MODIFIED EXPONENTIAL ESTIMATORS USING AUXILIARY INFORMATION UNDER RESPONSE AND NON-RESPONSE

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ABSTRACT
This paper suggests a class of modified exponential estimators for estimating the population mean of the study variable by using the information on the auxiliary variable under two situations: i) when information on the study variable and the auxiliary variable is available; ii) when there is non-response on the study as well as on auxiliary variable. Various estimators are obtained from the proposed class of estimators. The expressions for the bias and mean square errors (MSE) of the proposed estimator are derived up to the first degree of approximation. Theoretical comparisons are made with existing estimators and conditions are developed, under which proposed estimators are efficient. Also, these theoretical findings are supported by the simulation and empirical study by considering three real data sets.

KEYWORDS: Exponential estimators, non-response, bias, mean squared errors, percent relative efficiency (PRE).

MSC: 62D05

RESUMEN
Este paper sugiere una clase de estimadores exponenciales modificados para estimar la media de la población de la variable de estudio usando información sobre la variable auxiliar bajo dos situaciones: i) cuando la información sobre la variable de estudio y la auxiliar están disponibles; ii) cuando hay no-respuesta en ambas variables. Varios estimadores son obtenidos para la clase de estimadores propuesta. Las expresiones del sesgo y el error cuadrático medio (MSE) del propuesto estimador son derivados hasta el grado uno de aproximación. comparaciones teóricas se desarrollan con estimadores existentes y se desarrollan las condiciones bajo las cuales los propuestos estimadores son eficientes. Además, estos hallazgos teóricos fueron suportados por imulación y estudios d empíricos usando tres conjuntos de datos reales.

PALABRAS CLAVE: Estimadoresexponenciales, sesgo de no-respuesta, sesgo, error cuadrático medio, porciento de eficiencia relativa (PRE).

1. INTRODUCTION

In general, it is assumed that in sampling theory, the true value of each unit in the population $U = \{U_1, U_2, \dots, U_n\}$ can be obtained and tabulated without any errors. Unfortunately, in real life, this assumption may be violated due to several reasons and practical constraints which results in terms of the existence of some missing observations. The existence of non-response suggests that the population 'U' is divided into two strata U_1 and U_2 belongs to the responding units and non-responding units respectively, which is so, called 'Response strata' and was proposed by Hansen and Hurwitz (1946).
To estimate the population parameters like mean, total or ratio, sample survey experts sometimes use auxiliary information to improve the precision of the estimates. Ratio, Product and Regression methods of estimation are good examples in this regard. Cochran (1977) and Rao (1983, 1986) suggested the use of the ratio method of estimation for the population mean $\overline{Y}$of the study variable y with sub-sampling the non-respondents. Singh et al. (2009) suggested an exponential type estimator in the presence of non-response on the study as well as auxiliary variable by following the exponential estimator by Bahl and Tuteja (1991). Kumar (2013) suggested an improved exponential type estimator by using some known values of the population parameter(s) of the auxiliary variable X such as coefficient of variation (C_x), coefficient of kurtosis ($\beta_2(x)$) and correlation coefficient (ρ_{yx}). Singh et al. (2016) proposed a product and ratio type exponential estimators motivated by Sahai (1979). Yadav and Kadilar (2013), Singh and Pal (2015), Sinha and Kumar (2017), etc studied the exponential estimators for estimating the population mean of the study variable using auxiliary information in the presence of non-response. Further, Unal and Kadilar (2019) proposed families of estimators using the exponential function for the population mean by following Yadav and Kadilar (2013) and Singh and Pal (2015).

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In the present study, we propose a new class of modified exponential estimator with full response and in the presence of non-response on the study and auxiliary variable for the estimation of population mean of the study variable by following Unal and Kadilar (2019) and Zaman and Kadilar (2019) and studied their properties in section 3. Theoretical and numerical comparisons are made between the proposed and existing estimators in sections 4, 5 and section 6, respectively. Finally, in section7, some concluding remarks are given.

2. SOME EXISTING ESTIMATORS

We know that when the population correlation between the study variable (y) and the auxiliary variable (x) is highly positive then one can use ratio estimator. In the ratio method of estimation, auxiliary information on a variable is available and which is linearly related to the variable under study and is utilized to estimate the population mean of the study variable.
Let $\bar{X}$ is the known population mean of the auxiliary variable and $\bar{x}$ and $\bar{y}$ refer as the sample mean of the auxiliary and study variables respectively, then the classical ratio type estimator was given by Cochran (1940) for estimating the population mean as follows:

$$T_R = \frac{\bar{y}}{\bar{x}}\bar{X} \tag{1}$$

Bahl and Tuteja (1991) first introduced an estimator using the exponential function for the estimation of the population mean as

$$T_{BT} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{2}$$

Following Bahl and Tuteja (1991), Yadav and Kadilar (2013) proposed a generalized exponential type estimator as

$$T_{YK} = k\bar{y}\exp\left(\frac{(a\bar{X}+b)-(a\bar{x}+b)}{(a\bar{X}+b)+(a\bar{x}+b)}\right) \tag{3}$$

where a and b are the chosen constants or the function of parameters of auxiliary variable.
Singh and Pal (2015) also proposed a new estimator by using the exponential function as

$$T_{SP} = \bar{y}\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right)\exp\left(\frac{a(\bar{X}-\bar{x})}{a(\bar{X}+\bar{x})+2b}\right) \tag{4}$$

where (a, b) are real constants or the functions of the parameters of auxiliary variable.
Cochran (1977) proposed the classical regression estimator as

$$T_{reg} = \bar{y} + b(\bar{X}-\bar{x}) \tag{5}$$

where b is the regression coefficient of y on x in the simple random sampling method.
The Mean Squared Error (MSE) of the above estimators (1-5) to the first degree of approximation as

$$MSE(T_R) = \lambda\bar{Y}^2\left(C_y^2 + C_x^2 - 2C_{yx}\right), \tag{6}$$

$$MSE(T_{BT}) = \lambda\bar{Y}^2\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right), \tag{7}$$

$$MSE_{min}(T_{YK}) = \bar{Y}^2\left(1 - \frac{\left(\lambda\left(2\xi^2C_x^2-\xi C_{yx}\right)+1\right)^2}{\lambda\left(C_y^2+5\xi^2C_x^2-4\xi C_{yx}\right)+1}\right), \tag{8}$$

$$MSE(T_{SP}) = \bar{Y}^2\lambda\left(C_y^2 + \frac{3\theta C_x^2}{4}\left(3\theta - 4\rho_{yx}\frac{C_y}{C_x}\right)\right), \tag{9}$$

$$MSE\left(T_{reg}\right) = \bar{Y}^2\lambda C_y^2\left(1-\rho_{yx}^2\right), \tag{10}$$

