate at a time t
$I(t)$	Inventory level at a time t in the interval $[0,T]$
TP	Total profit function per year(dollars/year)

3.2 Assumptions

- The system under consideration deals with a single item.
- Shortages are not allowed.
- The units in the inventory system deteriorate at a constant rate. Replacement of deteriorating items is not permissible.
- Initially, a company orders Q units of products at each cycle time T .
- The units are sold according to the age-dependent selling price $p(t)$.
- The study is related to the monopolistic market where a company sells a single kind of deteriorating object during the time horizon.
- When the deteriorating objects are stored in serviceable inventory, the damage takes place immediately.
- The demand function is given by
  $R(p(t),G(t)) = \alpha - p(t) + \gamma G(t)$
  where α represents market potential, $p(t)$ is the selling price, γ is the decay rate, and $G(t)$ is the goodwill effect

4. **MATHEMATICAL MODEL**

In this section, two types of dynamics are involved. Firstly, a goodwill effect is taken into consideration that is positively influenced by the advertisement effect. An advertisement is a promotional tool that helps to increase sales. The differential equation for the goodwill variable $G(t)$ at a time t is given by

$$\dot{G}(t) = E(t) - \mu G(t) \tag{1}$$

where $\mu > 0$ represents the decay rate and $E(t)$ is the advertisement investment rate. With the initial condition $G(0) = G_0$. The cost related to the advertisement is a quadratic function and is given by

$$AC(E(t)) = \frac{r}{2}E^2(t)\ , r > 0 \tag{2}$$

The demand rate for the objects depends not only on the goodwill effect but also on the selling price. Here, $R(t)$ denote the demand rate at the time t and $p(t)$ is the selling price for the time t . Hence, the differential equation for the demand rate is

$$R(t) = \alpha - p(t) + \gamma G(t) \tag{3}$$

where $\alpha > 0$ is the potential rate of the market, meaning the consumption rate of the products when selling price $p(t) = 0$. The market potential α is positively affected by the goodwill of the product with the goodwill coefficient $\gamma > 0$. The above equation represents the demand rate that decreases with respect to the selling price.

The company sells deteriorating objects over time $[0,T]$ with an initial inventory level Q. The inventory level $I(t)$ at a time t during time $[0,T]$ goes on decreasing due to the effect of demand and the rate of deterioration. Therefore, the second dynamic function is the inventory level $I(t)$ which is given by

$$\dot{I}(t) = -R(t) - \theta I(t) \tag{4}$$

With boundary conditions $I(0) = Q$ and $I(T) = 0$, where $\theta > 0$ is the constant rate of deterioration. It is clear from the above equation (4) that the inventory level is non-negative during the time interval $[0,T]$. Thus, no shortages take place during the planning horizon.

The holding cost is the linear function of the current inventory level and is given by,

Holding cost $(HC) = hI(t)$ (5)

Here, $h > 0$ represents the holding cost per unit.

The company's total profit is given by subtracting total sales revenue from different associated costs. The lists of costs are holding cost, ordering cost, purchase cost, and advertisement cost. Therefore, the total profit function per unit time is

$$TP = \frac{1}{T}\int_0^T (p(t)R(t) - HC - AC(E(t)))dt - \frac{CQ}{T} - \frac{A}{T} \tag{6}$$

where C represents the purchase cost, A is the ordering cost, and $Q = I(0)$ is the initial order quantity. Here, the order quantity $Q = I(0)$ is also a decision variable. Using the selling price $p(t)$ and the cycle time T, the inventory level at a time t is being calculated along with the order quantity Q given by $Q = I(0)$. So, explicitly Q is not a decision variable.

