PROPERTIES OF GENERALIZED ORDER STATISTICS OF TRUNCATED INVERTED KUMARASWAMY EXPONENTIAL DISTRIBUTION

Jaspreet Kour Sudan*and Parmil Kumar
Department of Statistics, University of Jammu, Jammu-180006, J&K, India

ABSTRACT

This paper deals with the estimation of simplified expressions of moments relation of the Order Statistics, Generalized Order Statistics and Dual Generalized Order Statistics of Truncated Inverted Kumaraswamy Exponential Distribution. Maximum Likelihood Estimators of Generalized Order Statistics of the unknown parameters of the distribution has been obtained. Bayesian Estimates of the unknown parameters have also been estimated under LINEX and Square-Error loss function using Progressive Type II Censoring Scheme. MCMC algorithms has been employed to obtain Bayesian estimates from unexplicit expression form. 95% confidence intervals and credible interval has been obtained for precisonal purposes. Finally, Simulation Study has been performed for numerical purposes and further, a real life data is used to obtain numerical results.

KEYWORDS: Order Statistics, Generalized Order Statistics, Dual Generalized Order Statistics, Bayesian Estimation, Progressive Type II Censoring Scheme.

MSC: 62F15, 62F99

RESUMEN

Este artículo trata sobre la estimación de expresiones simplificadas de la relación de momentos de la Estadística de Orden, la Estadística de Orden Generalizada y la Estadística de Orden Generalizada Dual de la Distribución Exponencial de Kumaraswamy Invertida Truncada. Estimadores de Máxima Verosimilitud de Orden Generalizado Se han obtenido estadísticas de los parámetros desconocidos de la distribución. Las estimaciones bayesianas de los parámetros desconocidos también se han estimado bajo LINEX y la función de pérdida de error cuadrático utilizando el esquema de censura progresiva de tipo II. Se han empleado algoritmos MCMC para obtener estimaciones bayesianas a partir de formas de expresión no explícitas. Se han obtenido intervalos de confianza de 95% e intervalo creíble con fines precisos. Finalmente, el estudio de simulación se ha realizado con fines numéricos y, además, se utilizan datos de la vida real para obtener resultados numéricos.

PALABRAS CLAVE: Estadísticas de orden, Estadísticas de orden generalizadas, Estadísticas de orden generalizadas duales, Estimación bayesiana, Esquema de censura progresivo tipo II.

1. INTRODUCTION

[2] introduced a new general family of distributions, called Truncated Inverted Kumaraswamy Family of Distribution. The Family of Distribution is derived from the Truncated Inverted Kumaraswamy distribution on the unit interval. It has been used to generate various distributions with decent statistical and mathematical properties and flexible with lifetime data. From this family of distributions,

*jaspreet3.kour@gmail.com

we can simply obtain the three parameter distribution with exponential distribution use as a baseline distribution, named Truncated Inverted Kumaraswamy Exponential Distribution. Its hazard rate function is monotonically increasing, decreasing and upside-down bathtub shapes. Different researches introduced various family of distributions with truncation on semi finite interval $(0, +\infty)$ such as, [1] introduced Truncated Frechet G generator of distributions on the unit interval $(0, 1)$, [7] generated Burr G family of distributions, [12] introduced Truncated Weibull G family of distribution etc. Many new families of distributions have been generated during the recent few years and as a result new parametric distributions have also been introduced in the more advanced ways with many more unique statistical properties with the applications to flexible lifetime data.
The Cumulative Distribution Function of Truncated Inverted Kumaraswamy Exponential Distribution is given as

$$F(x; a, b, \theta) = \frac{1}{(1 - 2^{-a})^b} \left[1 - (2 - e^{-\theta x})^{-a}\right]^b; \; x > 0, \; a, b, \theta > 0 \tag{1.1}$$

The Corresponding Probability Distribution Function is

$$f(x; a, b, \theta) = \frac{ab}{(1 - 2^{-a})^b} \theta e^{-\theta x} (2 - e^{-\theta x})^{-a-1} \left[1 - (2 - e^{-\theta x})^{-a}\right]^{b-1}; \; x > 0, \; a, b, \theta > 0 \tag{1.2}$$

Order Statistics remains a fresh aspect for research for many years now. [4] studied the distributional properties of entropy of order statistics and proved that the Kullback-Leibler functions based on order statistics are non-parametric or distribution-free. [10] derived mathematical expressions for the entropy of the Generalized Feller-Pareto (GFP) probability distribution as well as order statistics of subfamilies of GFP. [8] introduced the concept of generalized order statistics which is represented as the integrated approach to several models of the ordered data such as order statistics, progressively type-II censored samples, type II censored order statistics, record values, and sequential order statistics among others. In recent years, various researches are carried out to study the properties of GOS for distributional purposes such as [6] studied the estimation of GOS from the Burr models, [9] derived exact moments expression of GOS for Type II exponentiated Log-Logistic distribution, [11] studied Generalized Pareto distribution based on generalized order statistics and associated inference. [3] have provided the relation for moments of DGOS for new inverse Kumaraswamy distribution.
In this paper, we have computed the single and product order moments from the Order Statistics, GOS and DGOS for TIKEx distribution. Further, we have derived the MLE and Bayesian estimates for the unknown parameters for GOS of TIKEx distribution under progressive Type II censoring scheme. Markov Chain Monte Carlo can be used to obtain Bayesian estimates of the parameters.

2. SINGLE AND PRODUCT MOMENTS FOR TIKEx DISTRIBUTION

In this section, we have derived the expression for the single and product moments of order statistics, GOS and DGOS for TIKEx distribution.

2.1. MOMENTS FOR ORDER STATISTICS

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from the pdf of given distribution $f(x)$ and let $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \cdots \leq X_{(n-1)} \leq X_{(n)}$ be its corresponding order statistics. The probability density function for the i^{th} order statistic is given as

$$f_{X_i}(x) = \frac{(n-1)}{(i-1)!(n-i)!} nF(x)^{i-1} [1 - F(x)]^{(n-i)} f(x); \;\; x > 0, \;\; 1 \geq i \geq n \tag{2.1}$$

The Joint pdf for i^{th} and s^{th} order statistics is given by

$$f_{X_i,X_s}(x) = \frac{(n-1)}{(i-1)!(s-i-1)!(n-s)!} nF(x)^{i-1}[F(y)-F(x)]^{s-1-i}[1-F(x)]^{(n-s)} f(x)f(y)$$
$$; \quad x>0, 0<x<y, 1<i,s<n \tag{2.2}$$

