A FAMILY OF BIAS REDUCED RATIO-TYPE ESTIMATORS: SIMPLE AND STRATIFIED RANDOM SAMPLING

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ABSTRACT
We consider applying simultaneously two methods for bias reduction for a family of ratio-type estimators with order $O(n^{-2})$, where n is the sample size. The methods considered are (a) a linear transformation to the auxiliary variable, and (b) use of Beale (1962) and Tin (1965) techniques. Expressions for the biases and variances have been derived. Numerical experiments serve as illustrations of the behavior of biases and efficiencies . The results concerning transformed ratio estimators are extended to stratified random sampling with applications to real life populations.

KEYWORDS: Bias; Efficiency; Simple Random Sampling, Stratified Random Sampling, Ratio Estimator ,Almost Unbiased Ratio Estimator .

MSC: 62D05.

RESUMEN
Consideramos aplicar simultáneamente dos métodos para reducir el sesgo de una familia de estimadores del tipo-razón con orden $O(n^{-2})$, n es el tamaño de la muestra. Los métodos considerados son (a) una transformación lineal de la variable auxiliar, y (b) usar las técnicas de Beale (1962) y Tin (1965). Expresiones para sesgos y varianzas han sido derivados. Experimentosnuméricossonilustrativos de como se comportan sesgos y varianzas. También derivamos un nuevo estimador detipo razóncompuesto que reduce el sesgo del estimadory sehace una comparaciónde los existentesmétodos. Modelos muestreo Simple yestratificadoson propuestos.

PALABRAS CLAVE: Sesgo; Eficiencia; Muestreo Simple Aleatorio, MuestreoEstratificado, Muestreo Aleatorio Simple, Estimador de Razón, Estimadores Casi Insesgados, Estimador de Razón Compuesto.

1. INTRODUCTION

Commonly sample surveys are conducted to assess development in different sectors of economy. It is a common practice to collect information on correlated auxiliary variables along with the study variable. Including them in the modeling increase the precision of the estimation of the finite population parameters. The correlation between the study variable with an auxiliary variable plays a key role. In the case of estimating the finite population mean/total, the correlation being positive, the use of the ratio method of estimation may be appropriate when looking for increasing the precision of the estimator. See some examples as Alomari et el (2015, 2016) and Subzar et al (2019) of appliacations.

The classical ratio estimator is biased, having first order bias of $O(1/n)$,n being the sample size. However, the bias may be substantially large for small sample sizes. The use of stratification with the ratio method of estimation in each stratum commonly deals with no large small sample sizes. Hence, the accumulation of the bias over strata may be large. To avoid such discerning feature of the ratio estimator the researchers in

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sampling theory have developed some bias reduction techniques. They make the bias of the ratio estimator completely removed or diminished. Recently, Lui (2020) has examined the bias reduction in a composite ratio estimator. The proposal used a weighted combination of the sample mean and the classic ratio estimator of the population mean with known optimal weight. An estimation of the optimal weight was obtained using simulations and natural populations.
Hartley-Ross (1954) used a novel technique to form an unbiased ratio estimator of the population mean/total/ratio by subtracting the estimate of bias from the mean of the ratio estimator which has constant bias. Goodman and Hartley (1958) and Robson (1957) have derived results for the variance of the Hartley-Ross unbiased ratio estimator. Quenouille (1956) reduced the bias of the ratio estimator to the order $O\left(\frac{1}{n^2}\right)$through the use of half-sample estimators. Different papers have dealt with this problem. Recently Rao and Swain (2014) constructed an alternative Hartley-Ross unbiased ratio estimator when related auxiliary variable is available. Kadilar and Cingi (2004,2006), Gupta and Shabbir (2008),Singh and Tailor (2003) and Khoshnevisan et al.(2007), among many others,have constructed transformed ratio-type estimators. These improved ratio-type estimators are less biased and more efficient than the usual ratio estimator in certain zones of preference. However, the possibility of having asignificant bias in these estimators in case of small samples still exists. As the Hartley-Ross type unbiased estimator happens to be less efficient than simple ratio estimator of the population mean under certain situations, many researchers have innovated several almost unbiased ratio estimators, whose first order biases were removed. Singh et al. (2014) derived Hartley-Ross type estimators for the population mean of the study variable, using known population parameters of auxiliary variable. Cekim and Kadilar (2016) also suggested Hartley-Ross type estimators for Khoshnevisan et al. (2007) family of estimators. Cekim and Kadilar (2017) discussed the Hartley-Ross type unbiased estimation when using stratified random sampling. For large samples, these estimators having very little bias, which may be safely ignored for practical purposes.
In this paper a family of transformed almost unbiased ratio-type estimators is derived, following both the transformation and almost unbiased ratio estimator techniques, proposed by Beale (1962) and Tin (1965) simultaneously. They reduce the bias to a lower order, that is, to $O(1/n^2)$. The results are extended to stratified sampling.

2. PRELIMINARIES AND NOTATIONS

Consider a finite population$U = \{u_1, u_2, \ldots\ldots u\}$ with N distinct and identifiable units indexed by the values of characteristic y (study variable) and x (auxiliary variable) as $\{Y_i, X_i\}, i = 1,2, \ldots, N$. Consider the selection of a simple random sample without replacement$s = \{u_{*1}, u_{*2}, \ldots\ldots, u_{*n}\}$ of size n. Denote the observed values of the characteristics on the sample units by $\{y_i, x_i\}, i = 1,2, \ldots, n$.
The involved population parameters are:
$\bar{Y} = \frac{1}{N}\sum_1^N Y_i$=population mean of y, $\bar{X} = \frac{1}{N}\sum_1^N X_i$=population mean of x;$R = \frac{\bar{Y}}{\bar{X}}$Population Ratio;
$S_y^2 = \frac{1}{N-1}\sum_1^N (Y_i - \bar{Y})^2$=finite population variance of y, $S_x^2 = \frac{1}{N-1}\sum_1^N (X_i - \bar{X})^2$=finite population variance of x.
$S_{xy} = \frac{1}{N-1}\sum_1^N (X_i - \bar{X})(Y_i - \bar{Y})$ = finite population covariance between x and y,
$C_{xy} = \frac{S_{xy}}{\bar{X}\bar{Y}}$=$\rho_{xy} C_x C_y$=Relative covariance of y and x, $C_y = \frac{S_y}{\bar{Y}}$=coefficient of variation of y, $C_x = \frac{S_x}{\bar{X}}$=coefficient of variation of x, ρ_{xy} =correlation coefficient between x and y, $\beta = \frac{S_{xy}}{S_x^2}$=population regression coefficient of$y$ on x.
The involved sample measures are:
$\bar{y} = \frac{1}{n}\sum_1^n y_i$=sample mean of y, $\bar{x} = \frac{1}{n}\sum_1^n x_i$=sample mean of x, $r = \frac{\bar{y}}{\bar{x}}$ = sample ratio