5. Structural results of optimal functions

In this section, dynamic pricing and advertisement investment policy are calculated through cycle time T. The objective function with associated dynamic functions is defined as follows:

$$TP = \frac{1}{T}\int_0^T (p(t)R(t) - HC - AC(E(t)))dt - \frac{CQ}{T} - \frac{A}{T}$$

$$\dot{I}(t) = -R(t) - \theta I(t)\,,\, I(0) = Q \text{ and } I(T) = 0$$

$$\dot{G}(t) = E(t) - \mu G(t)\,,\; G(0) = G_0 \tag{7}$$

where $E(t) \geq 0$, $p(t) \geq 0$ and $Q \geq 0$

The model uses Pontryagin's maximum principle to solve the optimality problem. Introducing two adjoint variables λ_1 and λ_2, the Hamiltonian function obtained is as follows:

$$H(p,E,I,G,\lambda_1,\lambda_2,t) = p(\alpha - p + \gamma G) - hI - \frac{1}{2}rE^2(t) + \lambda_1(-(\alpha - p + \gamma G) - \theta I) + \lambda_2(E - \mu G) \tag{8}$$

The maximum principle law explains that the required conditions for the functions (p^*, E^*) related to the trajectories (I^*, G^*) to be feasible for the optimal problem (7) are as follows:

The trajectory (I^*, G^*) will satisfy

$$\dot{I}^*(t) = -(\alpha - p^*(t) + \gamma G^*(t)) - \theta I^*(t), I^*(T) = 0 \tag{9}$$

$$\dot{G}^*(t) = E^*(t) - \mu G^*(t) \tag{10}$$

The two adjoint variables λ_1, λ_2 will satisfy the adjoint equations.

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial I} = h + \lambda_1\theta, \lambda_1(0) = 0 \tag{11}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial G} = \gamma(\lambda_1 - p) + \lambda_2\mu, \lambda_2(T) = 0 \quad (12)$$

The control function (p^*, E^*) has to maximize the Hamiltonian function. Differentiating given Hamiltonian function with respect to p and E, yields:

$$\frac{\partial H}{\partial p} = (\alpha - 2p + \gamma G + \lambda_1) \quad (13)$$

$$\frac{\partial H}{\partial E} = -rE + \lambda_2 \quad (14)$$

Equation (13) and (14) yield:

$$p = \frac{1}{2}(\alpha + \gamma G + \lambda_1) \quad (15)$$

$$E = \frac{\lambda_2}{r} \quad (16)$$

Using transversality conditions $\lambda_1(0) = 0$ and $\lambda_2(T) = 0$, the adjoint equations (11) and (12) gives:

$$\lambda_1 = \frac{h}{\theta}(\exp(\theta t) - 1) \quad (17)$$

$$\lambda_2 = \frac{\gamma h}{2\theta(\theta - \mu)}(\exp(\theta t) - \exp(\theta T - \mu T + \mu t)) + \frac{\gamma h S}{2\theta\mu} + \frac{1}{2}\frac{\gamma\alpha S}{\mu} + \frac{\gamma^2 SG}{2\mu} \quad (18)$$

Here, $S = (1 - \exp(-\mu T + \mu t))$.

The value of λ_2 is in terms of goodwill effect $G(t)$. Substituting the value of λ_2 in (16), the solution is

$$E = \frac{1}{r}\left(\frac{\gamma h}{2\theta(\theta - \mu)}(\exp(\theta t) - \exp(\theta T - \mu T + \mu t)) + \frac{\gamma h S}{2\theta\mu} + \frac{1}{2}\frac{\gamma\alpha S}{\mu} + \frac{\gamma^2 GS}{2\mu}\right) \quad (19)$$

Substituting equation (19) in (10), the goodwill effect is

$$\begin{aligned} G^*(t) = G_0 \exp(-\mu t) + \frac{\gamma h}{2r\theta(\theta - \mu)}\left(\frac{\exp(\theta t)}{(\theta + \mu)} - \frac{T}{2\mu} - \frac{\exp(-\mu t)}{(\theta + \mu)} + \frac{V}{2\mu}\right) \\ + \left(\frac{\gamma}{2\mu r}\right)\left(\alpha + \frac{h}{\theta}\right)\left(\frac{1}{\mu} - \frac{\exp(-\mu T + \mu t)}{2\mu} - \frac{\exp(-\mu t)}{\mu} + \frac{\exp(-\mu T - \mu t)}{2\mu}\right) \end{aligned} \quad (20)$$

For simplicity of the above equation, certain notations are being used:

$T = \exp(\theta T - \mu T + \mu t), V = \exp(\theta T - \mu T - \mu t)$.