where $\lambda = \frac{1}{N} - \frac{1}{n}$, $C_x^2 = \frac{S_x^2}{\bar{X}^2}$, $C_y^2 = \frac{S_y^2}{\bar{Y}^2}$, $C_{yx} = \rho_{yx}C_yC_x$, $\xi = \frac{a\bar{X}}{2(a\bar{X}+b)}$, $\theta = \frac{a\bar{X}}{a\bar{X}+b}$, $\bar{Y}$ is the population mean of the study variable and ρ_{yx} is the population correlation coefficient between the study and auxiliary variables, C_y and C_x are the coefficient of variation of 'Y' and 'X' respectively.
Hansen and Hurwitz (1946) introduced the estimation method to deal with non-response and also a new technique of sub-sampling the non-respondents. In this method, suppose that $S = (S_1, S_2, \dots, S_N)$ consists of N units. From S, a sample of size n is drawn without replacement (SRSWOR) and (y_i, x_i) are the values of the study and auxiliary variables for the i^{th} unit $(i = 1,2,\dots,11)$ of the population, respectively. Population of size N $(N_1 + N_2 = N)$ is composed of N_1 and N_2 belonging to the responding units and non-responding units, respectively. Similarly, sample of size n $(n_1 + n_2 = n)$ is divided in two parts n_1 as responding units and n_2 non-responding units. A sub-sample of size $r = {n_2}/{h}$ $(h > 1)$ units is randomly drawn from n_2 where h is the inverse sampling rate at the second phase sample of size$n$. Also, $W_1 = \frac{N_1}{N}$ and $W_2 = \frac{N_2}{N}$ are the proportions of the responding and the non-responding for the population, respectively.
Hansen and Hurwitz (1946) were the first to proposed the unbiased estimator for estimating the population mean in the presence of non-response as follows

$$T_{HH} = w_1\bar{y}_1 + w_2\bar{y}_{2(r)}$$

where $w_1 = \frac{n_1}{n}$ and $w_2 = \frac{n_2}{n}$ are the proportions of the responding and the non-responding for the sample and $\bar{y}_1$ and $\bar{y}_{2(r)}$ represents the sample mean of the study variable depending on n_1 and r units, respectively. The variance of T_{HH} is given by,

$$V(T_{HH}) = \bar{Y}^2\left(\lambda C_y^2 + \frac{W_2(h-1)}{n}C_{y(2)}^2\right)$$

When non-response exists only on the study variable and the population mean of the auxiliary variable is known, using the Hansen and Hurwitz (1946) technique, Rao (1986) modified the ratio and the regression estimators introduced by Cochran (1940) as

$$T_R^* = \frac{\bar{y}^*}{\bar{x}}\bar{X}, \tag{11}$$

$$T_{reg}^* = \bar{y}^* + b^*(\bar{X} - \bar{x}), \tag{12}$$

where $\bar{y}^*$ represents the sample mean of the study variable in the case of non-response and $b^* = \frac{S_{xy}^*}{S_x^{*2}}$.

The expression of the MSE for the estimators shown in (11) and (12) are given by,

$$MSE(T_R^*) = \bar{Y}^2\left\{\lambda\left(C_y^2 + C_x^2 - 2C_{yx}\right) + \frac{W_2(h-1)}{n}C_{y(2)}^2\right\}, \tag{13}$$

$$MSE\left(T_{reg}^*\right) = \bar{Y}^2\left\{\lambda C_y^2\left(1-\rho_{yx}^2\right) + \frac{W_2(h-1)}{n}C_{y(2)}^2\right\}. \tag{14}$$

Similarly, using the technique of Hansen and Hurwitz (1946), Singh et al. (2009) proposed an exponential type estimator when the non-response occurs only on the study variable by adapting the estimator introduced by Bahl and Tuteja (1991) as follows

$$T_{BT}^* = \bar{y}^*\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right). \tag{15}$$

The MSE of the estimator in given in (15) to the first order of approximation, is given by

$$MSE(T_{BT}^*) = \bar{Y}^2\left\{\lambda\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right) + \frac{W_2(h-1)}{n}C_{y(2)}^2\right\}. \tag{16}$$

When non-response occurs on both the study and auxiliary variables and the population mean of the auxiliary variable is known, Cochran (1977) adapts the estimator given by (1) is as follows:

$$T_R^{**} = \frac{\bar{y}^*}{\bar{x}^*}\bar{X}, \tag{17}$$

where $\bar{x}^*$ refers the sample mean of the study variable in the case of non-response and its MSE is

$$MSE(T_R^{**}) = \bar{Y}^2\left\{\lambda\left(C_y^2 + C_x^2 - 2C_{yx}\right) + \frac{W_2(h-1)}{n}\left(C_{y(2)}^2 + C_{x(2)}^2 - 2\rho_{yx(2)}C_{y(2)}C_{x(2)}\right)\right\}, \tag{18}$$

where $C_{x(2)}^2 = \frac{S_{x(2)}^2}{\bar{X}^2}$ and $C_{yx(2)} = \rho_{yx(2)}C_{y(2)}C_{x(2)}$.

Using the technique of Hansen and Hurwitz (1946), Singh et al. (2009) adapted exponential type estimator provided in (2) to the case non response on the study and auxiliary variables as

$$T_{BT}^{**} = \bar{y}^*\exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right). \tag{19}$$

The MSE is given by

$$MSE(T_{BT}^{**}) = \bar{Y}^2\left\{\lambda\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right) + \frac{W_2(h-1)}{n}\left(C_{y(2)}^2 + \frac{C_{x(2)}^2}{4} - \rho_{yx(2)}C_{y(2)}C_{x(2)}\right)\right\} \tag{20}$$

The classical regression estimator T_{reg}^{**} in the presence of non-response is defined as

$$T_{reg}^{**} = \bar{y}^* + b^*(\bar{X} - \bar{x}^*). \tag{21}$$

The equation of MSE to the first order of approximation is given by

$$MSE(T_{reg}^{**}) = \bar{Y}^2\left\{\lambda C_y^2\left(1-\rho_{yx}^2\right) + \frac{W_2(h-1)}{n}\left(C_{y(2)}^2 + \rho_{yx}^2\frac{C_y^2}{C_x^2}C_{x(2)}^2 - 2\rho_{yx}\frac{C_y}{C_x}C_{yx(2)}\right)\right\}. \tag{22}$$

Further, Unal and Kadilar (2019) motivated by Yadav and Kadilar (2013) and Singh and Pal (2015) proposed a new exponential family of estimators for the population mean of the study variable under the two situations:

Situation1: When non-response occurs only on the study variable and the population mean of the auxiliary variable is known, then a general class of estimator is

$$T_{1,i} = k\bar{y}^*\left(\frac{a_i\bar{X}+b_i}{a_i\bar{x}+b_i}\right)^{\alpha}\exp\left(\frac{a_i(\bar{X}-\bar{x})}{a_i(\bar{X}+\bar{x})+2b_i}\right), \quad i = 1,2,\dots,10, \tag{23}$$

where k is a suitably chosen constant.