$s_y^2 = \frac{1}{n-1}\sum_1^n (y_i - \bar{y})^2$=sample variance of $y, s_x^2 = \frac{1}{n-1}\sum_1^n (x_i - \bar{x})^2$=sample variance of x,
$s_{xy} = \frac{1}{n-1}\sum_1^n (x_i - \bar{x})(y_i - \bar{y})$ =sample covariance between x and y;
$c_y = \frac{s_y}{\bar{y}}$ = sample coefficient of variation of y; $c_x = \frac{s_x}{\bar{x}}$ =
sample coefficient of variation of x; $c_{xy} = \frac{s_{xy}}{\bar{x}\bar{y}}$ = sample relative covariance of y and x.
The classical ratio estimator of the population mean $\bar{Y}$ is given by
$\hat{\bar{Y}}_r = \frac{\bar{y}}{\bar{x}}\bar{X}$, (1)
where $\bar{X}$ is known in advance.
Expanding (1) in power series (see any standard book on sampling as Sukhatme et al.,1984), keeping terms up to second degree and denoting $\theta = \left(\frac{1}{n} - \frac{1}{N}\right)$,using the results $V(\bar{y}) = \theta C_y^2$, $V(\bar{x}) = \theta C_x^2$, and $Cov(\bar{x}, \bar{y}) = \theta C_{xy}$, We have to $O(1/n)$,
$E(\hat{\bar{Y}}_r) = \bar{Y}\left[1 + \theta(C_x^2 - C_{xy})\right]$ (2)
Hence , to $O\left(\frac{1}{n}\right)$ the bias is given by

$$B(\hat{\bar{Y}}_r) = \theta\bar{Y}(C_x^2 - C_{xy})$$

$\hat{\bar{Y}}_r$is a biased but consistent estimator of $\bar{Y}$and its biasis negligible in large samples.
Alternatively, we may write
$E(\hat{\bar{Y}}_r) = \bar{Y} + \bar{Y}\theta_1(C_{20} - C_{11})$, (3)
Where$C_{ij} = \frac{\mu_{ij}}{\bar{X}^i\bar{Y}^j} = \frac{\frac{1}{N}\sum(X_i-\bar{X})^i(Y_i-Y)^j}{\bar{X}^i\bar{Y}^j}$, and $\theta_1 = \frac{N-n}{(N-1)n}$
For very large$N$, $\theta_1 \approx \theta$, and the variance of $\hat{\bar{Y}}_r$, when is valid that the order is $O(1/n)$
$V(\hat{\bar{Y}}_r) = \theta\bar{Y}^2(C_y^2 + C_x^2 - 2C_{xy})$ (4)
Alternatively is possible to write
$V(\hat{\bar{Y}}_r) = \theta\bar{Y}^2(C_{02} + C_{20} - 2C_{11})$ (5)
$\hat{\bar{Y}}_r$is more efficient than the sample mean $\bar{y}$ if
$\rho_{xy} > \frac{1}{2}\frac{C_x}{C_y}$, (6)

3. MODIFIED RATIO TYPE ESTIMATORS (ALMOST UNBIASED)

Bias reduction methods due to Beale(1962) and Tin (1965) are widely discussed in sampling theory literature. Their methods remove the bias up to$O(1/n)$ and the ultimate bias is reduced to $O(1/n^2)$.
Beale (1962) suggested estimator
$\hat{\bar{Y}}_{rb} = \hat{\bar{Y}}_r\left[\frac{1+\theta s_{xy}/\bar{x}\bar{y}}{1+\theta s_x^2/\bar{x}^2}\right]$ (7)
reduces the bias to $O(1/n^2)$.
Assuming the validity of having a positive correlation between flux and flow, Richards and Holloway(1987) and Richards (1998) used Beale's estimator for flux in different parts of U.S. Their results showed that Beale's estimator generally exhibited greater estimation accuracy and lower bias.

Tin(1965) derived another method of obtaining an almost unbiased estimator by subtracting the estimate of first order bias of $O(1/n)$ from the estimator $\hat{\bar{Y}}_r$ to get an estimator whose bias of $O(1/n)$ is removed, Thus , Tin's estimator is given by

$$\hat{\bar{Y}}_{rt} = \hat{\bar{Y}}_r \left[1 + \theta(\frac{s_{xy}}{\bar{x}\bar{y}} - \frac{s_x^2}{\bar{x}^2})\right] \quad (8)$$

Note that Beale's estimator is reduced to Tin's form after obtaining its asymptotic expansion and retaining terms up to $O(1/n)$.Considering terms up to$O(1/n^2)$, Tin (1965) has shown that Beale's estimator is less biased and equally efficient when compared to his estimator.Unlike Beale's estimator, Tin's estimator is dominated by $s_x^2/\bar{x}^2$,which is extremely large when $\bar{x}$ is small, resulting in an inflated value for the estimator. On rare occasions it may take negative values for a positive population ratio (Tin, 1965). Empirically, Tin(1965) found that Beale's estimator performs better in terms of the reduction of bias . In large samples seems that there is a marginal loss of efficiency compared to his estimator.

Linear transformations of the auxiliary variable x are often used to construct transformed ratio estimators which are less biased and more efficient than the simple ratio estimatorin certain zones of preference.

Let $z_i = Ax_i + B$, where A and B are constants or known parameters of the auxiliary variable such as C_x, S_x, ρ, among others, see Singh et al. (2014).

The transformed ratio estimator of $\bar{Y}$

$$\hat{\bar{Y}}_{tr} = \bar{y}\frac{A\bar{X} + B}{A\bar{x} + B} = \bar{y}(\frac{\bar{Z}}{\bar{z}})$$

and the transformed Hartley-Ross ratio estimator

$$\hat{\bar{Y}}_{thr} = \bar{r}_t\bar{Z} + \frac{n(N-1)}{N(n-1)}(\bar{y} - \bar{r}_t\bar{z}),$$

where $\bar{z}$ and$\bar{Z}$ are the sample mean and population mean of z respectively;

$$\bar{r}_t = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{Ax_i+B} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{z_i}.$$

Some other transformations are stated as follows:

$t_s = \bar{y}(\frac{\bar{X}}{\bar{x}})^g$, where$g$ is a suitably chosen real number, (Srivastava,1967).