Substituting value of goodwill variable into equations (18) and (19) gives

$$\begin{aligned} \lambda_2 = \frac{\gamma h}{2\theta(\theta - \mu)}(\exp(\theta t) - \exp(\theta T - \mu T + \mu t)) + \frac{\gamma h S}{2\theta\mu} + \frac{1}{2}\frac{\gamma\alpha S}{\mu} + \frac{\gamma^2 S G_0 \exp(-\mu t)}{2\mu} \\ + \frac{\gamma^3 h S}{4\mu r\theta(\theta - \mu)}\left(\frac{\exp(\theta t)}{(\theta + \mu)} - \frac{T}{2\mu} - \frac{\exp(-\mu t)}{(\theta + \mu)} + \frac{V}{2\mu}\right) + \frac{\gamma^2 S}{2\mu}\left(\frac{\gamma}{2\mu r}\right)\left(\alpha + \frac{h}{\theta}\right)\left(\frac{1}{\mu}\right) \\ + \frac{\gamma^2 S}{2\mu}\left(\frac{\gamma}{2\mu r}\right)\left(\alpha + \frac{h}{\theta}\right)\left(\frac{\exp(-\mu T - \mu t)}{2\mu} - \frac{\exp(-\mu T + \mu t)}{2\mu} - \frac{\exp(-\mu t)}{\mu}\right) \end{aligned} \quad (21)$$

$$E^*(t)=\frac{\gamma h}{r2\theta(\theta-\mu)}(\exp(\theta t)-\exp(\theta T-\mu T+\mu t))+\frac{\gamma hS}{r2\theta\mu}+\frac{1}{2}\frac{\gamma\alpha S}{r\mu}+\frac{\gamma^2 SG_0\exp(-\mu t)}{2\mu r}$$
$$+\frac{\gamma^3 hS}{4\mu r^2\theta(\theta-\mu)}\left(\frac{\exp(\theta t)}{(\theta+\mu)}-\frac{T}{2\mu}-\frac{\exp(-\mu t)}{(\theta+\mu)}+\frac{V}{2\mu}\right)+\frac{\gamma^2 S}{2\mu r}\left(\frac{\gamma}{2\mu r}\right)\left(\alpha+\frac{h}{\theta}\right)\left(\frac{1}{\mu}\right) \quad (22)$$
$$+\frac{\gamma^2 S}{2\mu r}\left(\frac{\gamma}{2\mu r}\right)\left(\alpha+\frac{h}{\theta}\right)\left(\frac{\exp(-\mu T-\mu t)}{2\mu}-\frac{\exp(-\mu T+\mu t)}{2\mu}-\frac{\exp(-\mu t)}{\mu}\right)$$

Replacing values of λ_1 and goodwill function in equation (15), the selling price $p^*(t)$ is

$$p^*(t)=\frac{\alpha}{2}+\frac{h}{2\theta}\left(\exp(\theta t)-1\right)+\frac{\gamma}{2}G_0\exp(-\mu t)+\frac{\gamma^2 h}{4r\theta(\theta-\mu)}\left(\frac{\exp(\theta t)}{(\theta+\mu)}-\frac{T}{2\mu}-\frac{\exp(-\mu t)}{(\theta+\mu)}+\frac{V}{2\mu}\right)$$
$$+\left(\frac{\gamma^2}{4\mu r}\right)\left(\alpha+\frac{h}{\theta}\right)\left(\frac{1}{\mu}-\frac{\exp(-\mu T+\mu t)}{2\mu}-\frac{\exp(-\mu t)}{\mu}+\frac{\exp(-\mu T-\mu t)}{2\mu}\right) \quad (23)$$

The demand rate $R(t)$ thus obtained is

$$R(t)=\frac{\alpha}{2}-\frac{h}{2\theta}\left(\exp(\theta t)-1\right)+\frac{\gamma}{2}G_0\exp(-\mu t)+\frac{\gamma^2 h}{4r\theta(\theta-\mu)}\left(\frac{\exp(\theta t)}{(\theta+\mu)}-\frac{T}{2\mu}-\frac{\exp(-\mu t)}{(\theta+\mu)}+\frac{V}{2\mu}\right)$$
$$+\left(\frac{\gamma^2}{4\mu r}\right)\left(\alpha+\frac{h}{\theta}\right)\left(\frac{1}{\mu}-\frac{\exp(-\mu T+\mu t)}{2\mu}-\frac{\exp(-\mu t)}{\mu}+\frac{\exp(-\mu T-\mu t)}{2\mu}\right) \quad (24)$$