The expression for Bias and MSE of the estimator $T_{1,i}$, $i = 1,2,\dots10$ are as follows

$$B\left(T_{1,i}\right) = \bar{Y}(k-1) + \bar{Y}\lambda\left\{C_x^2\frac{k\theta_i^2}{2}\left(\frac{3}{4}+\alpha^2\right) - C_{yx}k\theta_i\left(\frac{1}{2}+\alpha\right)\right\}, \quad i = 1,2,\dots,10 \tag{24}$$

and

$$MSE\left(T_{1,i}\right) = \bar{Y}^2\left\{(k-1)^2 + k^2\left(\lambda C_y^2 + \frac{W_2(h-1)}{n}C_{y(2)}^2\right) + \lambda\theta_i^2C_x^2\left(k^2 + 2k^2\alpha^2 - k\alpha^2 + \alpha k^2 - \frac{3}{4}k\right) + \lambda\theta_iC_{xy}(k + 2k\alpha - 2k^2 - 4k^2\alpha)\right\}, \quad i = 1,2,\dots,10 \tag{25}$$

under the optimal value of k as $k^* = \frac{A_1}{A_2}$ when

$A_2 = \lambda\left(2C_y^2 + 2\theta_i^2 C_x^2 + 4\alpha^2\theta_i^2 C_x^2 + 2\alpha\theta_i^2 C_x^2 - 4\theta_i C_{yx} + 8\alpha\theta_i C_{yx}\right) + 2\frac{W_2(h-1)}{n}C_{y(2)}^2 + 2;$

$A_1 = \lambda\left(C_x^2\theta_i^2\left(\alpha^2 + \frac{3}{4}\right) - C_{yx}\theta_i(1+2\alpha)\right) + 2;\ \theta_i = \frac{a_i\bar{X}}{a_i\bar{X}+b_i}.$

The expression for minimum MSE of $T_{1,i}$ is

$$MSE_{min}(T_{1,i}) = \bar{Y}^2\left(1 - \frac{A_1^2}{2A_2}\right),\quad i = 1,2,\ldots,10. \tag{26}$$

Situation2: When non-response occurs on both the study and the auxiliary variables and the population mean of the auxiliary variable is known, then the family of estimators are as follows

$$T_{2,i} = k\bar{y}^*\left(\frac{a_i\bar{X}+b_i}{a_i\bar{x}^*+b_i}\right)^{\alpha}\exp\left(\frac{a_i(\bar{X}-\bar{x}^*)}{a_i(\bar{X}+\bar{x}^*)+2b_i}\right),\ i = 1,2,\ldots,10. \tag{27}$$

The expression for Bias and MSE of the estimator $T_{2,i}$, $i = 1,2,\ldots 10$ are as follows

$$B(T_{2,i}) = \bar{Y}\left\{(k-1) + \frac{k\theta_i^2}{2}\left(\lambda C_x^2 + \frac{W_2(h-1)}{n}C_{x(2)}^2\right)\left(\frac{3}{4}+\alpha^2\right) - k\theta_i\left(\frac{1}{2}+\alpha\right)\left(\lambda C_{yx} + \frac{W_2(h-1)}{n}\rho_{yx(2)}C_{y(2)}C_{x(2)}\right)\right\},\quad i = 1,2,\ldots,10 \tag{28}$$

and

$$MSE(T_{2,i}) = \bar{Y}^2\left\{\begin{array}{c}(k-1)^2 + k^2\left(\lambda C_y^2 + \frac{W_2(h-1)}{n}C_{y(2)}^2\right) + k\theta_i^2\left(k - \frac{3}{4} - \alpha^2 + 2k\alpha^2 + k\alpha\right)\\ \left(\lambda C_x^2 + \frac{W_2(h-1)}{n}C_{x(2)}^2\right)\\ +k\theta_i(1+2\alpha-2k-4k\alpha)\left(\lambda C_{yx} + \frac{W_2(h-1)}{n}\rho_{yx(2)}C_{y(2)}C_{x(2)}\right)\end{array}\right\},\ i = 1,2,\ldots,10 \tag{29}$$

which is optimal, when the value of k is $k^{**} = \frac{A_3}{A_4}$, where

$$A_3 = \left\{\theta_i^2\left(\alpha^2 + \frac{3}{4}\right)\left(\lambda C_x^2 + \frac{W_2(h-1)}{n}C_{x(2)}^2\right) - \theta_i(1+2\alpha)\left(\lambda C_{yx} + \frac{W_2(h-1)}{n}C_{yx(2)}\right) + 2\right\}$$

$$A_4 = \left\{\begin{array}{c}2\theta_i^2(2\alpha^2+\alpha+1)\left(\lambda C_x^2 + \frac{W_2(h-1)}{n}C_{x(2)}^2\right) - \theta_i(4+8\alpha)\\ \left(\lambda C_{yx} + \frac{W_2(h-1)}{n}\rho_{yx(2)}C_{y(2)}C_{x(2)}\right) + 2\left(\lambda C_x^2 + \frac{W_2(h-1)}{n}C_{y(2)}^2\right) + 2\end{array}\right\}.$$

The expression for minimum MSE of the proposed family of estimators as

$$MSE_{min}(T_{2,i}) = \bar{Y}^2\left(1 - \frac{A_3^2}{2A_4}\right),\quad i = 1,2,\ldots,10. \tag{30}$$

3. THE PROPOSED FAMILY OF ESTIMATORS

Motivated by Unal and Kadilar (2019) and Zaman and Kadilar (2019), we propose a new class of modified exponential estimator by following Singh et al. (2009) and Sinha and Kumar (2017) in order to estimate the population mean of the study variable by using auxiliary information possessing different attributes viz C_x, ρ_{yx} and $\beta_2(x)$, respectively when population mean of auxiliary variable is known.

Situation 1: When there is complete information on study as well as auxiliary variables and the population mean of auxiliary variable is known, we propose the following class of estimator as

$$\tau_{1,i} = \delta\bar{y}\left(\frac{(1-c_i)\bar{x}+c_i\bar{X}}{c_i\bar{x}+(1-c_i)\bar{X}}\right)^{\eta}\exp\left(\frac{c_i(\bar{x}-\bar{X})}{(c_i\bar{x}+d_i)+(c_i\bar{X}+d_i)}\right), i = 1,2,\ldots,11 \tag{31}$$

where $\delta(\neq 0)$ and η are suitably chosen constants, and c_i, d_i are either real numbers or the functions of the known parameters of the auxiliary variable, such as the coefficient of variation, coefficient of kurtosis, correlation coefficients, etc. Various estimators of the population mean can be generated by taking suitable choices of constants δ, η, c_i, d_i, respectively. Some of the estimators are presented in the Table 1.

Table 1: Members of the proposed estimators for different values of $c_i's$ and $d_i's$.