$t_w = \bar{y}\frac{\bar{X}}{\alpha\bar{x}+(1-\alpha)\bar{X}}$ where α is real positive fraction, (Walsh,1970)

$t_{red} = \bar{y}\frac{\bar{X}}{\bar{x}+\alpha(\bar{x}-\bar{X})}$ where α is a positive real number, (Reddy,1974).

$t_{kh} = \bar{y}\left[\frac{A\bar{X}+B}{\alpha(A\bar{x}+B)+(1-\alpha)(A\bar{X}+B)}\right]^g$ (Khoshnevisan et al.,2007),where α is a real positive fraction and g is a real number;A and B are real constants and may be known functions of population parameters of the auxiliary variable.

Remark 3.1. $\hat{\bar{Y}}_{tr}$,t_s, t_w, and t_{red} are special cases of $t_{kh}$for suitable choices for A, B, α and$g$.

4. A PROPOSED FAMILY OF ALMOST UNBIASED RATIO-TYPE ESTIMATORS

Consider a family of transformed ratio-type estimators suggested by Khoshnevisan, et al. (2007) given by

$$t_{kh} = \bar{y}\left[\frac{A\bar{X}+B}{\alpha(A\bar{x}+B)+(1-\alpha)(A\bar{X}+B)}\right]^g, \quad (9)$$

where α, A, B and g are defined earlier. This estimator incorporates both linear and exponential transformations.

Let us determine the expected value and variance of the family of ratio-type estimators with kernel t_{kh} and their almost unbiased ratio-type estimators

Define

$$\bar{y}=\bar{Y}(1+e_o),\bar{x}=\bar{X}(1+e_1),s_{xy}=S_{xy}(1+e_2),s_x^2=S_x^2(1+e_3);\ E(e_i)=0,i=0,1,2,3$$

$$\hat{\hat{\bar{Y}}}_{rb}=\hat{\hat{\bar{Y}}}_r\left[\frac{1+\theta s_{xy}/\bar{x}\bar{y}}{1+\theta s_x^2/\bar{x}^2}\right]$$

Assume that for all possible samples

$$|e_i|<1,i=0,1,2,3$$

Expanding t_{kh} in power series and retaining terms up to degree four, we have

$$t_{kh}=\bar{Y}(1-\delta\alpha g e_1+(\delta\alpha)^2\frac{g(g+1)}{2}e_1^2-(\delta\alpha)^3\frac{g(g+1)(g+2)}{6}e_1^3+(\delta\alpha)^4\frac{g(g+1)(g+2)(g+3)}{24}e_1^4$$
$$+e_0-(\delta\alpha)ge_1e_0+(\delta\alpha)^2\frac{g(g+1)}{2}e_1^2e_0-(\delta\alpha)^3\frac{g(g+1)(g+2)}{6}e_1^3e_0+\ldots\ldots)$$
$$=\bar{Y}(1-\lambda_1e_1+\lambda_2e_1^2-\lambda_3e_1^3+\lambda_4e_1^4+e_0-\lambda_1e_1e_0+\lambda_2e_1^2e_0-\lambda_3e_1^3e_0+\ldots\ldots.),$$

Where

$$\lambda_1=(\delta\alpha)g,\lambda_2=(\delta\alpha)^2\frac{g(g+1)}{2},\lambda_3=(\delta\alpha)^3\frac{g(g+1)(g+2)}{6},\lambda_4=(\delta\alpha)^4\frac{g(g+1)(g+2)(g+3)}{24}$$

and $\delta=\frac{A\bar{X}}{A\bar{X}+B}$.

Take the classic asymptotic expansion of the ratio estimator and approximating the finite population correction factors for large N:θ, we have to $O(1/n^2)$

$$E(t_{kh})=\bar{Y}+\bar{Y}[\theta(\lambda_2C_{20}-\lambda_1C_{11})+\theta^2(-\lambda_3C_{30}+3\lambda_4C_{20}^2+\lambda_2C_{21}-3\lambda_3C_{20}C_{11})] \quad (10)$$

Now,

$$E(t_{kh}^2)=\bar{Y}^2+\bar{Y}^2\theta[(2\lambda_2+\lambda_1^2)C_{20}-4\lambda_1C_{11}+C_{02}]$$
$$+\bar{Y}^2\theta^2[(4\lambda_2+2\lambda_1^2)C_{21}-(2\lambda_3+2\lambda_1\lambda_2)C_{30}-2\lambda_1C_{12}]$$
$$+\bar{Y}^2\theta^2[3(2\lambda_4+\lambda_2^2+2\lambda_1\lambda_3)C_{20}^2-(12\lambda_3+10\lambda_1\lambda_2)C_{20}C_{11}+(4\lambda_2+2\lambda_1^2)C_{11}^2+(2\lambda_2+\lambda_1^2)C_{20}C_{02}]$$

$$V(t_{kh})=\bar{Y}^2\theta(\lambda_1^2C_{20}-2\lambda_1C_{11}+C_{02}+\bar{Y}^2\theta^2\{C_{20}^2(2\lambda_2^2+6\lambda_1\lambda_3)+C_{11}^2(\lambda_1^2+4\lambda_2)+C_{20}C_{02}(\lambda_1^2+2\lambda_2)+C_{30}(-2\lambda_1\lambda_2)\}$$
$$+\bar{Y}^2\theta^2\{C_{21}(2\lambda_1^2+2\lambda_2)+C_{20}C_{11}(-6\lambda_3-10\lambda_1\lambda_2)-2\lambda_1C_{12}\} \quad (11)$$

when $\lambda_i=1$ for $i=1,2,3,4$,

$$E\left(\hat{\hat{\bar{Y}}}_r\right)=\bar{Y}+\bar{Y}[\theta(C_{20}-C_{11})+\theta^2(C_{30}+3C_{20}^2+C_{21}-3C_{20}C_{11})] \quad (12)$$

$$V(\hat{\hat{\bar{Y}}}_r)=\bar{Y}^2\theta\{C_{20}-2C_{11}+C_{02}\}+\bar{Y}^2\theta^2\{8C_{20}^2+5C_{11}^2+3C_{20}C_{02}-2C_{30}+4C_{21}-16C_{20}C_{11}-2C_{12}\} \quad (13)$$

The expressions for $E(\hat{\hat{\bar{Y}}}_r)$and $V(\hat{\hat{\bar{Y}}}_r)$ are the same as those derived by Tin(1965) . Beale-type (1962) almost unbiased ratio estimator of $\bar{Y}$ using t_{kh} is given by

$$t_{khb}=t_{kh}\left[\frac{1+\lambda_1\theta\frac{s_{xy}}{\bar{x}\bar{y}}}{1+\lambda_2\theta\frac{s_x^2}{\bar{x}^2}}\right] \quad (14)$$