Going through all the decision variables, the inventory level $I(t)$ at a time t is

$$I(t)=\frac{h}{4\theta^2}\left(\exp(\theta t)-\exp(2\theta T-\theta t)\right)-\frac{h}{2\theta^2}(1-X)-\frac{\alpha}{2\theta}(1-X)-\frac{\gamma G_0}{2(\theta-\mu)}\left(\exp(-\mu t)-W\right)$$
$$-\frac{\gamma^2 h}{4r\theta(\theta-\mu)}\left(\frac{1}{2\theta(\theta+\mu)}(U)-\frac{1}{2\mu(\theta+\mu)}\left(T-\exp(-\theta t)\right)-\frac{1}{(\theta^2-\mu^2)}\left(\exp(-\mu t)-W\right)\right) \quad (25)$$
$$-\frac{\gamma^2 h}{8r\theta\mu(\theta-\mu)^2}(V-L)-\frac{\gamma^2 Z}{4\mu r}\left(\frac{1}{\mu\theta}(1-X)-\frac{1}{2\mu(\theta+\mu)}(M-X)-\frac{1}{\mu(\theta-\mu)}\left(\exp(-\mu t)-W\right)\right)$$
$$-\frac{\gamma^2 Z}{8\mu^2 r(\theta-\mu)}\left(\exp(-\mu T-\mu t)-N\right)$$

where $T=\exp(\theta T-\mu T+\mu t)$, $V=\exp(\theta T-\mu T-\mu t)$, $W=\exp(\theta T-\mu T-\theta t)$,
$X=\exp(\theta T-\theta t)$ $U=\exp(\theta t)-\exp(2\theta T-\theta t)$, $L=\exp(2\theta T-2\mu T-\theta t)$,
$N=\exp(-2\mu T+\theta T-\theta t)$, $M=\exp(-\mu T+\mu t)$ and $Z=\left(\alpha+\frac{h}{\theta}\right)$

The order quantity Q is given by

$$Q=I(0)=\frac{h}{4\theta^2}\left(1-\exp(2\theta T)\right)-\frac{h}{2\theta^2}\left(1-\exp(\theta T)\right)-\frac{\alpha}{2\theta}\left(1-\exp(\theta T)\right)-\frac{\gamma G_0}{2(\theta-\mu)}(1-B)$$
$$-\frac{\gamma^2 h}{4r\theta(\theta-\mu)}\left(\frac{1}{2\theta(\theta+\mu)}\left(1-\exp(2\theta T)\right)-\frac{1}{2\mu(\theta+\mu)}(B-1)-\frac{1}{(\theta^2-\mu^2)}(1-B)\right) \quad (26)$$
$$-\frac{\gamma^2 h}{8r\mu\theta(\theta-\mu)^2}(B-D)-\frac{\gamma^2 Z}{4\mu r}\left(\frac{1}{\mu\theta}\left(1-\exp(\theta T)\right)-\frac{1}{2\mu(\theta+\mu)}\left(\exp(-\mu T)-\exp(\theta T)\right)\right)$$
$$-\frac{\gamma^2 Z}{4\mu r}\left(\frac{1}{2\mu(\theta-\mu)}\left(\exp(-\mu T)-\exp(-2T\mu+T\theta)\right)-\frac{1}{\mu(\theta-\mu)}(1-B)\right)$$

Here, $B = \exp(-T\mu + T\theta)$, $D = \exp(-2T\mu + 2T\theta)$.

By assigning all the values in equation (6), it is being observed that the total profit TP is a function of T. The first derivative of the total profit function is given by

$$\dot{TP}(T) = \frac{1}{T}\left(p^*(T)R(T) - hI^*(T) - \frac{1}{2}rE^*(T)E^*(T) - C\dot{Q}(T) \right) - \frac{1}{T^2}\left(\int_0^T \left(p^*(T)R(T) - hI^*(T) - \frac{1}{2}rE^*(T)E^*(T) \right) dt - CQ - A \right) \tag{27}$$

The optimal cycle time T^* can be obtained by calculating the first-order derivative of the total profit function TP i.e. $\dot{TP}(T) = 0$, given by

$$T\left(p^*(T)R(T) - hI^*(T) - \frac{1}{2}rE^*(T)E^*(T) - C\dot{Q}(T) \right) - \left(\int_0^T \left(p^*(T)R(T) - hI^*(T) - \frac{1}{2}rE^*(T)E^*(T) \right) dt - CQ - A \right) = 0 \tag{28}$$

Hence, the total profit function TP^* can be calculated through objective functions.