S. No.	Values		Estimators
	c_i	d_i	
1	1	1	$\tau_{1,1} = \delta\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\eta}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2}\right)$
2	1	C_x	$\tau_{1,2} = \delta\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\eta}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2C_x}\right)$
3	1	ρ_{yx}	$\tau_{1,3} = \delta\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\eta}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2\rho_{yx}}\right)$

4	1	0	$\tau_{1,4} = \delta\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\eta} \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$
5	C_x	ρ_{yx}	$\tau_{1,5} = \delta\bar{y}\left(\frac{\bar{x}+C_x(\bar{X}-\bar{x})}{\bar{X}+C_x(\bar{x}-\bar{X})}\right)^{\eta} \exp\left(\frac{C_x(\bar{x}-\bar{X})}{C_x(\bar{x}+\bar{X})+2\rho_{yx}}\right)$
6	ρ_{yx}	C_x	$\tau_{1,6} = \delta\bar{y}\left(\frac{\bar{x}+\rho_{yx}(\bar{X}-\bar{x})}{\bar{X}+\rho_{yx}(\bar{x}-\bar{X})}\right)^{\eta} \exp\left(\frac{\rho_{yx}(\bar{x}-\bar{X})}{\rho_{yx}(\bar{x}+\bar{X})+2C_x}\right)$
7	ρ_{yx}	1	$\tau_{1,7} = \delta\bar{y}\left(\frac{\bar{x}+\rho_{yx}(\bar{X}-\bar{x})}{\bar{X}+\rho_{yx}(\bar{x}-\bar{X})}\right)^{\eta} \exp\left(\frac{\rho_{yx}(\bar{x}-\bar{X})}{\rho_{yx}(\bar{x}+\bar{X})+2}\right)$
8	1	$\beta_2(x)$	$\tau_{1,8} = \delta\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\eta} \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2\beta_2(x)}\right)$
9	$\beta_2(x)$	1	$\tau_{1,9} = \delta\bar{y}\left(\frac{\bar{x}+\beta_2(x)(\bar{X}-\bar{x})}{\bar{X}+\beta_2(x)(\bar{x}-\bar{X})}\right)^{\eta} \exp\left(\frac{\beta_2(x)(\bar{x}-\bar{X})}{\beta_2(x)(\bar{x}+\bar{X})+2}\right)$
10	$\beta_2(x)$	C_x	$\tau_{1,10} = \delta\bar{y}\left(\frac{\bar{x}+\beta_2(x)(\bar{X}-\bar{x})}{\bar{X}+\beta_2(x)(\bar{x}-\bar{X})}\right)^{\eta} \exp\left(\frac{\beta_2(x)(\bar{x}-\bar{X})}{\beta_2(x)(\bar{x}+\bar{X})+2C_x}\right)$
11	C_x	$\beta_2(x)$	$\tau_{1,11} = \delta\bar{y}\left(\frac{\bar{x}+C_x(\bar{X}-\bar{x})}{\bar{X}+C_x(\bar{x}-\bar{X})}\right)^{\eta} \exp\left(\frac{C_x(\bar{x}-\bar{X})}{C_x(\bar{x}+\bar{X})+2\beta_2(x)}\right)$

To obtain the Bias and MSE of $\tau_{1,i}; i = 1,2,\dots,11$ in (31), we assume

$$\bar{y} = \bar{Y}(1+\epsilon_0)\ ;\ \bar{x} = \bar{X}(1+\epsilon_1)$$

then we have,

$E(\epsilon_0) = E(\epsilon_1) = 0;\ E(\epsilon_0^2) = \lambda C_y^2;\ E(\epsilon_1^2) = \lambda C_x^2;\ E(\epsilon_0\epsilon_1) = \lambda C_{yx}$.

Now, the family of estimators $\tau_{1,i};\ i = 1,2,\dots,11$ can be expressed in terms of ϵ_0 and ϵ_1, we get

$$\tau_{1,i} = \delta\bar{Y}(1+\epsilon_0)\left(\frac{(1-c_i)(\bar{X}(1+\epsilon_1))+c_i\bar{X}}{c_i(\bar{X}(1+\epsilon_1))+(1-c_i)\bar{X}}\right)^{\eta} \exp\left(\frac{c_i(\bar{X}(1+\epsilon_1)-\bar{X})}{c_i(\bar{X}(1+\epsilon_1)+d_i+c_i\bar{X}+d_i)}\right) \quad (32)$$

Expanding the right-hand side of (32) to the first degree of approximation, we have

$$\tau_{1,i} = \delta\bar{Y}(1+\epsilon_0)\left\{1+\eta(1-c_i)\epsilon_1+\frac{\eta(\eta-1)}{2}(1-c_i)^2{\epsilon_1}^2+\cdots\right\}\left\{1-\eta c_i\epsilon_1+\frac{\eta(\eta+1)}{2}c_i^2\epsilon_1^2 + \cdots\right\}\exp\left\{\frac{\omega_i\epsilon_1}{1+\omega_i\epsilon_1}\right\}$$

$$\tau_{1,i} = \delta\bar{Y}(1+\epsilon_0)\left\{1+\eta(1-c_i)\epsilon_1-\eta c_i\epsilon_1+\frac{\eta(\eta+1)}{2}c_i^2\epsilon_1^2-\eta^2(1-c_i)c_i\epsilon_1^2 + \frac{\eta(\eta-1)}{2}(1-c_i)^2\epsilon_1^2\right\}\left\{1+\omega_i\epsilon_i-\frac{\omega_i^2\epsilon_1^2}{2}\right\}$$

$$\tau_{1,i} = \delta\bar{Y}\left\{1+\epsilon_0+(\omega_i+\eta(1-c_i)-\eta c_i)\epsilon_1+\left(\eta(1-c_i)\omega_i-\frac{{\omega_i}^2}{2}-\eta c_i\omega_i-\eta^2(1-c_i)c_i+\frac{\eta(\eta+1)}{2}c_i^2+\frac{\eta(\eta-1)}{2}(1-c_i)\right)\epsilon_1^2+(\omega_i+\eta(1-c_i)-\eta c_i)\epsilon_0\epsilon_1\right\}$$

$$\tau_{1,i} = \delta\bar{Y}(1+\epsilon_0+\phi_i\epsilon_1+\psi_i\epsilon_1^2+\phi_i\epsilon_0\epsilon_1)$$

$$\tau_{1,i}-\bar{Y}=\bar{Y}\{(\delta-1)+\delta\epsilon_0+\delta\phi_i\epsilon_1+\delta\psi_i\epsilon_1^2+\delta\phi_i\epsilon_0\epsilon_1\} \quad (33)$$

where $\phi_i = \omega_i+\eta(1-c_i)-\eta c_i;\ \omega_i = \frac{c_i\bar{X}}{2(c_i\bar{X}+d_i)}$

and $\psi_i = \eta(1-c_i)\omega_i-\frac{{\omega_i}^2}{2}-\eta c_i\omega_i-\eta^2(1-c_i)c_i+\frac{\eta(\eta+1)}{2}c_i^2+\frac{\eta(\eta-1)}{2}(1-c_i)^2$.