After expanding t_{khb} in power series with the assumptions$|e_i|<1,i=1,2,3,4$ for all possible samples and some mathematical simplifications, we write the expressions for the expected value and variance of t_{khB} to$O(1/n^2)$ as correcting the small b in place of B

$$E(t_{khb})=\bar{Y}+\bar{Y}\theta^2[(-\lambda_1-\lambda_1^2)C_{21}+(-\lambda_3+2\lambda_2+\lambda_1\lambda_2)C_{30}]+\bar{Y}\theta^2[(3\lambda_4-2\lambda_1\lambda_2-3\lambda_2)C_{20}^2+(-3\lambda_3+\lambda_1\lambda_2+2\lambda_2+\lambda_1+\lambda_1^2)C_{20}C_{11}] \quad (15)$$

$$E(t_{khb}^2) = \bar{Y}^2 + \bar{Y}^2\theta(\lambda_1^2 C_{20} - 2\lambda_1 C_{11} + C_{02}) + \bar{Y}^2\theta^2[2(-\lambda_3 + \lambda_1\lambda_2 + 2\lambda_2)C_{30} - 2(\lambda_1^2 + \lambda_1)C_{21}] + \bar{Y}^2\theta^2[(4\lambda_2 - 2\lambda_1 - \lambda_1^2)C_{11}^2 + \lambda_1^2 C_{20}C_{02} + (6\lambda_4 + 6\lambda_1\lambda_3 + 2\lambda_2^2 - 8\lambda_1\lambda_2 - 6\lambda_2 - 2\lambda_1^2\lambda_2)C_{20}^2] + \bar{Y}^2\theta^2(-12\lambda_3 - 4\lambda_1\lambda_2 + 4\lambda_1^2 + 2\lambda_1 + 8\lambda_2 + 2\lambda_1^3)C_{20}C_{11}$$

$$V(t_{khb}) = \bar{Y}^2\theta(\lambda_1^2 C_{20} - 2\lambda_1 C_{11} + C_{02}) + \bar{Y}^2\theta^2[(4\lambda_2 - 2\lambda_1 - \lambda_1^2)C_{11}^2 + \lambda_1^2 C_{20}C_{02}] + \bar{Y}^2\theta^2[(6\lambda_1\lambda_3 - 4\lambda_1\lambda_2 + 2\lambda_2^2 - 2\lambda_1^2\lambda_2)C_{20}^2 + (2\lambda_1^3 + 2\lambda_1^2 + 4\lambda_2 - 6\lambda_3 - 6\lambda_1\lambda_2)C_{20}C_{11}] \quad (16)$$

Substituting,

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1,$$

$$E(\hat{\bar{Y}}_{rb}) = \bar{Y} + \bar{Y}\theta^2\{2C_{30} - 2C_{20}^2 + 2C_{20}C_{11} - 2C_{21}\} \quad (17)$$

$$V(\hat{\bar{Y}}_{rb}) = \bar{Y}^2[\theta(C_{20} - 2C_{11} + C_{02}) + \theta^2(C_{11}^2 + C_{20}C_{02} + 2C_{20}^2 - 4C_{20}C_{11})] \quad (18)$$

which are given in Tin(1965).

Further, following Tin (1965) another almost unbiased ratio-type estimator is

$$t_{kht} = t_{kh}(1 + \lambda_1\theta\frac{s_{xy}}{\bar{x}\bar{y}} - \lambda_2\theta\frac{s_x^2}{\bar{x}^2} = t_{kh}[1 + \lambda_1\theta C_{11}(1+e_0)^{-1}(1+e_1)^{-1}(1+e_2) - \lambda_2\theta C_{20}(1+e_3)(1+e_1)^{-2}]$$

$$= t_{kh}[1 + \lambda_1\theta C_{11}(1 - e_1 + e_1^2 - e_0 + e_1e_0 + e_0^2 + e_2 - e_1e_2 - e_0e_2) - \lambda_2\theta C_{20}(1 - 2e_1 + 3e_1^2 - 4e_1^3 + e_3 - 2e_1e_3)] \quad (19)$$

$$E(t_{kht}) = \bar{Y} + \bar{Y}\theta^2[(-\lambda_1 - \lambda_1^2)C_{21} + (-\lambda_3 + 2\lambda_2 + \lambda_1\lambda_2)C_{30}] + \bar{Y}\theta^2[(3\lambda_4 - \lambda_2^2 - 2\lambda_1\lambda_2 - 3\lambda_2)C_{20}^2 + (-3\lambda_3 + 2\lambda_1\lambda_2 + 2\lambda_2 + \lambda_1 + \lambda_1^2)C_{20}C_{11}] \quad (20)$$

$$E(t_{kht}^2) = \bar{Y}^2 + \bar{Y}^2\theta(\lambda_1^2 C_{20} - 2\lambda_1 C_{11} + C_{02}) + \bar{Y}^2\theta^2[2(-\lambda_3 + \lambda_1\lambda_2 + 2\lambda_2)C_{30} - 2(\lambda_1^2 + \lambda_1)C_{21}] + \bar{Y}^2\theta^2[(4\lambda_2 - 2\lambda_1 - \lambda_1^2)C_{11}^2 + \lambda_1^2 C_{20}C_{02} + (6\lambda_4 + 6\lambda_1\lambda_3 - 6\lambda_2 - 8\lambda_1\lambda_2 - 2\lambda_1^2\lambda_2)C_{20}^2] + \bar{Y}^2\theta^2(-12\lambda_3 - 2\lambda_1\lambda_2 + 4\lambda_1^2 + 2\lambda_1 + 8\lambda_2 + 2\lambda_1^3)C_{20}C_{11}$$

$$V(t_{kht}) = \bar{Y}^2\theta(\lambda_1^2 C_{20} - 2\lambda_1 C_{11} + C_{02}) + \bar{Y}^2\theta^2[(4\lambda_2 - 2\lambda_1 - \lambda_1^2)C_{11}^2 + \lambda_1^2 C_{20}C_{02}] + \bar{Y}^2\theta^2[(6\lambda_1\lambda_3 - 4\lambda_1\lambda_2 + 2\lambda_2^2 - 2\lambda_1^2\lambda_2)C_{20}^2 + (2\lambda_1^3 + 2\lambda_1^2 + 4\lambda_2 - 6\lambda_3 - 6\lambda_1\lambda_2)C_{20}C_{11}] \quad (21)$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$$

when

$$E(\hat{\bar{Y}}_{rt}) = \bar{Y} + \bar{Y}\theta^2(2C_{30} - 3C_{20}^2 + 3C_{20}C_{11} - 2C_{21}) \quad (22)$$

$$V(\hat{\bar{Y}}_{rt}) = V\left(\hat{\bar{Y}}_{rb}\right) \quad (23)$$

Remark 4.2: The expressions for $E(\hat{\bar{Y}}_{rb})$,$E(\hat{\bar{Y}}_{rt})$, $V(\hat{\bar{Y}}_{rb})$ and $V\left(\hat{\bar{Y}}_{rt}\right)$ are derived by Tin (1965) using bivariate cumulants and by De-Graft Johnson (1969) using bivariate moments.