The r^{th} moment for the i^{th} order statistics obtain as

$$\begin{aligned} E({X^r}_{(i)}) &= \int_0^\infty x^{(r)} f(x)dx \\ &= \int_0^\infty x^{(r)} \frac{(n-1)}{(i-1)!(n-i)!} nF(x)^{i-1}[1-F(x)]^{(n-i)} f(x)dx \end{aligned}$$

Using negative binomial theorem, we get

$$\int_0^\infty x^r F(x)^{a_1} f(x)dx = \frac{1}{(1-2^{-a})^{ba_1}} \frac{ab\theta}{(1-2^{-a})^b} \sum_{k_1=0}^{b(a_1+1)-1} \binom{b(a_1+1)-1}{k_1} (-1)^{k_1}$$
$$\int_0^\infty x^r e^{-\theta x}((1-e^{-\theta x})+1)^{-(a(k_1+1)+1)} dx$$

So the single order moment for order statistics of TIKEx distribution is obtained as

$$= \frac{(n-1)}{(i-1)!(n-i)!} n \frac{ab\theta}{(1-2^{-a})^b} \sum_{k=0}^{n-i} \sum_{k_1=0}^{b(a_1+1)-1} \sum_{k_2=0}^{(a(k_1-1)+1)+k_2-1} \sum_{k_3=0}^{k_2} (-1)^{(k+k_1+k_3}(1)^{-(a(k_1+1)+1)-k_2}$$
$$\binom{n-i}{k} \frac{1}{(1-2^{-a})^{b(k+i-1)}} \binom{b((k+i)-1}{k_1} \binom{(a(k_1-1)+1)+k_2-1}{k_2} \binom{k_2}{k_3} \frac{\Gamma(r+1)}{(\theta(k_3+1))^{r+1}}$$

The product order moment i^{th} and s^{th} order statistics obtain as

$$\begin{aligned} E({X^r}_{(i)}, {X^q}_{(s)}) &= \int_0^\infty \int_x^\infty x^{(r)} y^{(q)} f(x)f(y)dxdy \\ &= \int_0^\infty \int_x^\infty x^{(r)} y^{(q)} \frac{(n-1)}{(i-1)!(s-i-1)!(n-s)!} nF(x)^{i-1}[F(y)-F(x)]^{s-1-i} \\ &\quad [1-F(x)]^{(n-s)} f(x)f(y)dxdy \end{aligned}$$

Consider $s-1-i-p = a_2$ and $i+p+p_1-1 = a_3$ and solving the integral, we get

$$\int_0^\infty \int_x^\infty x^{(r)} y^{(q)} F^{a_3}(x) F^{a_2}(y) f(x)f(y)dxdy$$
$$= (ab\theta)^2 \sum_{p_2=0}^{b(a_2+1)-1} \sum_{p_3=0}^{(a(p_2+1)+1)+p_3-1} \sum_{p_4=0}^{p_3} \sum_{p_5=0}^{b(a_3+1)-1} \sum_{p_6=0}^{(a(p_5+1)+1)+p_6-1} \sum_{p_7=0}^{p_6} \binom{b(a_2+1)-1}{p_2}$$
$$\binom{(a(p_2+1)+1)+p_3-1}{p_3} \binom{p_3}{p_4} \binom{b(a_3+1)-1}{p_5} \binom{(a(p_5+1)+1)+p_6-1}{p_6} \binom{p_6}{p_7}$$
$$(-1)^{p_2+p_4+p_5+p_7}(1)^{(-(a(p_2+1)+1)-p_3)+(-(a(p_5+1)+1)-p_6)} \frac{1}{(\theta(p_4+1))^{q+1}} \frac{1}{(1-2^{-a})^{b((a_2+1)+(a_3+1))}}$$
$$\frac{(\theta(p_4+1))^{q+1}\Gamma((r+1)+(q+1))}{(r+1)((\theta(p_4+1))+(\theta(p_7+1)))^{((r+1)+(q+1))}}$$
$${}_2F_1\Big(1, ((r+1)+(q+1)), ((q+1)+1); \frac{\theta(p_4+1)}{(\theta(p_4+1)+\theta(p_7+1))}\Big)$$

The product moment for the i^{th} and s^{th} order statistics of TIKEx distribution is obtained as

$$E(X^r{}_{(i)}, X^q{}_{(s)})$$
$$= \frac{(n-1)}{(i-1)!(s-i-1)!(n-s)!} n(ab\theta)^2 \sum_{p=0}^{s-i-1} \sum_{p_1=0}^{n-s} \sum_{p_2=0}^{b(a_2+1)-1} \sum_{p_3=0}^{(a(p_2+1)+1)+p_3-1} \sum_{p_4=0}^{p_3} \sum_{p_5=0}^{b(a_3+1)-1}$$
$$\sum_{p_6=0}^{(a(p_5+1)+1)+p_6-1} \sum_{p_7=0}^{p_6} \binom{s-1-i}{p} \binom{n-s}{p_1} \binom{b(s-i-p)-1}{p_2} \binom{(a(p_2+1)+1)+p_3-1}{p_3}$$
$$\binom{p_3}{p_4} \binom{b(i+p+p_1)-1}{p_5} \binom{(a(p_5+1)+1)+p_6-1}{p_6} \binom{p_6}{p_7}$$
$$(-1)^{p+p_1+p_2+p_4+p_5+p_7} (1)^{(-(a(p_2+1)+1)-p_3)+(-(a(p_5+1)+1)-p_6)}$$
$$\frac{1}{(\theta(p_4+1))^{q+1}} \frac{1}{(1-2^{-a})^{b((s-i-p)+(i+p+p_1))}}$$
$$\frac{(\theta(p_4+1))^{q+1}\Gamma((r+1)+(q+1))}{(r+1)((\theta(p_4+1))+(\theta(p_7+1)))^{((r+1)+(q+1))}}$$
$${}_2F_1\Big(1, ((r+1)+(q+1)), ((q+1)+1); \frac{\theta(p_4+1)}{(\theta(p_4+1)+\theta(p_7+1))}\Big)$$

where ${}_2F_1\Big(1, ((r+1)+(q+1)), ((q+1)+1); \frac{\theta(p_4+1)}{(\theta(p_4+1)+\theta(p_7+1))}\Big)$ is the Gauss Hypergeometric Function [13] .