Taking expectation on both sides of (33), we get the Bias of $\tau_{1,i}; i = 1,2,\dots,11$ as

$$\mathrm{Bias}(\tau_{1,i}) = \bar{Y}\{(\delta-1)+\delta\lambda\phi_iC_{yx}+\delta\lambda\psi_i{C_x}^2\}. \quad (34)$$

Squaring both sides of (33) and then taking expectation, we get the MSE of $\tau_{1,i}$ as

$$\mathrm{MSE}(\tau_{1,i}) = \bar{Y}^2\{(\delta-1)^2+\lambda\delta^2C_y^2+\lambda C_x^2(\delta^2\phi_i^2+2\delta^2\psi_i-2\delta\phi_i)+\lambda C_{yx}(4\delta^2\phi_i-2\delta\phi_i)\};\ i = 1,2,\dots,11. \quad (35)$$

Theorem 1: To the first degree of approximation

$$\mathrm{MSE}_{\min}(\tau_{1,i}) \geq \bar{Y}^2 \left\{\frac{\lambda(C_y^2 + \phi_i^2 C_x^2 + \psi_i C_x^2 + 3\phi_i C_{yx})}{1 + \lambda(C_y^2 + \phi_i^2 C_x^2 + 2\psi_i C_x^2 + 4\phi_i C_{yx})}\right\}$$

with equality holding if $\delta = \delta_0$(say).

Proof: Differentiating $\mathrm{MSE}(\tau_{1,i})$ in (35) with respect to δ and equating it to zero, we obtain the optimum value of δ as

$$\delta = \frac{1+B}{1+A} = \delta_0(\text{say}) \tag{36}$$

where $B = \lambda(\psi_i C_x^2 + \phi_i C_{yx})$ and $A = \lambda(C_y^2 + \phi_i^2 C_x^2 + 2\psi_i C_x^2 + 4\phi_i C_{yx})$.

Replacing the value of δ from (36) in (35), we have the minimum MSE of the proposed family of estimators as

$$\mathrm{MSE}_{\min}(\tau_{1,i}) = \bar{Y}^2 \left(\frac{A-B}{1+A}\right); \qquad i = 1,2,\dots,11$$

$$\mathrm{MSE}_{\min}(\tau_{1,i}) = \bar{Y}^2 \left\{\frac{\lambda(C_y^2+\phi_i^2 C_x^2+\psi_i C_x^2+3\phi_i C_{yx})}{1+\lambda(C_y^2+\phi_i^2 C_x^2+2\psi_i C_x^2+4\phi_i C_{yx})}\right\} \tag{37}$$

It is observed from Table 1 that each c_i and d_i are different. For this reason, all ω_i's are different from each other. Therefore, it is clear that the values of minMSEare different for each estimator,$\tau_{1,1},\dots,\tau_{1,11}$.

Situation 2: When there is non-response on study variable as well as auxiliary variable and population mean of the auxiliary variable is known, we propose the following class of estimator as follows:

$$\tau_{2,i} = \delta\bar{y}^* \left(\frac{(1-c_i)\bar{x}^*+c_i\bar{X}}{c_i\bar{x}^*+(1-c_i)\bar{X}}\right)^{\eta} \exp\left(\frac{c_i(\bar{x}^*-\bar{X})}{(c_i\bar{x}^*+d_i)+(c_i\bar{X}+d_i)}\right), \qquad i = 1,2,\dots,11 \tag{38}$$

Similarly, as in Table 1, one can also write the members of the proposed family of estimators in under the case of non-response existing both on the study variable and the auxiliary variable for different values of c_i's and d_i's, respectively.

The expression of bias and MSE of $\tau_{2,i}; i = 1,2,\dots,11$ to the first degree of approximation are obtained as

$$B(\tau_{2,i}) = \bar{Y}\{(\delta-1) + \delta\phi_i(\lambda C_{yx} + \theta C_{yx(2)}) + \delta\psi_i(\lambda {C_x}^2 + \theta C_x^2)\} \tag{39}$$

and

$$\mathrm{MSE}(\tau_{2,i}) = \bar{Y}^2\{(\delta-1)^2 + \delta^2(\lambda {C_y}^2 + \theta C_y^2) + (\lambda {C_x}^2 + \theta C_x^2)(\delta^2\phi_i^2 + 2\delta^2\psi_i - 2\delta\phi_i) + (\lambda C_{yx} + \theta C_{yx(2)})(4\delta^2\phi_i - 2\delta\phi_i)\}, i = 1,2,\dots,11 \tag{40}$$

Theorem 2: To the first degree of approximation

$$\mathrm{MSE}_{\min}(\tau_{2,i}) \geq \bar{Y}^2 \left\{\frac{A_1 + 2\phi_i C_1 + \phi_i^2 B_1 - (\psi_i B_1 + \phi_i C_1)^2}{1 + A_1 + \phi_i^2 B_1 + 2\psi_i B_1 + 4\phi_i C_1}\right\}$$

with equality holding if $\delta = \delta_{00}$(say).

Proof: Differentiating $\mathrm{MSE}(\tau_{2,i})$ in (40) with respect to δ and equating it to zero, we obtain the optimum value of δ as

$$\delta = \frac{1+\psi_i A_1+\phi_i C_1}{1+A_1+(\phi_i^2+2\psi_i)B_1+4\phi_i C_1} = \delta_{00}(\text{say}) \tag{41}$$

where $A_1 = \lambda C_y^2 + \theta C_{y(2)}^2$, $B_1 = \lambda C_x^2 + \theta C_{x(2)}^2$ and $C_1 = \lambda C_{yx} + \theta C_{yx(2)}$.

Substituting the optimum value of δ from (41) in (40), we get the minimum MSE of $\tau_{2,i}$ as

$$\mathrm{MSE}_{\min}(\tau_{2,i}) = \bar{Y}^2 \left\{\frac{A_1+2\phi_i C_1+\phi_i^2 B_1-(\psi_i B_1+\phi_i C_1)^2}{1+A_1+\phi_i^2 B_1+2\psi_i B_1+4\phi_i C_1}\right\}; \qquad i = 1,2,\dots,11 \tag{42}$$

We would like to note that the values of min MSE are also different for each estimator,$\tau_{2,1},\dots,\tau_{2,11}$, as in Situation 1 because ϕ_i and ψ_i in (42) are computed by using the values of ω_i.

4. THEORETICAL COMPARISON OF THE ESTIMATORS

In this section, we obtain the efficiency conditions for the proposed exponential estimators by comparing the MSE equations of the proposed family of estimators $\tau_{1,i}$ and $\tau_{2,i}$, $i = 1,2,\dots,11$ with the other existing estimators.