Some of the higher order bivariate moments neglecting finite population correction factor for large finite population mentioned by De-Graft Johnson (1969) are given below:

$$E(e_1^2 e_0^2) = \frac{1}{n^2}(2C_{11}^2 + C_{20}C_{02});\ E(e_1^4) = \frac{3C_{20}^2}{n^2};\ E(e_1^3 e_0) = \frac{1}{n^2}3C_{20}C_{11};E(e_1e_2) = \frac{1}{n}\frac{C_{21}}{C_{11}};E(e_1e_3) = \frac{1}{n}\frac{C_{30}}{C_{20}};E(e_0e_2) = \frac{1}{n}\frac{C_{12}}{C_{11}}.$$

4.2 Comparison of bias and variance of the class of Beale type and Tin type almost unbiased ratio-type estimators.

Consider the situation when both y and x follow a bivariate symmetric distribution with zero-odd order moments We have:

$$B(t_{khb}) = \bar{Y}\theta^2[(3\lambda_4 - 2\lambda_1\lambda_2 - 3\lambda_2)C_{20}^2 + (-3\lambda_3 + \lambda_1\lambda_2 + 2\lambda_2 + \lambda_1 + \lambda_1^2)C_{20}C_{11}] = \bar{Y}\theta^2(B_1C_{20}^2 + B_2C_{20}C_{11})$$

where $B_1 = (3\lambda_4 - 2\lambda_1\lambda_2 - 3\lambda_2)$ and $B_2 = (-3\lambda_3 + \lambda_1\lambda_2 + 2\lambda_2 + \lambda_1 + \lambda_1^2)$

$$B(t_{kht}) = \bar{Y}\theta^2[(3\lambda_4 - \lambda_2^2 - 2\lambda_1\lambda_2 - 3\lambda_2)C_{20}^2 + (-3\lambda_3 + 2\lambda_1\lambda_2 + 2\lambda_2 + \lambda_1 + \lambda_1^2)C_{20}C_{11}] = \bar{Y}\theta^2(T_1C_{20}^2 + T_2C_{20}C_{11}),$$

where $T_1 = (3\lambda_4 - \lambda_2^2 - 2\lambda_1\lambda_2 - 3\lambda_2)$ and $T_2 = (-3\lambda_3 + 2\lambda_1\lambda_2 + 2\lambda_2 + \lambda_1 + \lambda_1^2)$

Hence, t_{khb} will be less biased than t_{kht} if

$$\left|B_1 + B_2\frac{\beta}{R}\right| < \left|T_1 + T_2\frac{\beta}{R}\right| \quad (24)$$

Further,

$$V(t_{khb}) = V(t_{kht}) \quad (25)$$

4.3 Comparison of transformed and untransformed ratio estimators:

Accepting the validity of the assumed first order of approximation,

$$E(\hat{\bar{Y}}_r) = \bar{Y} + \bar{Y}[\theta(C_{20} - C_{11})]$$

$$E(t_{kh}) = \bar{Y} + \bar{Y}[\theta(\lambda_2C_{20} - \lambda_1C_{11})],$$

Therefore, transformed ratio estimator is less biased than simple ratio estimator if

$$\left|\lambda_2 - \lambda_1\frac{\beta}{R}\right| < \left|1 - \frac{\beta}{R}\right| \quad (26)$$

Further, the transformed ratio estimator is more efficient than both the untransformed simple ratio estimator and the simple mean per unit estimator if

$$\frac{1}{2}\lambda_1 < \frac{C_{11}}{C_{20}} < \frac{1}{2}(\lambda_1 + 1) \text{ or } \frac{1}{2}\lambda_1\sqrt{\frac{C_{20}}{C_{02}}} < \rho_{xy} < \frac{1}{2}(\lambda_1 + 1)\sqrt{\frac{C_{20}}{C_{02}}} \quad (27)$$

4.4 Special Cases

Let us consider special cases of t_{kh} when $g = 1, \alpha = 1$ and $A = 1$, and B may be either (i) S_x or (ii) C_x or (iii) ρ_{xy}. These special cases of t_{kh}, in the form of transformed ratio estimators, are given as follows:

$$t_1 = \bar{y}\frac{\bar{X}+S_x}{\bar{x}+S_x};\ t_2 = \bar{y}\frac{\bar{X}+C_x}{\bar{x}+C_x};\ t_3 = \bar{y}\frac{\bar{X}+\rho_{xy}}{\bar{x}+\rho_{xy}};\ t_4 = \bar{y}(\frac{\bar{X}}{\bar{x}})$$

Define $\delta_1 = \frac{\bar{X}}{\bar{X}+S_x}, \delta_2 = \frac{\bar{X}}{\bar{X}+C_x}, \delta_3 = \frac{\bar{X}}{\bar{X}+\rho}, \delta_4 = 1$