2.2. MOMENTS FOR GENERALIZED ORDER STATISTICS(GOS)

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d random variables with the cumulative density function $F(x)$ and probability density function $f(x)$. Let $n \in N$, $n \geq 2, k > 0$, $(\tilde{m}) = (m_1, m_2, ..., m_{n-1}), M_i = \sum_{j=1}^{n-1} m_i$ such that $\gamma_i = k + n - i + M_i > 0$ for all $i \in (1, 2, \ldots, n-1)$. Then $X(i, n, \tilde{m}, k)$, where $i = 1, 2, \ldots, n$ are said to be GOS and their joint probability distribution function is given by

$$f_{X(i,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n) = k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{j=1}^{n-1} [1-F(x_i)]^{m_i} f(x_i)\Big)[1-F(x_n)]^{k-1} f(x_n) \tag{2.3}$$

The pdf of the i^{th} GOS is given by

$$f_{X(i,n,\tilde{m},k)}(x) = \frac{C_{i-1}}{(i-1)!}[1-F(x)]^{\gamma_i - 1} g_m^{i-1}(F(x)) f(x), \quad x > 0 \tag{2.4}$$

where $C_{i-1} = \prod_{j=1}^{n-1} \gamma_j$ and $g_m(x) = \frac{1}{m+1}[1-(1-x)^{m+1}]$. The Joint pdf of i^{th} and s^{th} GOS is given as

$$f_{X(i,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x) = \frac{C_{s-1}}{(i-1)!(s-i-1)!}[1-F(x)]^m [1-F(y)]^{\gamma_i - 1} g_m^{i-1}(F(x))$$
$$[g_m(F(y)) - g_m(F(x))]^{s-i-1} f(x) f(y), \quad x > 0 \tag{2.5}$$

The Single order moment of i^{th} GOS for TIKEx distribution can be obtained as

$$E(X^r{}_{(i,n,\tilde{m},k)}) = \int_0^\infty x^{(r)} f(x) dx$$
$$= \int_0^\infty x^{(r)} \frac{C_{i-1}}{(i-1)!}[1-F(x)]^{\gamma_i - 1} g_m^{i-1}(F(x)) f(x) dx$$

$$= \frac{C_{i-1}(m+1)^{1-r}}{(i-1)!} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{(m+1)l_1+\gamma_i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{(m+1)l_1+\gamma_i-1}{l_2} (-1)^{l_1+l_2}$$
$$\int_0^\infty x^{(r)} [F(x)]^{l_2} f(x)$$

Now solving the integral by using the procedure applied in the obtaining the moment of single order statistics, we get the resultant expression as

$$E({X^r}_{(i,n,\tilde{m},k)}) = \frac{C_{i-1}(m+1)^{1-r}}{(i-1)!} ab\theta \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{(m+1)l_1+\gamma_i-1} \sum_{l_3=0}^{b(l_2+1)-1} \sum_{l_4=0}^{(a(l_3+1)+1)+l_4-1} \sum_{l_5=0}^{l_4}$$
$$(-1)^{l_1+l_2+l_3+l_5} (1)^{-(a(l_3+1)+1)-l_4} \binom{i-1}{l_1} \binom{(m+1)l_1+\gamma_i-1}{l_2} \binom{b(l_2+1)-1}{l_3}$$
$$\binom{(a(l_3+1)+1)+l_4-1}{l_4} \binom{l_4}{l_5} \frac{1}{(1-2^{-a})^{b(l_2+1)}} \frac{\Gamma(r+1)}{(\theta(l_5+1))^{r+1}} \tag{2.6}$$

The product order moment of i^{th} and s^{th} GOS for TIKEx distribution would be obtained as

$$E({X^r}_{(i)}, {X^q}_{(s)}) = \int_0^\infty \int_x^\infty x^{(r)} y^{(q)} f(x) f(y) dx dy$$
$$= \int_0^\infty \int_x^\infty x^{(r)} y^{(q)} \frac{C_{i-1}}{(i-1)!(s-i-1)!} [1-F(x)]^m [1-F(y)]^{\gamma_i - 1} g_m^{i-1}(F(x))$$
$$[g_m(F(y)) - g_m(F(x))]^{s-i-1} f(x) f(y) dx dy$$

By using the above result, the product order moment i^{th} and s^{th} GOS for TIKEx distribution can be obtained as

$$E({X^r}_{(i)}, {X^q}_{(s)})$$
$$= \frac{C_{i-1}}{(i-1)!(s-i-1)!} (m+1)^{2-s} (ab\theta)^2 \sum_{l_1=0}^{s-i-1} \sum_{l_2=0}^{i+l_1-1} \sum_{l_3=0}^{s-i-1-l_1} \sum_{l_4=0}^{m+(m+1)l_2} \sum_{l_5=0}^{\gamma_i-1+(m+1)l_3} \sum_{l_6=0}^{b(l_5+1)-1}$$
$$\sum_{l_7=0}^{(a(l_6+1)+1)+l_7-1} \sum_{l_8=0}^{l_7} \sum_{l_9=0}^{b(l_4+1)-1} \sum_{l_{10}=0}^{(a(l_9+1)+1)+l_{10}-1} \sum_{l_{11}=0}^{l_10} (-1)^{l_1+l_2+l_3+l_4+l_5+l_6+l_9+l_11}$$
$$(1)^{((-(a(l_6+1)+1)-l_7)+(-(a(l_9+1)+1)-l_{10}))} \binom{s-i-1}{l_1} \binom{i+l_1-1}{l_2} \binom{s-i-1-l_1}{l_3}$$
$$\binom{m+(m+1)l_2}{l_4} \binom{\gamma_i-1+(m+1)l_3}{l_5} \binom{b(l_5+1)-1}{l_6} \binom{(a(l_6+1)+1)+l_7-1}{l_7}$$
$$\binom{l_7}{l_8} \binom{b(l_4+1)-1}{l_9} \binom{(a(l_9+1)+1)+l_{10}-1}{l_{10}} \binom{l_10}{l_{11}}$$
$$\frac{1}{(1-2^{-a})^{b((l_4+1)+(l_5+1))}} \frac{(\theta(l_8+1))^{q+1} \Gamma((r+1)+(q+1))}{(r+1)((\theta(l_8+1)) + (\theta(l_{11}+1)))^{((r+1)+(q+1))}}$$
$${}_2F_1\Big(1, ((r+1)+(q+1)), ((q+1)+1); \frac{\theta(l_8+1)}{(\theta(l_8+1)+\theta(l_{11}+1))}\Big) \tag{2.7}$$

Where ${}_2F_1\Big(1, ((r+1)+(q+1)), ((q+1)+1); \frac{\theta(l_8+1)}{(\theta(l_8+1)+\theta(l_{11}+1))}\Big)$ is the Gauss Hypergeometric Function.