6. NUMERICAL EXAMPLE

To illustrate the above theory, a numerical example is presented with managerial insights.

Example 1: Take α =90 units, C =\$10/unit, h =\$.02/unit, θ =0.3, A =\$70/order, μ =0.2, γ =0.6, r=1, G_0 =2.5. Through equation (28), the optimal cycle time T^* obtained is T^* =2.424 years. The selling price and the investment are

$$p^*(t) = 247.62 - 202.02e^{-0.2t} + 0.15e^{0.3t} - 0.15e^{0.24+0.2t} + 0.15e^{0.24-0.2t} - 101.33e^{-0.48+0.2t} + 101.33e^{-0.48-0.2t}$$

$$E^*(t) = 0.2e^{0.3t} - 0.2e^{0.24+0.2t} + 135.1 - 135.1e^{-0.48+0.2t} + 0.9\left(1 - e^{-0.48+0.2t}\right)\left(-673.4e^{-0.2t} + 0.4e^{0.3t}\right) + 0.9\left(-0.5e^{0.24+0.2t} + 0.5e^{.24-.2t} + 675.5 - 337.75e^{-0.48+0.2t} + 337.75e^{-0.48-0.2t}\right)\left(1 - e^{-0.48+0.2t}\right)$$

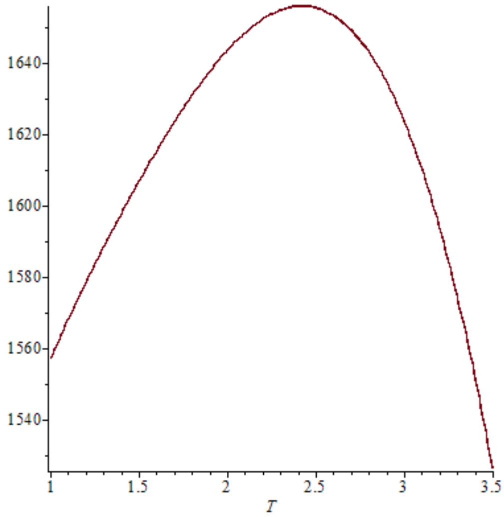

Figure1. Concavity of total profit related to cycle time

With the above values of p^* and E^*, the related goodwill function and inventory level is

$$G^*(t) = -673.4e^{-0.2t} + 0.4e^{0.3t} - 0.5e^{0.1T}\left(e^{0.2t} - e^{-0.2t}\right) + 675.5 - 337.75e^{-0.2T}\left(e^{0.2t} - e^{-0.2t}\right)$$

$$I^*(t) = 2020.2\left(e^{-0.2t} - e^{-0.3t+0.24}\right) + 0.3\left(e^{0.24+0.2t} - e^{-0.3t}\right) + 1.5\left(e^{-0.3t+0.48} - e^{0.24-0.2t}\right) - 825.61 + 1013.25\left(e^{-0.3t-0.24} - e^{-0.48-0.2t}\right) + 0.14\left(e^{-0.3t+1.45} - e^{0.3t}\right) + 622.96e^{-0.3t+0.73} + 202.65e^{-0.48+0.2t}$$

The optimal order quantity Q^*=206.42. And the total profit function is TP^*=\$1655.87.

A sensitivity analysis is done related to different inventory parameters by changing the value of one parameter from -20%,-10%, 10%, and 20% and the other remains unchanged.