To $O(1/n^2)$, the expressions for $E(t_i), V(t_i), E(t_{ib}), V(t_{ib}), E(t_{it})$ and $V(t_{it})$ are obtained by substituting $\lambda_i = \delta_j^i, j = 1,2,3,4; i = 1,2,3,4$ in the relevant expressions in (10),(11),(15),(16), (20) and (21) respectively. Thus, assuming a bivariate symmetric distribution for (y , x),we have

$$B(t_i) = \bar{Y}[\theta(\delta_i^2C_{20} - \delta_iC_{11}) + \theta^2(3\delta_i^4C_{20}^2 - 3\delta_i^3C_{20}C_{11})],\ i=1 \quad (28)$$

$$V(t_i) = \bar{Y}^2\theta(\delta_i^2C_{20} - 2\delta_iC_{11} + C_{02}) + \bar{Y}^2\theta^2\{8\delta_i^4C_{20}^2 + 5\delta_i^2C_{11}^2 + 3\delta_i^2C_{20}C_{02} - 16\delta_i^3C_{20}C_{11}\} \quad (29)$$

$$B(t_{it}) = \bar{Y}\theta^2[(2\delta_i^4 - 2\delta_i^3 - 3\delta_i^2)C_{20}^2 + (-\delta_i^3 + 3\delta_i^2 + \delta_i)C_{20}C_{11}] \quad (30)$$

$$V(t_{it}) = \bar{Y}^2\theta(\delta_i^2 C_{20} - 2\delta_i C_{11} + C_{02}) + \bar{Y}^2\theta^2[(3\delta_i^2 - 2\delta_i)C_{11}^2 + \delta_i^2 C_{20}C_{02}] + \bar{Y}^2\theta^2[(6\delta_i^4 - 4\delta_i^3)C_{20}^2 + (6\delta_i^2 - 10\delta_i^3)C_{20}C_{11}] \quad (31)$$

$$B(t_{ib}) = \bar{Y}\theta^2[(3\delta_i^4 - 2\delta_i^3 - 3\delta_i^2)C_{20}^2 + (-2\delta_i^3 + 3\delta_i^2 + \delta_i)C_{20}C_{11}] \quad (32)$$

$$V(t_{ib}) = \bar{Y}^2\theta(\delta_i^2 C_{20} - 2\delta_i C_{11} + C_{02}) + \bar{Y}^2\theta^2[(3\delta_i^2 - 2\delta_i)C_{11}^2 + \delta_i^2 C_{20}C_{02}] + \bar{Y}^2\theta^2[(6\delta_i^4 - 4\delta_i^3)C_{20}^2 + (-10\delta_i^3 + 6\delta_i^2)C_{20}C_{11}] \quad (33)$$

$$V(t_{ib}) = V(t_{it}) \quad (34)$$

5. EMPIRICAL ILLUSTRATIONS

To compare the almost unbiased ratio -type estimators corresponding to t_1, t_2, t_3, t_4 we are considering four examples
given by Murthy (1967), and Kadilar and Cingi (2004) respectively. We assume that (y, x) has a bivariate symmetric distribution. The comparison of biases and variances of Beale type and Tin type almost unbiased estimators are given in Table-1(excepting constant multipliers).

Example 1, (Murthy,1967)

X=Geographical area of village (in acres), Y=Area under paddy (in acres)

N=108 and n=20 ,$\theta = \frac{N-n}{Nn}$ $\delta_1 = \frac{\bar{X}}{\bar{X}+S_x}$ $\delta_2 = \frac{\bar{X}}{\bar{X}+C_x}$ $\delta_3 = \frac{\bar{X}}{\bar{X}+\rho}$ $\delta_4 = 1$

$\bar{X} = 460.9259$,$\bar{Y} = 172.1481$,$Var(x) = 101645.27$,$Var(y) = 18187.62$

$C_{02=0.6077}$,$C_{20} = 0.4740$,$C_{11} = 0.4211$,$S_x = 318.82$,$\rho = 0.7845$

Example 2. (Murthy,1967)

X=Number of workers, Y=Number of absentees

N=43 and n=10, $\theta = \frac{N-n}{Nn}$ $\delta_1 = \frac{\bar{X}}{\bar{X}+S_x}$ $\delta_2 = \frac{\bar{X}}{\bar{X}+C_x}$ $\delta_3 = \frac{\bar{X}}{\bar{X}+\rho}$ $\delta_4 = 1$

$\bar{X} = 79.4651$,$\bar{Y} = 9.6512$,$Var(x) = 1330.2597$,$Var(y) = 43.1373$

$C_{02=0.4523}$,$C_{20} = 0.2058$,$C_{11} = 0.2016$,$C_x = 0.4628$,$S_x = 36.2726$,$\rho = 0.6607$

Example 3 (Murthy, 1967)

Y =Output in factory, X=Number of workers

N =80, n =20,$\theta = \frac{N-n}{Nn}$ $\delta_1 = \frac{\bar{X}}{\bar{X}+S_x}$ $\delta_2 = \frac{\bar{X}}{\bar{X}+C_x}$ $\delta_3 = \frac{\bar{X}}{\bar{X}+\rho}$ $\delta_4 = 1$

$\bar{Y} = 51.8264, \bar{X} = 2.8513, C_x = 0.3542, C_y = 0.9484, S_x = 1.0099, \rho = 0.9150$

$C_{02} = 0.8995, C_{20} = 0.1254, C_{11} = 0.3074$

Example 4 (Kadilar and Cingi, 2004)

Y =Level of apple production, X =Number of apple trees

$\delta_1 = \frac{\bar{X}}{\bar{X}+S_x}$ $\delta_2 = \frac{\bar{X}}{\bar{X}+C_x}$ $\delta_3 = \frac{\bar{X}}{\bar{X}+\rho}$ $\delta_4 = 1$

N=106, n =20,

$\bar{Y} = 2212.59, \bar{X} = 27421.70, C_x = 2.10, C_y = 5.22, S_x = 57585.57, \rho = 0.86$

$C_{02} = 27.2484, C_{20} = 4.41, C_{11} = 9.4273$

Table-1. Comparison of biases and variances of Transformed ratio estimators corresponding to Beale type and Tin type almost unbiased ratio-type estimators

(Excepting constant multipliers)

Estimators	$\lvert B(t_i)\rvert$	$V(t_i)$	$\lvert B(t_{ib})\rvert$	$V(t_{ib})$	$\lvert B(t_{it})\rvert$	$V(t_{it})$
Pop-1						
t_1	0.996538	0.011508	0.2109E-05	0.011326	0.2079E-4	0.011326
t_2	1.002246	0.010337	0.83727E-04	0.009943	0.12461E-03	0.009943
t_3	1.002241	0.01335	0.83794E-04	0.009942	0.12457E-03	0.009942
t_4	1.002280	0.010345	0.832580E-04	0.009950	0.12486E-03	0.009950
Pop-2						
t_1	0.996751	0.021429	0 .7219E-05	0.021089	0.3087E-04	0.021089

t_2	1.000240	0.024752	0.1155E-04	0.019864	0.1513E-04	0.019864
t_3	1.000199	0.020469	0.1209E-04	0.019862	0.1505E-04	0.019862
t_4	1.000338	0.020489	0.1018E-04	0.019871	0.1527E-04	0.019871
Pop-3						
t_1	0.994006	0.019596	0.5079E-04	0.019341	0.6605E-04	0.019341
t_2	0.993394	0.017347	0.5850E-04	0.017045	0.8281E-04	0.017045
t_3	0.993919	0.019308	0.5178E-04	0.019047	0.6084E-04	0.019047
t_4	0.993079	0.011583	0.6419E-04	0.015498	0.9628E-04	0.015498
Pop-4						
t_1	0.889402	0.981100	0.027734	0.866657	0.029684	0.866657
t_2	0.687254	1.005283	0.072817	0.653764	0.109222	0.653764
t_3	0.687243	1.005282	0.072820	0.653752	0.109229	0.653752
t_4	0.687235	1.005282	0.072822	0.653743	0. 092331	0.653743