2.3. MOMENTS FOR DUAL GENERALIZED ORDER STATISTICS(DGOS)

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d random variables with the cumulative density function $F(x)$ and probability density function $f(x)$. Let $n \in N$, $n \geq 2, k > 0$, $(\tilde{m}) = (m_1, m_2, ..., m_{n-1}), M_i = \sum_{j=1}^{n-1} m_i$ such that $\gamma_i = k + n - i + M_i > 0$ for all $i \in (1, 2, \ldots, n-1)$. Then $X(i, n, \tilde{m}, k)$, where $i = 1, 2, \ldots, n$ are said to be dual GOS and their joint probability distribution function is given by

$$f_{X(i,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n) = k\big(\prod_{j=1}^{n-1} \gamma_j\big)\big(\prod_{i=1}^{n-1} [F(x_i)]^{m_i} f(x_i)\big)[F(x_n)]^{k-1} f(x_n) \tag{2.8}$$

The pdf of i^{th} dual GOS is given as

$$f_{X(i,n,\tilde{m},k)}(x) = \frac{C_{i-1}}{(i-1)!}[F^{\gamma_i - 1}(x)] g_m^{i-1}(F(x)) f(x), \quad x > 0 \tag{2.9}$$

where $C_{i-1} = \prod_{j=1}^{n-1} \gamma_j$ and $g_m(x) = \frac{1}{m+1}[1 - x^{m+1}]$. The Joint pdf of i^{th} and s^{th} GOS is given as

$$\begin{aligned} f_{X(i,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = & \frac{C_{s-1}}{(i-1)!(s-i-1)!}[F^m(x)][F^{\gamma_i-1}(y)] g_m^{i-1}(F(x)) \\ & [g_m(F(y)) - g_m(F(x))]^{s-i-1} f(x) f(y), \quad x > 0 \end{aligned} \tag{2.10}$$

The Single order moment of i^{th} GOS for TIKEx distribution can be obtained as

$$\begin{aligned} E(X^r{}_{(i,n,\tilde{m},k)}) &= \int_0^\infty x^{(r)} f(x) dx \\ &= \int_0^\infty x^{(r)} \frac{C_{i-1}}{(i-1)!}[F^{\gamma_i-1}(x)] g_m^{i-1}(F(x)) f(x) \\ &= \frac{C_{i-1}(m+1)^{1-i}}{(i-1)!} \sum_{w_1=0}^{i-1} (-1)^{w_1} \binom{i-1}{w_1} \int_0^\infty x^{(r)} F(x)^{(m+1)w_1 + \gamma_i - 1} f(x) \end{aligned}$$

Now by solving the integral, we get the resultant expression of moment of i^{th} DGOS for TIKEx distribution as

$$\begin{aligned} E(X^r{}_{(i)}) = & \frac{C_{i-1}(m+1)^{1-i}}{(i-1)!} ab\theta \sum_{w_1=0}^{i-1} \sum_{w_2=0}^{b((m+1)w_1+\gamma_i)-1} \sum_{w_3=0}^{-(a(w_2+1)+1)+w_3-1} \sum_{w_4=0}^{w_3} (-1)^{w_1+w_2+w_4} \\ & (-1)^{-(a(w_2+1)+1)-w_3} \binom{i-1}{w_1} \binom{b((m+1)w_1+\gamma_i)-1}{w_2} \\ & \binom{(a(w_2+1)+1)+w_3-1}{w_3} \binom{w_3}{w_4} \frac{\Gamma r+1)}{(\theta(w_4+1))^{r+1}} \end{aligned} \tag{2.11}$$

The product order moment of i^{th} and s^{th} GOS for TIKEx distribution would be obtained as

$$\begin{aligned} E(X^r{}_{(i)}, X^q{}_{(s)}) = & \int_0^\infty \int_x^\infty x^{(r)} y^{(q)} f(x) f(y) dx dy \\ = & \frac{C_{s-1}}{(i-1)!(s-i-1)!}(m+1)^{2-s} \sum_{w_1=0}^{s-i-1} \binom{s-i-1}{l_1} \\ & \sum_{w_2=0}^{i+w_1-1} \binom{i+w_1-1}{w_2} \sum_{w_3=0}^{s-i-1-w_1} \binom{s-i-1-w_1}{w_3} \\ & \int_0^\infty \int_x^\infty x^{(r)} y^{(q)} F(x)^{m+(m+1)w_2} F(y)^{(m+1)w_3+\gamma_s-1} f(x) f(y) dx dy \end{aligned}$$

By solving the integral, we get

$$\int_x^\infty y^{(q)} F(y)^{(m+1)w_3+\gamma_s-1} f(y)dy = ab\theta \sum_{w_4=0}^{b((m+1)w_3+\gamma_s)-1} \sum_{w_5=0}^{(a(w_4+1)+1)+w_5-1} \sum_{w_6=0}^{w_5} \binom{b((m+1)w_3+\gamma_s)-1}{w_4} \binom{-(a(w_4+1)+1)+w_5-1}{w_5} \binom{w_5}{w_6} (-1)^{w_4+w_6} (1)^{-(a(w_4+1)+1)-w_5} \frac{1}{(1-2^{-a})^{b(m+1)w_3+\gamma_s)}} \frac{1}{(\theta(w_6+1))^{q+1}} \Gamma\big((q+1),(\theta(w_6+1)x)\big)$$

By using the above result, the product order moment i^{th} and s^{th} DGOS for TIKEx distribution can be obtained as

$$E(X^r{}_{(i)}, X^q{}_{(s)}) = \frac{C_{s-1}(m+1)^{2-s}}{(i-1)!(s-i-1)!} (ab\theta)^2 \sum_{w_1=0}^{s-i-1} \sum_{w_2=0}^{i+w_1-1} \sum_{w_3=0}^{s-i-1-w_1} \sum_{w_4=0}^{b((m+1)w_3+\gamma_s)-1} \sum_{w_5=0}^{(a(w_4+1)+1)+w_5-1} \sum_{w_6=0}^{w_5} \sum_{w_7=0}^{b(m+(m+1)w_2+1)-1} \sum_{w_8=0}^{(a(w_7+1)+1)+w_8-1} \sum_{w_9=0}^{w_8} \binom{s-i-1}{l_1} \binom{i+w_1-1}{w_2} \binom{s-i-1-w_1}{w_3} \binom{b((m+1)w_3+\gamma_s)-1}{w_4} \binom{(a(w_4+1)+1)+w_5-1}{w_5} \binom{w_5}{w_6} \binom{b(m+(m+1)w_2+1)-1}{w_7} \binom{(a(w_7+1)+1)+w_8-1}{w_8} \binom{w_8}{w_9} (-1)^{w_1+w_2+w_3+w_4+w_6+w_7+w_9} (1)^{((-(a(w_4+1)+1)-w_5)+(-(a(w_7+1)+1)-w_8))} \frac{1}{(1-2^{-a})^{b(((m+1)w_3+\gamma_s)+(m+(m+1)w_2+1))}} \frac{(\theta(w_6+1))^{q+1}\Gamma((r+1)+(q+1))}{(r+1)((\theta(w_6+1))+(\theta(w_9+1)))^{((r+1)+(q+1))}} {}_2F_1\Big(1,((r+1)+(q+1)),((q+1)+1); \frac{\theta(w_6+1)}{(\theta(w_6+1)+\theta(w_9+1))}\Big) \quad (2.12)$$

Where ${}_2F_1\Big(1,((r+1)+(q+1)),((q+1)+1); \frac{\theta(w_6+1)}{(\theta(w_6+1)+\theta(w_9+1))}\Big)$ is the Gauss Hypergeometric Function [5].