Situation 1

In this section, we compare the MSE equation of the proposed family of estimators $\tau_{1,i}$, $i = 1,2,\dots,11$ with the MSE of the mentioned estimators, viz Cochran classical ratio (T_R) and regression estimators (T_{reg}), Bahl and Tuteja (1991) estimator using exponential function (T_{BT}), Yadav and Kadilar (2013) exponential type estimator (T_{YK}), Singh and Pal (2015) estimator using exponential function (T_{SP}) and Unal and Kadilar (2019) proposed family of estimators using exponential function ($T_{1,i}$), mentioned in section 2, respectively. From equation (6), (7), (8), (9), (10), (26) and (37), we find the efficiency comparisons of the proposed family of estimators as:

i) $\mathrm{MSE}_{\min}(\tau_{1,i}) < \mathrm{MSE}(T_R), i = 1,2,\dots,11$

$$\left\{\frac{\lambda(C_y^2+\phi_i^2 C_x^2+\psi_i C_x^2+3\phi_i C_{yx})}{1+\lambda(C_y^2+\phi_i^2 C_x^2+2\psi_i C_x^2+4\phi_i C_{yx})}\right\} < \lambda(C_y^2 + C_x^2 - 2C_{yx}) \tag{43}$$

ii) $\mathrm{MSE}_{\min}(\tau_{1,i}) < \mathrm{MSE}(T_{BT}), i = 1,2, \dots, 11$

$$\left\{\frac{\lambda(C_y^2+\phi_i^2C_x^2+\psi_iC_x^2+3\phi_iC_{yx})}{1+\lambda(C_y^2+\phi_i^2C_x^2+2\psi_iC_x^2+4\phi_iC_{yx})}\right\} < \lambda\left(C_y^2+\frac{C_x^2}{4}-C_{yx}\right) \quad (44)$$

iii) $\mathrm{MSE}_{\min}(\tau_{1,i}) < \mathrm{MSE}(T_{YK}), i = 1,2, \dots, 11$

$$\left\{\frac{\lambda(C_y^2+\phi_i^2C_x^2+\psi_iC_x^2+3\phi_iC_{yx})}{1+\lambda(C_y^2+\phi_i^2C_x^2+2\psi_iC_x^2+4\phi_iC_{yx})}\right\} < \lambda\left\{1-\frac{\left(\lambda(2\xi^2C_x^2-\xi C_{yx})+1\right)^2}{\lambda(C_y^2+5\xi^2C_x^2-4\xi C_{yx})+1}\right\} \quad (45)$$

iv) $\mathrm{MSE}_{\min}(\tau_{1,i}) < \mathrm{MSE}(T_{SP}), i = 1,2, \dots, 11$

$$\left\{\frac{\lambda(C_y^2+\phi_i^2C_x^2+\psi_iC_x^2+3\phi_iC_{yx})}{1+\lambda(C_y^2+\phi_i^2C_x^2+2\psi_iC_x^2+4\phi_iC_{yx})}\right\} < \lambda\left\{C_y^2+\frac{3\theta C_x^2}{4}\left(3\theta-4\rho_{yx}\frac{C_y}{C_x}\right)\right\} \quad (46)$$

v) $\mathrm{MSE}_{\min}(\tau_{1,i}) < \mathrm{MSE}(T_{reg}), i = 1,2, \dots, 11$

$$\left\{\frac{\lambda(C_y^2+\phi_i^2C_x^2+\psi_iC_x^2+3\phi_iC_{yx})}{1+\lambda(C_y^2+\phi_i^2C_x^2+2\psi_iC_x^2+4\phi_iC_{yx})}\right\} < \lambda C_y^2(1-\rho_{yx}^2) \quad (47)$$

For the Situation 1, when the conditions (43)-(47) are satisfied, we infer that the proposed family of estimators $\tau_{1,i}, i = 1,2, \dots, 11$ is more efficient than the compared estimators $T_R, T_{BT}, T_{YK}, T_{SP}, T_{reg}$ and $T_{1,i}$.

Situation 2

In this section, we compare the MSE equation of the proposed family of estimators $\tau_{2,i}, i = 1,2, \dots, 11$ for the Situation 2 with the MSE equation of the mentioned estimators, such as Rao (1986) ratio (T_R^*) and regression estimators (T_{reg}^*), Singh et al. (2009) exponential type estimator (T_{BT}^*) and (T_{BT}^{**}), Cochran (1977) classical ratio (T_R^{**}) and regression estimators (T_{reg}^{**}), Unal and Kadilar (2019) proposed family of estimators using exponential function ($T_{2,i}$), mentioned in section 2, respectively.

From (13), (14), (16), (18), (20), (22), (30) and (37), we find the efficiency of the proposed family of estimators as follows:

i) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}(T_R^*), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \lambda(C_y^2+C_x^2-2C_{yx})+\frac{W_2(h-1)}{n}C_{y(2)}^2 \quad , \quad (48)$$

ii) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}(T_{BT}^*), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \lambda\left(C_y^2+\frac{C_x^2}{4}-C_{yx}\right)+\frac{W_2(h-1)}{n}C_{y(2)}^2, \quad (49)$$

iii) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}(T_{reg}^*), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \lambda C_y^2(1-\rho_{yx}^2)+\frac{W_2(h-1)}{n}C_{y(2)}^2, \quad (50)$$

iv) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}(T_R^{**}), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \lambda(C_y^2+C_x^2-2C_{yx})+\frac{W_2(h-1)}{n}\left(C_{y(2)}^2+C_{x(2)}^2-2\rho_{yx(2)}C_{y(2)}C_{x(2)}\right), \quad (51)$$

v) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}(T_{BT}^{**}), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \lambda\left(C_y^2+\frac{C_x^2}{4}-C_{yx}\right)+\frac{W_2(h-1)}{n}\left(C_{y(2)}^2+\frac{C_{x(2)}^2}{4}-\rho_{yx(2)}C_{y(2)}C_{x(2)}\right), \quad (52)$$

vi) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}(T_{reg}^{**}), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \lambda C_y^2(1-\rho_{yx}^2)+\frac{W_2(h-1)}{n}\left(C_{y(2)}^2+\rho_{yx}^2\frac{C_y^2}{C_x^2}C_{x(2)}^2-2\rho_{yx}\frac{C_y}{C_x}C_{yx(2)}\right), \quad (53)$$

vii) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}_{\min}(T_{1,i}), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \left(1-\frac{A_1^2}{2A_2}\right), \quad (54)$$

vii) $\mathrm{MSE}_{\min}(\tau_{2,i}) < \mathrm{MSE}_{\min}(T_{2,i}), i = 1,2, \dots, 11$

$$\left\{\frac{A_1+2\phi_iC_1+\phi_i^2B_1-(\psi_iB_1+\phi_iC_1)^2}{1+A_1+\phi_i^2B_1+2\psi_iB_1+4\phi_iC_1}\right\} < \left(1-\frac{A_3^2}{2A_4}\right), \quad (55)$$

For the Situation 2, when the conditions (48)-(55) are satisfied, we infer that the proposed family of estimators $\tau_{2,i}, i = 1,2, \dots, 11$ is more efficient than the compared estimators $T_R^*, T_{BT}^*, T_{reg}^*, T_R^{**}, T_{BT}^{**}, T_{reg}^{**}$ and $T_{2,i}$.