The computations in Table1show that there have been a serious reduction of bias and an increase in efficiency for the transformed ratio estimators. Note that we have an advance knowledge of the population parameters (standard deviations, coefficients of variation of x, and of the coefficient of correlation). Further, a substantial reduction in bias and variance has also been achieved by the application of the almost unbiased estimator techniques proposed by Beale and Tin. Beale estimator was less biased and equally efficient compared to Tin´s one.

6. STRATIFIED EXTENSION WITH THE SEPARATE ESTIMATOR

We will develop a stratification extension for using the estimators developed in the previous section in each stratum. Stratified sampling considers the selection of independent samples from each stratum

$$U_i = \{u_{i0}, \ldots, u_{iN_i}\}, i = 1, \ldots, K; N = \sum_{i=1}^{K} N_i$$

Independent samples

$$s_i = \{u_{ij}^*,\ \ j = 1, \ldots, n_i\};\ s_i \subset U_i;\ i = 1, \ldots, K;$$

Are selected using SRSWR and denoting Z=Y, X the means of the population may be written as

$$\bar{Z} = \frac{1}{N}\sum_{1}^{N} Z_i = \sum_{i=1}^{K} W_i \bar{Z}_i$$

$where\ \bar{Z}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} Zij,\ W_i = \frac{N_i}{N};\ Z_i = \sum_{i=1}^{K} N_i \bar{Z}_i$

We will consider valid that $\bar{X}_i$ is known in advance and that is valid the approximation order $O(1/n_i)$ for each i=1,..,K. Then

$R_i = \frac{\bar{Y}_i}{\bar{X}_i}$ is the i´th stratum Population Ratio;

$S_{Z(i)}^2 = \frac{1}{N_i - 1}\sum_{j=1}^{N_i}(Z_{ij} - \bar{Z}_i)^2$ is the finite population variance of z *in* U_i

$S_{xy(i)} = \frac{1}{N_i - 1}\sum_{j=1}^{N_i}(X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i)$ *is the* finite population covariance between X and y *in* U_i,

$C_{xy(i)} = \frac{S_{xy(i)}}{\bar{X}_i \bar{Y}_i} = \rho_{xy(i)} C_{x(i)} C_{y(i)}$=Relative covariance of y and x in U_i

$C_{z(i)} = \frac{S_{z(i)}}{\bar{Z}_i}$=coefficient of variation of z *in* U_i,

$\rho_{xy(i)}$ =correlation coefficient between x and y, *in* U_i

$\beta_i = \frac{S_{xy(i)}}{S^2_{x(i)}}$=population regression coefficient of y on x *in* U_i.

The involved sample measures are :

$\bar{z}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} z_{ij}$=sample mean of z *in* U_i,

$r_i = \frac{\bar{y}_i}{\bar{x}_i}$ = sample ratio *in* U_i

$s^2_{z_i} = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(z_{ij} - \bar{z}_i)^2$=sample variance of z, *in* U_i

$s_{xy(i)} = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)(y_{ij} - \bar{y}_i)$ =sample covariance between x and y *in* U_i

The classic ratio estimator of the population mean *in* U_i is given by

$$\hat{\bar{Y}}_{r_i} = \frac{\bar{y}_i}{\bar{x}_i}\bar{X}_i,$$

$$E(\hat{\bar{Y}}_{r_i}) = \bar{Y}_i\left[1 + \theta_i(C^2_{x(i)} - C_{xy(i)})\right] = \bar{Y}_i + \bar{Y}_i\left[\theta_i(C^2_{x(i)} - C_{xy(i)})\right] = \bar{Y}_i + B_i = \bar{Y}_i + \bar{Y}_i\theta_{1(i)}(C_{20(i)} - C_{11(i)})$$

$where\ C_{th(i)} = \frac{\mu_{th}}{\bar{X}^t{}_i\bar{Y}^h{}_i} = \frac{\frac{1}{N_i}\sum_{j=1}^{N_i}(X_{ij}-\bar{X}_i)^t(Y_{ij}-Y)^h}{\bar{X}^t{}_i\bar{Y}^h{}_i}$, and $\theta_{1(i)} = \frac{N_i-n_i}{(N_i-1)n_i}$

For very large N_i, $\theta_{1(i)} \approx \theta_i$

$$V(\hat{\bar{Y}}_{r_i}) = \theta_i\,\bar{Y}^2{}_i(C^2_{y(i)} + C^2_{x(i)} - 2C_{xy(i)})$$

Then the efficiency of each $\hat{\bar{Y}}_{r_i}$ is supported by $\rho_{xy(i)} > \frac{1}{2}\frac{C_{x(i)}}{C_{y(i)}}$,

We will consider extending the estimators with the better behavior. The extensions to Stratified Sampling yields that for the transformed separate type ratio estimators models are:

S1:

$$t_{S1} = \sum_{i=1}^{K} W_i\left(\bar{y}_i\frac{\bar{X}_i + S_{x_i}}{\bar{x}_i + S_{x_i}}\right)$$

Under bivariate symmetric distributions,

$$B(t_{S1}) = \sum_{i=1}^{K} W_i\left(\bar{Y}_i\left[\theta_i(\delta_1^2C_{20(i)} - \delta_1C_{11(i)}) + \theta^2{}_i(3\delta_1^4C^2_{20(i)} - 3\delta_1^3C_{20(i)}C_{11(i)})\right]\right)$$

$$V(t_{S1}) = \sum_{i=1}^{K} W_i^2\left(\bar{Y}_i^2\theta_i\left(\delta_1^2C_{20(i)} - 2\delta_1C_{11(i)} + C_{02(i)}\right) + \bar{Y}_i^2\theta^2{}_1\left\{8\delta_1^4C^2_{20(i)} + 5\delta_1^2C^2_{11(i)} + 3\delta_1^2C_{20(i)}C_{02(i)} - 16\delta_1^3C_{20(i)}C_{11(i)}\right\}\right)$$