3. MAXIMUM LIKELIHOOD ESTIMATION OF GOS FOR TIKEx DISTRIBUTION

Let $X(1,m,n,k), X(2,m,n,k), \ldots, X(n,m,n,k)$ be a GOS sample of size n from the TIKEx distribution. The Joint PDF function for the GOS is given as

$$f_{(x(i,n,m,k))}(x_i) = k \prod_{j=1}^{n-1} \gamma_j \prod_{i=1}^{n-1} ([1-F(x_i)]^{m_i} f(x_i))([1-F(x_n)]^{k-1} f(x_n)) \quad (3.1)$$

By putting the pdf and cdf of TIKEx distribution, the likelihood function ignoring the constants can be written as follows

$$L(a,b,\theta)|\underline{x}) = \prod_{i=1}^{n-1} \Big[\frac{ab}{(1-2^{-a})^b} \theta e^{-\theta x_i} (2-e^{-\theta x_i})^{-a-1} \Big[1-(2-e^{-\theta x_i})^{-a}\Big]^{b-1}\Big] \prod_{i=1}^{n-1} \Big[1-\frac{1}{(1-2^{-a})^b}\Big[1-(2-e^{-\theta x_i})^{-a}\Big]^b\Big]^{m_i} \prod_{i=1}^{n-1} \Big[1-\frac{1}{(1-2^{-a})^b}\Big[1-(2-e^{-\theta x_n})^{-a}\Big]^b\Big] \quad (3.2)$$

The Log Likelihood Function can be written as

$$\begin{aligned}\log L &= n\log(a) + n\log(b) + n\log(\theta) - nb\log(1-2^{-a}) - \theta\sum_{i=1}^{n} x_i \\ &+ (-a-1)\sum_{i=1}^{n}\log(2-e^{-\theta x_i}) + (b-1)\sum_{i=1}^{n}\log\left[1-(2-e^{-\theta x_i})^{-a}\right] \\ &+ \sum_{i=1}^{n} m_i \log\left[1-\frac{1}{(1-2^{-a})^b}\left[1-(2-e^{-\theta x_i})^{-a}\right]^b\right] \\ &+ (k-1)\log\left[1-\frac{1}{(1-2^{-a})^b}\left[1-(2-e^{-\theta x_n})^{-a}\right]^b\right]\end{aligned} \tag{3.3}$$

Now differentiate with respect to a, b and θ, we have

$$\begin{aligned}\frac{\partial \log L}{\partial a} &= \frac{n}{a} - \sum_{i=1}^{n}\log(2-e^{(-\theta x_i)}) - \sum_{i=1}^{n}\frac{(\log(2-e^{(-\theta x_i)})(b-1))}{(-e^{(-\theta x_i)}+2)^a}\Big(\frac{1}{(2-e^{(-\theta x_i)})^a}-1)\Big) \\ &- \frac{b\Big(\frac{\big(\frac{1}{(2-e^{(-\theta x_n)})^a}-1\big)}{(\frac{1}{2^a}-1)}\Big)^{(b-1)}\Big(\frac{\log(2-e^{(-\theta x_n)})}{((-e^{(-\theta x_n)}+2)^a(\frac{1}{2^a}-1))} - \frac{(\frac{1}{2^a}\log(2)(\frac{1}{(2-e^{(-\theta x_n)})^a}-1))}{(\frac{1}{2^a}-1)^2}\Big)(k-1)}{\Big(\frac{\big(\frac{1}{(2-e^{(-\theta x_n)})^a}-1\big)}{(\frac{1}{2^a}-1))^b}-1\Big)} \\ &- \sum_{i=1}^{n}\frac{\Big(bm_i\Big(\frac{\big(\frac{1}{(2-e^{(-\theta x_i)})^a}-1\big)}{(\frac{1}{2^a}-1)}\Big)^{(b-1)} * \Big(\frac{\log(2-e^{(-\theta x_i)})}{((-e^{(-\theta x_i)}+2)^a(\frac{1}{2^a}-1))} - \frac{\big(\frac{1}{2^a}\log(2)(\frac{1}{(2-e^{(-\theta x_i)})^a}-1)\big)}{(\frac{1}{2^a}-1)^2}\Big)\Big)}{\Big((\frac{\big(\frac{1}{(2-e^{(-\theta x_i))^a}}-1\big)}{(\frac{1}{2^a}-1)})^b-1\Big)}\end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{\partial \log L}{\partial b} &= \log\Big(1-\frac{1}{(-e^{(-\theta x_i)}+2)^a}\Big) - n\log(1-\frac{1}{2^a}) + \frac{n}{b} \\ &+ \frac{(m_i\log(\frac{\big(\frac{1}{(2-e^{(-\theta x_i)})^a}-1)}{(\frac{1}{2^a}-1)})\Big(\frac{\big(\frac{1}{(2-e^{(-\theta x_i)})^a}-1\big)}{(1/2^a-1))^b}\Big)}{\Big(\frac{(\frac{1}{(2-e^{(-\theta x_i)})^a}-1)}{(\frac{1}{2^a}-1)^b-1)}\Big)} \\ &+ \frac{(\log(\frac{(\frac{1}{(2-e^{(-\theta x_n)})^a}-1)}{(\frac{1}{2^a}-1)})(\frac{(\frac{1}{(2-e^{(-\theta x_n)})^a}-1)}{(\frac{1}{2^a}-1)})^b * (k-1))}{(\frac{(\frac{1}{(2-e^{(-\theta x_n)})^a}-1)}{((\frac{1}{2^a}-1))^b-1)})}\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \theta} &= \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{(x_i e^{(-\theta x_i)}(a+1))}{(e^{(-\theta x_i)}-2)} - \sum_{i=1}^{n}\frac{(a x_i e^{(-\theta x_i)}(b-1))}{((-e^{(-\theta x_i)}+2)^{(a+1)}(\frac{1}{(2-e^{(-\theta x_i)})}^a-1))} \\ &- \sum_{i=1}^{n}\frac{(abm_i x_i e^{(-\theta x_i)}\frac{((\frac{1}{(2-e^{(-\theta x_i)})^a}-1)}{(\frac{1}{2^a}-1))}^{(b-1)})}{((-e^{(-\theta x_i)}+2)^{(a+1)}((\frac{(\frac{1}{(2-e^{(-\theta x_i)})^a}-1)}{(\frac{1}{2^a}-1)})^b-1)(\frac{1}{2^a}-1))} \\ &- \frac{(abx_n e^{(-\theta x_n)}(\frac{(\frac{1}{(2-e^{(-\theta x_n)})^a}-1)}{(\frac{1}{2^a}-1)})^{(b-1)}(k-1))}{((-e^{(-\theta x_n)}+2)^{(a+1)}(\frac{((\frac{1}{(2-e^{(-\theta x_n))^a}}-1)}{(1/2^a-1)})^b-1)(\frac{1}{2^a}-1))}\end{aligned} \tag{3.6}$$