Table1. Sensitivity analysis

Parameters	Values of parameters	Q	T (in months)	TP (in \$)
α	72	123.01	2.032	925.86
	81	163.11	2.247	1264.19
	90	206.42	2.424	1655.87
	99	251.76	2.570	2101.44
	108	298.50	2.690	2601.23
C	8	258.99	2.746	1835.26
	9	232.08	2.588	1743.27
	10	206.42	2.424	1655.87
	11	181.86	2.254	1572.95
	12	158.24	2.073	1494.45
h	0.016	206.69	2.427	1656.98
	0.018	206.55	2.426	1656.43
	0.02	206.42	2.424	1655.87
	0.022	206.30	2.423	1655.31
	0.024	206.38	2.421	1654.76
θ	0.24	236.76	2.744	1726.65
	0.27	222.81	2.591	1691.38
	0.30	206.42	2.424	1655.87
	0.33	187.89	2.247	1622.53
	0.36	167.77	2.060	1592.24
A	56	204.17	2.409	1661.66
	63	205.30	2.417	1658.76
	70	206.42	2.424	1655.87
	77	207.53	2.432	1652.99
	84	208.61	2.439	1650.11
μ	0.16	221.90	2.502	1678.71
	0.18	214.08	2.464	1666.95
μ	0.20	206.42	2.424	1655.87
	0.22	210.96	2.383	1645.34
	0.24	191.61	2.340	1635.15
	0.48	170.32	2.266	1560.99
γ	0.48	170.32	2.266	1560.99
	0.54	195.97	2.411	1607.94
	0.6	206.42	2.424	1655.87
	0.66	207.40	2.372	1701.61
	0.72	202.97	2.287	1743.87
r	0.8	207.69	2.364	1704.80
	0.9	208.22	2.405	1678.81
	1	206.42	2.424	1655.87
	1.1	199.56	2.426	1635.63
	1.2	197.20	2.412	1617.81

Through the sensitivity table, one can conclude the following results.

- With an increase in market potential, order quantity, as well as cycle time, increases that increase total profit function of retailers. Therefore, an increase is advisable as it helps to boost the consumption rate of the product.
- With an increase in holding cost, purchase cost, and ordering cost the total budget of the company increases and the total profit function decreases. Hence, the increase is not preferable.
- Deterioration rates have a huge impact on inventory models. More will be the deterioration rate, more will be the damage. The company faces financial loss which in turn reduces profit function. To reduce the rate of spoilage a firm should do preservation investments.
- Parameter μ plays a negative impact on the profitability of a company. With an increase in decay rate, the product losses its goodwill which reduces the total profit function.
- A large advertisement investment cost parameter i.e. r reduces the retailer's profit function.
- A large intensity coefficient γ will have a positive impact on the demand rate. It helps to increase the demand that in turn increases profit function.

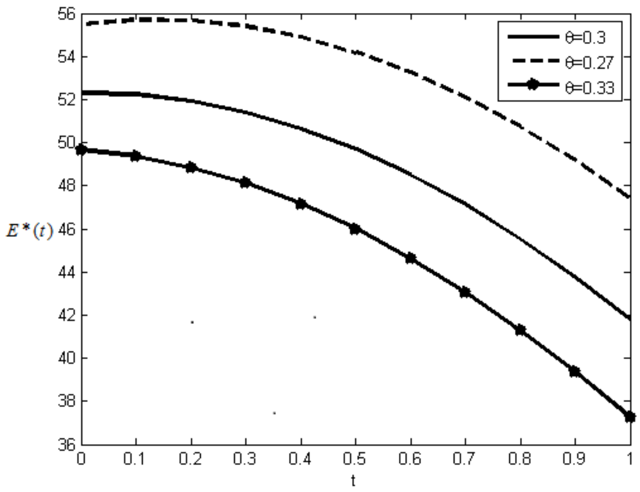

Figure. 2. Advertising investment effort via θ

The advertisement function via different θ is represented in Figure 2. One can observe that with an increase in deterioration rate, advertisement investment decreases. A company would not like to invest more for a product whose rate of deterioration increases with time. It will lead to a financial and economic loss that in turn reduces the company's total profit per unit time. For constant θ, advertisement investment decreases with time. As demand rate is highly influenced by the rate of deterioration. With an increase in deterioration, the consumption rate decreases.

The goodwill function via different θ is represented in Figure 3. It is obvious through the graph that the goodwill function decreases with an increase in deterioration.