5. EMPIRICAL STUDY

To see the performance of the suggested estimators of the population mean, we consider three natural datasets.

Population I: Khare and Srivastava (1995)

The population of 100 consecutive trips (after leaving 20 outlier values) measured by two fuel meters for a small family car in normal usage given by Lewisi et.al (1991) has been taken into consideration. The measurement of turbine meter (in ml.) is considered as main variable y and the measurement of displacement meter (in cm^3) is considered as auxiliary variable x. The values of the parameters are as follows

$$N = 100, \quad n = 30, \quad \bar{Y} = 3500.12, \quad \bar{X} = 260.84, \quad C_x = 0.5996, \quad C_y = 0.5941,$$
$$\rho_{yx} = 0.985, \quad \rho_{yx(2)} = 0.995, \quad C_{y(2)} = 0.5075, \quad C_{x(2)} = 0.5168, \quad W_2 = 0.25.$$

Population II: Khare and Srivastava (1993)

A list of 70 villages in India along their population in 1981 and cultivated areas (in acres) in the same year is considered (Singh and Choudhary, 1986). Here the cultivated area (in acres) is taken as the main study variable and the population of the village is taken as the auxiliary variable. The parameters of the population are as follows

$$N = 70, \quad n = 35, \quad \bar{Y} = 981.29, \quad \bar{X} = 1755.53, \quad C_x = 0.8009, \quad C_y = 0.6254,$$
$$\rho_{yx} = 0.778, \quad \rho_{yx(2)} = 0.445, \quad C_{y(2)} = 0.4087, \quad C_{x(2)} = 0.5739, \quad W_2 = 0.20.$$

Population III: Khare and Sinha (2009)

Here, 96 village wise population of rural area under Police-station – Singur, District -Hooghly, West Bengal from the District Census Handbook 1981. The 25% villages (i.e. 24 villages) whose area is greater than 160 hectares have been considered as non-response group of the population. The number of agricultural labours in the village is taken as study character (y) while the area (in hectares) of the village is taken as auxiliary variable x. The values of the different parameters of the population are as follows

$$N = 96, \quad n = 40, \quad \bar{Y} = 137.92, \quad \bar{X} = 144.87, \quad C_x = 0.81, \quad C_y = 1.32,$$
$$\rho_{yx} = 0.77, \quad \lambda = 0.01458, \quad C_{yx} = 0.8232, \quad \beta_2(x) = 1.19997, \quad \rho_{yx(2)} = 0.72$$
$$C_{y(2)} = 2.08, \quad C_{x(2)} = 0.94, \quad W_2 = 0.25$$

We have calculated the percent relative efficiency (PRE) of the estimators with respect to the usual unbiased estimator for both the situation 1 and 2 by using the formulae:

Situation I:

$$\mathrm{PRE}(T, \bar{y}) = \frac{\mathrm{MSE}(\bar{y})}{\mathrm{MSE}(T)} \times 100$$

where $T = T_R, T_{BT}, T_{reg}, T_{YK}, T_{SP}, T_{1,i}$ and $\tau_{1,i}$; $i = 1,2, \dots, 11$.

Situation II:

$$\mathrm{PRE}(T^*, \bar{y}^*) = \frac{\mathrm{MSE}(\bar{y}^*)}{\mathrm{MSE}(T^*)} \times 100$$

where $T^* = T_R^*, T_{BT}^*, T_{reg}^*, T_R^{**}, T_{BT}^{**}, T_{reg}^{**}, T_{2,i}$ and $\tau_{2,i}$; $i = 1,2, \dots, 11$.

Table 2 represents the PRE of ratio estimator(T_R), Bahl and Tuteja estimator (T_{BT}) and the usual regression estimator (T_{reg}) for all populations. Table 3 represents the PRE of proposed class of estimators($\tau_{1,i}$), Yadav and Kadilar estimator (T_{YK}) and Singh and Pal estimator (T_{SP}) respectively, with respect to $\bar{y}$ for different values of the constants under the situation 1 in which complete information is available for both study and auxiliary variables.

Table 3 and 4 represents the PRE of the estimators with respect to the usual unbiased estimator $\bar{y}^*$ for situation 2 in which non-response is present on study and auxiliary variables.

Table 2: PRE of existing estimators with respect to $\bar{y}$ for situation 1.

	T_R	T_{BT}	T_{reg}
Pop I	3293.4350	383.8308	3358.5222
Pop II	154.4775	241.7359	253.3467
Pop III	231.7231	160.8654	245.6398

Table 3: PRE of proposed estimators with respect to $\bar{y}$ for different values of c_i and d_i; $i = 1, 2, \dots, 11$ for situation 1.

	c_i	d_i	$\mathrm{PRE}(\tau_{1,i})$	$\mathrm{PRE}(T_{YK})$	$\mathrm{PRE}(T_{SP})$
Population I	1	1	7488.0534	382.0060	329.5830
	1	0.5996	7840.9427	383.0923	326.9732
	1	0.985	7500.6871	382.0466	329.4848
	1	0	8437.9969	384.7336	323.1059
	0.5996	0.985	*	380.2784	333.8188
	0.985	0.5996	27232.1225	383.0674	327.0325
	1	0.985	7500.6871	382.0466	329.4848

Population II	1	1	190.6711	243.4425	*
	1	0.8009	190.6837	243.4549	*
	1	0.778	190.6851	243.4564	*
	1	0	190.7344	243.5048	*
	0.8009	0.778	110.0036	243.4443	*
	0.778	0.8009	100.6851	243.4407	*
	1	0.778	190.6852	243.4564	*
Population III	1	1	303.1200	162.5198	233.7116
	1	0.81	303.2374	162.6161	233.5233
	1	0.77	303.2622	162.6363	233.4834
	1	0	303.7408	163.0296	232.7006
	0.81	0.77	302.9038	162.5448	233.6628
	0.77	0.81	303.0018	162.4935	233.7628
	1	0.77	303.2622	162.6363	233.4834
	1	1.19997	302.9967	162.4188	233.9080
	1.19997	1	302.2291	162.6042	233.5465
	1.19997	0.81	302.2935	162.6846	233.3884
	0.81	1.19997	302.5555	162.277216	234.1812

* The percent relative efficiency is less than 100%.