As we can observe that the equations are not in the closed form, so we have to apply numerical method in order to obtain MLE for unknown parameters a,b and θ

4. BAYESIAN ESTIMATION FOR GENERALIZED ORDER STATISTICS OF TIKEx DISTRIBUTION

Bayesian estimation plays a vital role in Inferential Statistics. This section present the estimation of Bayesian estimators for unknown parameters a, b and θ of TIKEx distribution under Square Error and LINEX Loss Functions. Here, it is assumed that the parameters of the distribution are independent and follows gamma priors where hyperparameters are informative. Therefore, the prior density of a, b and θ can be written as

$$h(a,b,\theta) = \frac{{\alpha_2}^{\alpha_1}}{\Gamma\alpha_1} a^{\alpha_1-1} \frac{{\beta_2}^{\beta_1}}{\Gamma\beta_1} b^{\beta_1-1} \frac{{\tau_2}^{\tau_1}}{\Gamma\tau_1} \theta^{\tau_1-1} e^{-\alpha_2 a} e^{-\beta_2 b} e^{-\tau_2\theta} \tag{4.1}$$

Here, $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$ and θ_2 are hyperparameters to have priors knowledge about the parameters. The Joint Posterior Distribution of a,b and θ can be obtained as

$$\begin{aligned} h(a,b,\theta|\underline{\mathrm{x}}) \propto\ & a^{n+\alpha_1-1} b^{n+\beta_1-1} \theta^{n+\tau_1-1} (1-2^{-a})^{bn} e^{-\theta(\sum_{i=1}^{n} x_i + \tau_2)} \prod_{i=1}^{n-1} (2-e^{-\theta x_i})^{-a-1} \\ & \left[1-(2-e^{-\theta x_i})^{-a}\right]^{b-1} \left[1-\frac{1}{(1-2^{-a})^b}\left[1-(2-e^{-\theta x_i})^{-a}\right]^b\right]^{m_i} \\ & \prod_{i=1}^{n-1} \left[1-\frac{1}{(1-2^{-a})^b}\left[1-(2-e^{-\theta x_n})^{-a}\right]^b\right]^{k-1} e^{-\alpha_2 a} e^{-\beta_2 b} \end{aligned} \tag{4.2}$$

Now we can find the Bayesian estimate under any loss function but as we can observe the above equations do not provide any analytical solution, so here we are using Markov Chain Monte Carlo to get the desired estimates. Conditional densities obtained from the posterior distribution given in equation (4.2) can be used to generate samples and obtain Bayesian estimates under different loss functions. The Conditional posterior densities for a, b and θ are obtained as

$$\begin{aligned} h(a|\underline{\mathrm{x}},b,\theta) = & a^{n+\alpha_1-1} (1-2^{-a})^{bn} \prod_{i=1}^{n-1} (2-e^{-\theta x_i})^{-a-1} \left[1-(2-e^{-\theta x_i})^{-a}\right]^{b-1} \\ & \left[1-\frac{1}{(1-2^{-a})^b}\left[1-(2-e^{-\theta x_i})^{-a}\right]^b\right]^{m_i} \\ & \left[1-\frac{1}{(1-2^{-a})^b}\left[1-(2-e^{-\theta x_n})^{-a}\right]^b\right]^{k-1} e^{-\alpha_2 a} \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
h(b|\underline{x}, a, \theta) = b^{n+\beta_1-1}(1-2^{-a})^{bn} \prod_{i=1}^{n-1} \Big[1-(2-e^{-\theta x_i})^{-a}\Big]^{b-1} \\
\Big[1-\frac{1}{(1-2^{-a})^b}\Big[1-(2-e^{-\theta x_i})^{-a}\Big]^b\Big]^{m_i} \\
\Big[1-\frac{1}{(1-2^{-a})^b}\Big[1-(2-e^{-\theta x_n})^{-a}\Big]^b\Big]^{k-1} e^{-\beta_2 b}
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
h(\theta|\underline{x}, a, b) = \theta^{n+\tau_1-1} e^{-\theta(\sum_{i=1}^{n} x_i+\tau_2)} \prod_{i=1}^{n-1}(2-e^{-\theta x_i})^{-a-1}\Big[1-(2-e^{-\theta x_i})^{-a}\Big]^{b-1} \\
\Big[1-\frac{1}{(1-2^{-a})^b}\Big[1-(2-e^{-\theta x_i})^{-a}\Big]^b\Big]^{m_i} \\
\Big[1-\frac{1}{(1-2^{-a})^b}\Big[1-(2-e^{-\theta x_n})^{-a}\Big]^b\Big]^{k-1}
\end{aligned}
\tag{4.5}
$$

It can be seen that equations (4.3)-(4.5) do not follow any specific distribution, so we cannot generate samples from them directly. So in this situations, we use Metropolis Hastings algorithm to generate samples of a, b and θ by considering normal distribution as proposal distribution. The steps for the algorithm is given below

1. Consider the initials for a, b and θ. Take K=Burnin.
2. Initialize t=1.
3. Since the Conditional probabilities do not follow any particular distribution, so we can generate $a^{(t)}$, $b^{(t)}$ and $\theta^{(t)}$ from $h(a|\underline{x}, b, \theta)$, $h(b|\underline{x}, a, \theta)$ and $h(\theta|\underline{x}, a, b)$ by using $N(a^{(t-1)}, \sigma)$, $N(b^{(t-1)}, \sigma)$ and $N(\theta^{(t-1)}, \sigma)$.
4. Put $t = t + 1$.
5. Repeat the steps from 3-4 N times to obtain $a^{(t)}$, $b^{(t)}$ and $\theta^{(t)}$, $t = K+1 \dots N$.