S2:

$$t_{S2} = \sum_{i=1}^{K} W_i\left(\bar{y}_i\frac{\bar{X}_i + C_{x_i}}{\bar{x}_i + C_{x_i}}\right)$$

$$B(t_{S2}) = \sum_{i=1}^{K} W_i\left(\bar{Y}_i\left[\theta_i(\delta_2^2C_{20(i)} - \delta_2C_{11(i)}) + \theta^2{}_i(3\delta_2^4C^2_{20(i)} - 3\delta_2^3C_{20(i)}C_{11(i)})\right]\right)$$

$$V(t_{S2}) = \sum_{i=1}^{K} W_i^2\left(\bar{Y}_i^2\theta_i\left(\delta_2^2C_{20(i)} - 2\delta_2C_{11(i)} + C_{02(i)}\right) + \bar{Y}_i^2\theta^2{}_i\left\{8\delta_2^4C^2_{20(i)} + 5\delta_2^2C^2_{11(i)} + 3\,\delta_2^2\,C_{20(i)}C_{02(i)} - 16\delta_2^3C_{20(i)}C_{11(i)}\right\}\right)$$

S3:

$$t_{S3} = \sum_{i=1}^{K} W_i \left(\bar{y}_i \frac{\bar{X}_i + \rho_{x_i, y_i}}{\bar{x}_i + \rho_{x_i, y_i}} \right)$$

$$B(t_{S3}) = \sum_{i=1}^{K} W_i \left(\bar{Y}_i \left[\theta_i (\delta_3^2 C_{20(i)} - \delta_3 C_{11(i)}) + \theta^2{}_i (3\delta_3^4 C_{20(i)}^2 - 3\delta_3^3 C_{20(i)} C_{11(i)}) \right] \right)$$

$$V(t_{S3}) = \sum_{i=1}^{K} W_i^2 \left(\bar{Y}_i^2 \theta_i (\delta_3^2 C_{20(i)} - 2\delta_3 C_{11(i)} + C_{02(i)}) + \bar{Y}_i^2 \theta^2{}_i \{ 8\delta_3^4 C_{20(i)}^2 + 5\delta_3^2 C_{11(i)}^2 + 3\delta_3^2 C_{20(i)} C_{02(i)} - 16\delta_3^3 C_{20(i)} C_{11(i)} \} \right)$$

M4

$$t_{S4} = \sum_{i=1}^{K} W_i \bar{y}_i \left(\frac{\bar{X}_i}{\bar{x}_i} \right)$$

$$B(t_{S4}) = \sum_{i=1}^{K} W_i \left(\bar{Y}_i \left[\theta_i (C_{20(i)} - \delta_1 C_{11(i)})) + \theta^2{}_i (3C_{20(i)}^2 - 3C_{20} C_{11(i)}) \right] \right)$$

$$V(t_{S4}) = \sum_{i=1}^{K} W_i^2 \left(\bar{Y}_i^2 \theta_i (C_{20(i)} - 2C_{11(i)} + C_{02(i)}) + \bar{Y}_i^2 \theta^2{}_i \{ 8C_{20(i)}^2 + 5C_{11(i)}^2 + 3C_{20(i)} C_{02(i)} - 16C_{20(i)} C_{11(i)} \} \right)$$

We develop a study with the population data of 5 real life populations and computed biases and variances of them. The populations are:

- Population 1-4. Leaching of elements from solid waste compost. (plumb, magnesium , cadmium and others). x=report of the factories of the level of contaminants, y= Environmental Control laboratory results. N=1785. K=7.
- Population 5. Study of acute inhalation risk assessment. x, y= laboratory results. K=10
- Population 6. Measurement of serum-prevalence in a population of COVID-19 patients by age group. x=hospital report y=report after vaccination. N=3400, K=5.
- Population 7-9. Measurements of Dissolved Oxygen, BDO (Biological Oxygen Demand), pH, Nitrate, x=measures with a portable kit. Y=measures in a laboratory. N=780.
- Population 10. Evaluation of resilience. x=self-report, y=psychologic test. N=2318, K=10.

The sampling fraction was 0.1 for all the strata. We computed $RB(h) = \left| \frac{B(t_{Sh})}{t_{Sh}} \right|$ *and* $RV(b) = \left| \frac{V(t_{Sh})}{t_{Sh}} \right|$ for h =1,…,4. See the results in Table 2.

Table-2. Comparison of biases and variances of the estimation in stratified models

Estimators	t_{S1}	t_{S2}	t_{S3}	t_{S4}	t_{S1}	t_{S2}	t_{S3}	t_{S4}
		Relative	Absolute	Biases		Relative	Variances	
Population 1	0,532	0,289	0,427	**0,806**	0,857	0,695	0,597	**0,909**
Population2	0,050	0,620	0,886	**0,905**	0,200	1,094	0,066	**1,188**
Population3	0,329	0,725	0,843	**1,200**	0,470	**2,72**	0,670	2,006
Population4	0,596	0,390	0,687	**0,990**	0,963	1,059	0,936	**1,228**

Population5	0,075	0,097	0,192	**0,857**	0,063	0,673	0,688	**0,775**
Population6	0,243	0,088	0,890	**1,405**	0,063	0,695	0,988	**1,075**
Population7	0,479	0,238	0,560	**0,600**	0,884	0,707	0,796	**0,882**
Population8	0,442	0,206	0,849	**1,499**	0,376	0,732	0,106	**0,984**
Population9	0,470	0,672	0,770	**0,795**	0,843	0,769	0,948	**1,706**
Population10	0,463	0,059	0,536	**0,990**	0,687	0,596	0,390	**0,987**

See that t_{s4} is, as expected, the worse estimator in terms of the relative biases, and except in population 3 the same result is obtained in term o rel-variance. Both t_{S1} and t_{S2} have the smaller rel-biases in the 50% of the populations. For the 60% of the populations t_{s1} has the smallest rel-variances while t_{s2} never is the best alternative. Hence , considering the experiments is recommended preferring t_{S1}.

7. CONCLUSIONS

This paper deals with constructing a family of almost unbiased ratio estimators, which are hitherto not available in literature. The biases and variances of these estimators are derived up to $O(1/n^2)$. As special cases, we have considered transformed ratio estimators when the population standard deviation and the coefficient variation of the auxiliary variable x and the coefficient of correlation are known in advance or approximately known from the past surveys. For four natural populations under consideration the Beale type estimator is found to be least biased and more efficient than both transformed ratio estimators and simple ratio estimator. Further, an extension to stratified random sampling with transformed ratio estimator in each stratum has been suggested with applications to real life populations.

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