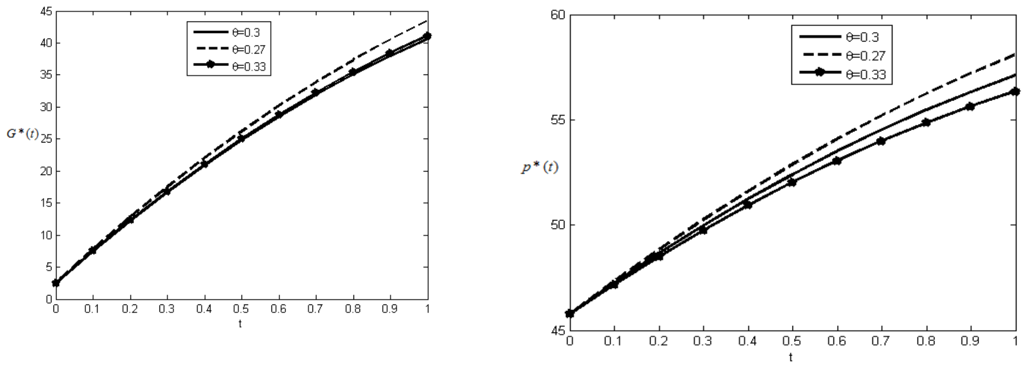

Figure. 3. Goodwill functions via θ

In figure. 4. Note that the greater the deterioration rate θ, the more the objects perished.

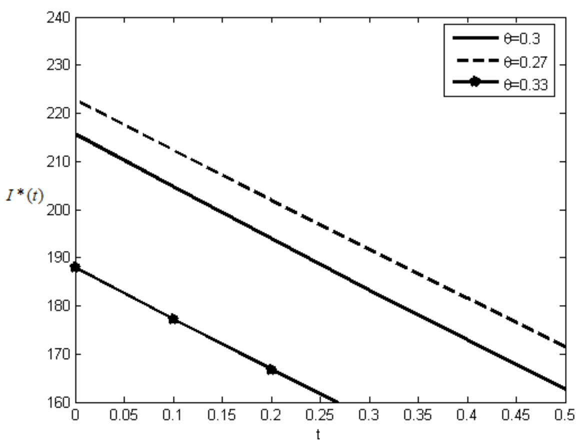

Figure. 4. Dynamic selling price for different values of θ

It shows for a given θ, the selling price increases with time. Also, more will be the deterioration, less will be the price. As with time objects lose their quality which in turn reduces their selling price which helps to sell the product before it gets completely spoiled.

Figure. 5. Represents the optimal inventory level behavior with reference to θ. The graph represents that the inventory level goes on decreasing for a given deterioration rate.

7. CONCLUSION

This article is concerned with the difficulty of adjusting advertising investment, selling price, and

dynamic goodwill effect for a monopolistic company that sells deteriorating objects to goodwill-and price-sensitive consumers. The model deals with a Figure. 5. Dynamic inventory level for different values of θ constant deterioration rate. The company adjusts the
selling price according to the consumers and invests more in advertisement to increase the goodwill effect and demand rate. A dynamic optimization problem is solved to set an optimal selling price, and advertising effort and decide, what should be the feasible order quantity to maximize the total profit function. The joint optimal strategy is obtained through Pontryagin's maximum principle. An example with sensitive analysis under different inventory parameters is performed to modify the model. This article provides important managerial implications. First, the model deals with perishable objects. Second, the solution obtained through pontryagin's maximum principle helps the firm in making decisions related to advertisement investment and selling price. This method is more precise as compared to discrete parameters. From optimal strategies, one can observe that it is more profitable to invest in the advertisement at the beginning of the inventory cycle. As it increases consumption of inventory level by increasing demand. Related to the selling price, a low sales price is advisable in the beginning to consume inventory promptly. Third, the deterioration has a crucial impact on order quantity, cycle time, selling price, advertising investment, goodwill effect, and the inventory level. The products with a higher rate of deterioration, a firm will reduce the total order quantity along with cycle time to reduce the loss due to spoilage. The work can further be extended by introducing competition between different companies and objects with respect to pricing and quality investments. Furthermore, promotional tools such as discount rates and trade-credit period may be used to make the situation more real.

ACKNOWLEDGMENT: The second author (Kavita Rabari) is funded by a Junior Research Fellowship from the Council of Scientific & Industrial Research having (file no.-09/070(0067)/2019-EMR-I) and all the authors are thankful to DST-FIST file # MSI-097 for their technical support to the Department of Mathematics, Gujarat University.