To neglect the effect cause by the selection of initial values of a, b and θ, the first K generated values should be removed or discarded. The remaining sample values from $a^{(t)}$, $b^{(t)}$ and $\theta^{(t)}$, $t = K+1 \dots N$, for sufficiently large N, forms an approximate posterior samples which can be used for the estimation of bayesian estimates of a, b and θ under SEL and LINEX loss function which are given as

$$
\hat{\phi}_{BS} = \frac{1}{N-K} \sum_{t=K+1}^{N} \phi^{(t)}
$$

$$
\hat{\phi}_{BLINEX} = -\frac{1}{c} \ln \Big[\frac{1}{N-K} \sum_{t=K+1}^{N} e^{-c*\phi^{(t)}}\Big]
$$

where N is the total samples generated from the conditional posterior distribution and K represents the burn-in samples. For the credible intervals of the parameters a, b and θ, first sort the samples in increasing order and the calculate the $100(1-\gamma)\%$ intervals of the mentioned parameters

$$
[a^{(N_K)(\gamma/2)}, a^{(N_K)(1-\gamma/2)}]
$$
$$
[b^{(N_K)(\gamma/2)}, b^{(N_K)(1-\gamma/2)}]
$$
$$
[\theta^{(N_K)(\gamma/2)}, \theta^{(N_K)(1-\gamma/2)}]
$$

5. SIMULATION STUDY

In this section, a simulation study has been performed to obtained means and variances of Order Statistics, GOS and DGOS for different sample sizes under the different values of parameter a and b with constant value of third parameter $\theta = 0.9$. The values calculated for n=2,3,4,8 and 10 for different order values. The relations obtained in the preceding sections allow us to evaluate the expected values and variances of all order statistics, GOS and DGOS for all sample sizes. A numerical study has been performed using MatLab version 9.0. The data should be generated using the quantile function of the distribution under consideration. Here, we have been utilizing the quantile function of TIKEx distribution which is obtained as

$$Q(u|a,b,\theta) = \left(\frac{-1}{\theta}\right)\log(2-(1-(u^{(\frac{1}{b})})(1-2^{(-a)}))^{(\frac{-1}{a})})$$

Here Table 1 and Table 2 represent the values for means and variances for Order statistics which can be obtained by using the moments of order statistics. Table 3 and Table 4 represent the means and variances of GOS and Table 5 and 6 represent the means and variances for DGOS. From the tables, we can observe that the means and variances have been decreasing with the increase in sample size for order statistics whereas both are increasing for GOS and DGOS for the different values of parameters a and b.

Table 1: Means of Order Statistics

n	i	a	0.02	0.03	0.04	0.05
$\theta = 0.9$		b	0.009	0.008	0.007	0.006
2	1		0.0004157	0.0005501	0.0006380	0.0006803
2	2		0.0004320	0.0005675	0.0006543	0.0006942
4	1		0.0004157	0.0005502	0.0006381	0.0006803
4	2		0.001296	0.001702	0.001963	0.002082
4	3		0.001347	0.001756	0.002013	0.002125
4	4		0.0004667	0.006039	0.0006881	0.0007229
5	1		0.0001732	0.0002292	0.0002658	0.0002834
5	2		0.0007201	0.0009458	0.001090	0.001157
5	4		0.0007778	0.001006	0.001146	0.001204
5	5		0.0002021	0.0002595	0.0002940	0.0003074
8	1		0.000002309	0.000003056	0.000003544	0.000003779
8	3		0.00005239	0.00006830	0.00007829	0.00008265
8	6		0.00005881	0.00007497	0.00008443	0.00008784
8	7		0.00002037	0.00002577	0.00002886	0.00002987
10	1		0.00000005155	0.00000006822	0.00000007912	0.00000008436
10	5		0.000007579	0.000009733	0.00001102	0.00001152
10	8		0.000002431	0.000003052	0.000003398	0.000003500

5.1. APPLICATION TO REAL LIFE DATA

This section comprises of the application of real life dataset for performing the progressive type II censoring scheme to estimate the maximum likelihood and Bayesian estimates of the TIKEx distribution.

Table 2: Variances of Order Statistics

n	i	a	0.02	0.03	0.04	0.05
$\theta = 0.9$		b	0.009	0.008	0.007	0.006
2	1		0.0009236	0.001222	0.001417	0.001511
2	2		0.0009599	0.001260	0.001453	0.001542
4	1		0.0009237	0.001223	0.001418	0.001511
4	2		0.002878	0.003780	0.004358	0.004624
4	3		0.002991	0.003899	0.004469	0.004718
4	4		0.0001036	0.001341	0.001528	0.001606
5	1		0.0003849	0.0005093	0.00005907	0.0006298
5	2		0.001599	0.002101	0.002422	0.002569
5	4		0.001728	0.002235	0.002547	0.002676
5	5		0.0004491	0.0005767	0.0006533	0.0006830
8	1		0.000005132	0.000006792	0.000007877	0.000008399
8	3		0.0001164	0.0001517	0.0001739	0.0001836
8	6		0.0001307	0.00016659	0.0001876	0.0001951
8	7		0.00004528	0.00005728	0.00006414	0.00006639
10	1		0.0000001145	0.0000001516	0.0000001758	0.0000001874
10	5		0.00001684	0.00002163	0.00002450	0.00002561
10	8		0.000005402	0.000006783	0.000007551	0.000007778

Table 3: Means of Generalized Order Statistics

n	i	a	0.02	0.03	0.04	0.05
$\theta = 0.9$		b	0.009	0.008	0.007	0.006
2	1		0.001247	0.006051	0.001914	0.002040
2	2		0.002494	0.003300	0.003828	0.004081
4	1		0.002078	0.002750	0.003190	0.003401
4	2		0.008314	0.01100	0.01276	0.01360
4	3		0.01247	0.01650	0.01914	0.02040
4	4		0.008314	0.01100	0.01276	0.01360
5	1		0.002494	0.003300	0.003828	0.004081
5	2		0.01247	0.01650	0.01914	0.02040
5	4		0.02494	0.03301	0.03829	0.04082
5	5		0.01247	0.01650	0.01914	0.02041
8	1		0.003741	0.004951	0.005742	0.006122
8	3		0.1047	0.1386	0.1608	0.1714
8	6		0.2095	0.2772	0.3216	0.3428
8	7		0.1048	0.1386	0.1608	0.1714
10	1		0.004573	0.001650	0.007019	0.007483
10	5		0.9603	1.2708	1.4740	1.5715
10	8		0.5487	0.7262	0.8422	1.1891

Table 4: Variances of Generalized Order Statistics

n	i	a	0.02	0.03	0.04	0.05
$\theta = 0.9$		b	0.009	0.008	0.007	0.006
2	1		0.002769	0.003665	0.004250	0.004531
2	2		0.005536	0.007324	0.008493	0.009054
4	1		0.004614	0.006105	0.007079	0.007547
4	2		0.01840	0.02433	0.02819	0.03005
4	3		0.02755	0.03640	0.04217	0.04493
4	4		0.01840	0.02433	0.02819	0.03005
5	1		0.005536	0.007324	0.008493	0.009058
5	2		0.02756	0.03641	0.04217	0.04493
5	4		0.05480	0.07226	0.08361	0.08904
5	5		0.02755	0.03640	0.04217	0.04494
8	1		0008300	0.01097	0.01272	0.01356
8	3		0.2218	0.2888	0.3314	0.3515
8	6		0.4217	0.5392	0.6112	0.6439
8	7		0.2218	0.2889	0.3314	0.3516
10	1		0.01014	0.01341	0.01554	0.01657
10	5		1.2118	1.2090	1.1028	1.0225
10	8		0.9183	1.0864	1.1623	0

Table 5: Means of Dual Generalized Order Statistics

n	i	a	0.02	0.03	0.04	0.05
$\theta = 0.9$		b	0.009	0.008	0.007	0.006
2	1		0.001512	0.001927	0.002171	0.002258
2	2		0.002800	0.003623	0.004129	0.004337
4	1		0.002941	0.003638	0.004002	0.004082
4	2		0.01089	0.01367	0.01522	0.01568
4	3		0.01512	0.01927	0.02171	0.02258
4	4		0.009334	0.01207	0.01376	0.01445
5	1		0.003812	0.004645	0.005051	0.005101
5	2		0.01764	0.02182	0.002401	0.02449
5	4		0.03024	0.03855	0.04342	0.04517
5	5		0.01400	0.01811	0.02064	0.02168
8	1		0.007207	0.008396	0.008813	0.008642
8	3		0.1729	0.2076	0.2231	0.2231
8	6		0.2744	0.3446	0.3836	0.3951
8	7		0.1270	0.1619	0.1823	0.1897
10	1		0.01027	0.1162	0.01191	0.01145
10	5		0.1.5855	1.9032	2.0452	2.0455
10	8		0.7188	0.9026	1.00474	1.03498

Table 6: Variances of Dual Generalized Order Statistics

n	i	a	0.02	0.03	0.04	0.05
$\theta = 0.9$		b	0.009	0.008	0.007	0.006
2	1		0.003358	0.004280	0.004820	0.005014
2	2		0.006215	0.008038	0.009158	0.009621
4	1		0.006527	0.008071	0.008878	0.009055
4	2		0.02408	0.03020	0.03359	0.03460
4	3		0.03381	0.04246	0.04777	0.04968
4	4		0.02065	0.02669	0.03039	0.03192
5	1		0.008457	0.01030	0.01119	0.01131
5	2		0.03890	0.04803	0.05279	0.05383
5	4		0.06630	0.08419	0.09461	0.09834
5	5		0.03091	0.03993	0.04545	0.04772
8	1		0.01596	0.01858	0.01950	0.01931
8	3		0.3544	0.4182	0.4460	0.4461
8	6		0.5345	0.6470	0.7053	0.7220
8	7		0.2661	0.3336	0.3720	0.3856
10	1		0.02273	0.02568	0.02633	0.02532
10	5		1.0094	0.6070	0.3619	0.3614
10	8		1.0806	1.1911	1.2232	1.2287

Since the equation obtained for ML estimation are in closed form, so here we apply Newton Raphson for non-linear equation and MCMC algorithm has been utilized to find out the bayesian estimates for the above said distribution. Let us utilized the dataset to the time between failures for repairable items. The data are: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40,1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17. This data has been recently utilized by [2] and follows TIKEx distribution.

By following the algorithm, we should generate data from the conditional densities and further proceed to find out the estimates for parameters. Here we use hyperparameters $\alpha_i' s = 1$, $\beta_i' s = 3$ where $i = 1, 2, 3$. The MLE and Bayesian parameters have been obtained under progressive type II censoring scheme by putting $m_i = R_i$ where $i = 1, 2, \ldots m-1$ and $K = R_m + 1$, m is the observed number of failure. we generated 1000 mcmc samples from which 200 generated samples to be burn in order to neglect the effect of initial values. From table 8, we can have the classical and bayesian estimates of the parameters. Confidence and credible intervals have also been provided. MSE's obtained represents the efficiency of Bayesian estimates over Maximum Likelihood Estimates.

Table 7: 95% Confidence Interval and Credible Interval For α, β, θ

Scheme	Parameters	Confidence Interval Lower Limit	Confidence Interval Upper Limit	Credible interval Lower Limit	Credible interval Upper Limit
Scheme I	α	0.1665	0.6134	0.5979	2.8195
	β	-0.0315	0.1115	0.1233	1.5166
	θ	0.0972	0.4827	0.8129	2.0554
Scheme II	α	14.1579	16.9820	0.4110	1.3980
	β	13.8239	16.6160	-1.0409	1.3534
	θ	14.0625	16.8774	-1.1141	0.9327

Table 8: MLE and Bayesian MCMC Estimates of Parameters

n	m	Scheme	Parameters	MLE	MCMC SEL	MCMC LINEX q=-2	MCMC LINEX q=0.1	MCMC LINEX q=3
30	20	$(11, 19^0)$	α	0.3900(0.1774)	2.9391(0.1054)	2.1331(0.1290)	1.3463(0.0295)	0.7026(0.0062)
			β	0.0400(0.0936)	1.9717(0.1459)	1.1287(0.1383)	0.4793(0.0802)	0.2675(0.0762)
			θ	0.2900(0.2754)	1.1160(0.0975)	1.4075(0.0673)	0.4219(0.1621)	0.5233(0.1885)
30	20	$(9^0, 11, 9^0)$	α	15.5700(0.3026)	0.4834(0.2436)	0.5625(0.1922)	0.2762(0.2789)	0.3268(0.3012)
			β	15.2200(0.3864)	0.4471(0.0671)	0.4623(0.0647)	-0.7808(0.1077)	0.3483(0.0815)
			θ	15.4700(0.2045)	1.0176(0.3327)	0.0541(0.2819)	-1.2793(0.3661)	-0.4580(0.3467)

6. CONCLUSIONS

Based on Generalized order statistics, we derived the single and product order moments for Truncated Inverted Kumaraswamy Exponential Distribution. We have also derived the single and product order moments of Order Statistics and Dual Generalized Order Statistics. Means and Variances of Order Statistics, Generalized Order Statistics and Dual generalized Order Statistics have been obtained from the moments derivations. From Table 1 to Table 6, we can observe that the mean and variance of OS, GOS and DGOS increases with the change in parametric values of a and b with θ being constant. Inferential properties of Generalized order Statistics deduced for estimation purposes. Classical and Bayesian inference have been utilised for estimation of parameters under progressive type ii censoring scheme by using different loss function. From Table 8, we observed that Bayesian estimates performs better than Classical ones and hence provide better understanding of observations.

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