RECEIVED: DECEMBER, 2021.
REVISED: APRIL, 2022

REFERENCES

[1] AKSEN, D. (2007). Loss of customer goodwill in the uncapacitated lot-sizing problem. . **Comput Oper. Res**. 34, 2805-2823.

[2] AVINADAV, T., CHERNONOG, T., LAHAV, Y., and U. SPIEGEL. (2017): Dynamic pricing and promotion expenditures in an EOQ model of perishable products. **Ann. Oper. Res**. 248, 75-91.

[3] BASS, F. M., BRUCE, N., MAJUMDAR, S., and B.P.S. MURTHI. (2007): Wear out effects of different advertising themes: A dynamic Bayesian model of the advertising-sales relationship. **Mark. Sci**. 26, 179-195.

[4] CHANDE, A., DHEKANE, S., HEMACHANDRA, N., and N.RANGARAJ. (2005): Perishable inventory management and dynamic pricing using RFID technology. **Sadhana**. 30, 445-462.

[5] CRETTEZ, B., HAYEK, N., and G. ZACCOUR. (2018): Existence and uniqueness of optimal dynamic pricing and advertising controls without concavity. **Oper. Res. Lett**. 46, 199-204.

[6] DUONG, L. N., WOOD, L. C., and W. Y. WANG. (2015): A multi-criteria inventory management system for perishable and substitutable products. **Procedia. Manuf**. 2, 66-76.

[7] DYE, C. Y., and C. T. YANG. (2016): Optimal dynamic pricing and preservation technology investment for deteriorating products with reference price effects. **Omega**. 62, 52-67.

[8] ERICKSON, G. M. (2009): An oligopoly model of dynamic advertising competition. **Eur. J. Oper. Res**. 197, 374-388.

[9] FENG, L., ZHANG, J., and W. TANG. (2015): A joint dynamic pricing and advertising model of perishable products. **J. Oper. Res. Soc**. 66, 1341-1351.

[10] GOYAL, S. K., and B. C. GIRI. (2001): Recent trends in modelling of deteriorating inventory. **Eur. J**. **Oper. Res**. 134, 1-16.

[11] HERBON, A., LEVNER, E., and T. C. E. CHENG. (2014): Perishable inventory management with dynamic pricing using time-temperature indicators linked to automatic detecting devices. **Int. J**. **Prod. Econ**. 147, 605-613.

[12] KESKIN, N. B., and A. ZEEVI. (2014): Dynamic pricing with an unknown demand model: Asymptotically optimal semi-myopic policies. **Oper. Res**. 62, 1142-1167.

[13] LIU, Y., and V. SHANKAR, V. (2015): The dynamic impact of product-harm crises on brand preference and advertising effectiveness: An empirical analysis of the automobile industry. **Manage. Sci.** 61, 2514-2535.

[14] LIU, G., ZHANG, J., and W. TANG. (2015): Strategic transfer pricing in a marketing–operations interface with quality level and advertising dependent goodwill. **Omega**. 56, 1-15.

[15] LU, L., GOU, Q., TANG, W., and J. ZHANG. (2016): Joint pricing and advertising strategy with reference price effect. **Int. J. Prod**. 54, 5250-5270.

[16] NAIR, A., and R. NARASIMHAN. (2006): Dynamics of competing with quality-and advertising-based goodwill. **Eur. J. Oper. Res**. 175, 462-474.

[17] PAL, B., SANA, S. S., and K. CHAUDHURI. (2015): Two echelon manufacturer–retailer supply chain strategies with price, quality, and promotional effort-sensitive demand. **Int. T. Oper. Res**. 22, 1071-1095.

[18] SCHLOSSER, R. (2015): Dynamic pricing and advertising of perishable products with inventory holding costs. **J. Econ. Dyn. Control**. 57, 163-181.

[19] XIE, J., and J. C. WEI. (2009): Coordinating advertising and pricing in a manufacturer–retailer channel. **Eur. J. Oper. Res**. 197, 785-791.

[20] ZHANG, J. L., CHEN, J., and C.Y. LEE. (2008): Joint optimization on pricing, promotion, and inventory control with stochastic demand. **Int. J. Prod. Econ**. 162, 190-198.