It is envisaged from table 2 and 3 that for complete response on study and auxiliary variables, our proposed class of estimators $\tau_{1,i}$ performs better than $T_R, T_{BT}, T_{reg}, T_{YK}$ and T_{SP} for populations I and III in terms of PRE with respect to $\bar{y}$. For population I, the estimator $\tau_{1,5}, c_i = C_x$ and $d_i = \rho_{yx}$, performs less efficient among the considered estimators. For population II, the proposed class of estimator $\tau_{1,i}, i = 1,2,\dots,7$, performs better than T_R and T_{SP} but less efficient than other estimators in terms of PRE with respect to the usual unbiased estimator $\bar{y}$. Also, for $c_i = \rho_{yx}$ and $d_i = C_x$, the proposed estimator $\tau_{1,6}$ performs most efficient among the other estimators considered in the study.

Table 4: PRE of existing estimators with respect to $\bar{y}^*$ for situation 2.

		$h = 2$	$h = 3$	$h = 4$	$h = 5$
Population I	T_R^*	433.24	275.79	219.38	190.38
	T_{BT}^*	241.89	194.59	170.94	156.75
	T_{reg}^*	434.11	276.08	219.54	190.49
	T_R^{**}	3807.94	4243.75	4617.64	4941.92
	T_{BT}^{**}	388.30	391.30	393.45	395.07
	T_{reg}^{**}	3893.79	4349.47	4742.10	5083.91
Population II	T_R^*	140.96	132.82	127.38	123.49
	T_{BT}^*	193.48	169.74	155.61	146.25
	T_{reg}^*	199.51	173.66	158.46	148.47
	T_R^{**}	119.55	103.20	*	*
	T_{BT}^{**}	202.58	181.93	169.18	160.51
	T_{reg}^{**}	201.74	176.54	161.61	151.74
Population III	T_R^*	138.01	122.21	115.69	112.13
	T_{BT}^*	122.44	113.76	109.92	107.76
	T_{reg}^*	140.30	123.38	116.47	112.71
	T_R^{**}	202.26	194.37	190.70	188.58
	T_{BT}^{**}	148.09	144.42	142.68	141.67
	T_{reg}^{**}	218.55	211.11	207.63	205.61

* The percent relative efficiency is less than 100%.

Table 5: PRE of proposed estimators with respect to $\bar{y}^*$ for different values of c_i and d_i; $i = 1, 2, \dots, 11$ for situation 2.

			$\tau_{2,i}$				$T_{2,i}$			
Population	c_i	d_i	h=2	h=3	h=4	h=5	h=2	h=3	h=4	h=5

	1	1	3895.7896	4352.2105	4743.6090	5086.0615	*	*	*	*
	1	0.5996	3896.0078	4352.4232	4744.5054	5087.0399	*	*	*	*
	1	0.985	3895.8010	4352.2224	4743.6473	5086.1036	*	*	*	*
	1	0	3896.9997	4352.3236	4745.3552	5087.939	*	*	*	*
	0.5996	0.985	3896.1167	4352.5824	4746.0404	5088.7236	*	*	*	*
	0.985	0.5996	3895.9869	4352.2879	4745.5672	5088.0624	*	*	*	*
	0.985	1	3896.1420	4352.5424	4745.9259	5088.5277	*	*	*	*
Population II	1	1	203.8707	183.4582	170.9766	162.58738	*	*	*	*
	1	0.5996	203.8744	183.4573	170.9773	162.58899	*	*	*	*
	1	0.985	203.8748	183.4572	170.9774	162.58918	*	*	*	*
	1	0	203.8893	183.4534	170.9799	162.59543	*	*	*	*
	0.5996	0.985	205.0797	183.4634	170.3529	161.59765	*	*	*	*
	0.985	0.5996	203.8887	183.4687	170.9637	162.56073	*	*	*	*
	0.985	1	203.8937	183.4699	170.9628	162.55861	*	*	*	*
Population III	1	1	224.3245	220.1681	219.0518	219.07625	209.1400	201.7912	198.7745	197.3475
	1	1.19997	224.3236	220.1781	219.0671	219.09495	200.2747	193.3975	190.6851	189.5007
	1	0.81	224.3252	220.15864	219.0372	219.05835	216.8027	209.5582	206.4915	204.9653
	1	0.77	224.3254	220.1566	219.0341	219.05457	218.1061	211.0070	207.9851	206.4692
	1	0	224.3270	220.1165	218.9733	218.98052	134.4618	143.9264	149.3330	152.9492
	1.19997	1	223.7595	220.4235	219.7644	220.0622	192.6047	190.7458	188.1541	187.0584
	1.19997	0.81	223.7483	220.4228	219.7587	220.0533	201.5875	198.9796	196.0418	194.6818
	0.81	0.77	224.2639	219.5383	219.8482	220.2732	213.7986	212.1212	207.8831	208.1906
	0.81	1.19997	224.2809	219.5806	219.8574	220.2936	218.4273	211.4002	211.2020	209.8146
	0.77	0.81	224.3508	220.0486	219.6543	219.8781	204.2981	205.3100	203.4763	204.2965
	0.77	1	224.3519	220.0627	219.6655	219.8940	217.6433	214.5446	212.5270	212.1277

* The percent relative efficiency is less than 100%. Also, for situation II $T_{1,i}$ is not efficient.

Following points are noted from table 4 and 5

- From table 4, Regression estimator T^{*}_{reg} (non-response on study variable only) and T^{**}_{reg} (non-response on both study and auxiliary variables) performs better to the Cochran's ratio estimators (T^{*}_{R} and T^{**}_{R}) and Singh et.al estimator's (T^{*}_{BT} and T^{**}_{BT}) in terms of PRE with respect to $\bar{y}^{*}$ in all three populations. For all populations, PRE of the estimators decreases with the increase in the value of h, except for population I, PRE of regression estimator T^{**}_{reg} increases with the increase in the value of h.
- From table 5, it is noted that for all three populations, the PRE of the proposed class of estimators $\tau_{2,i}; i = 1, 2, \ldots, 11$ is efficient in all cases as compared to the Cochran's ratio estimator's (T^{*}_{R} and T^{**}_{R}), Singh et.al estimator's (T^{*}_{BT} and T^{**}_{BT}), Classical regression estimators (T^{*}_{reg} and T^{**}_{reg}) and Unal and Kadilar estimators ($T_{1,i}$ and $T_{2,i}$).
- For population I, it is noted that
  - ➢ For $h = 2, c_i = \rho_{yx}$ and $d_i = 1$, the proposed estimator $\tau_{2,7}$ is more efficient among all estimators.
  - ➢ For $h = 3, 4\ 5, c_i = C_x$ and $d_i = \rho_{yx}$, the estimator $\tau_{2,5}$ performs efficient among others.

- PRE of the estimators $\tau_{2,i}$ increases with the increase in the value of h.

- For population II, it is noted that
  - For $h = 2$ and $3, c_i = \rho_{yx}$ and $d_i = 1$, the proposed estimator $\tau_{2,7}$ performs better in terms of PRE with respect to $\bar{y}^*$.
  - For $h = 4, 5, c_i = 1$ and $d_i = 1$, the estimator $\tau_{2,1}$ if efficient among others.
  - PRE of the estimators $\tau_{2,i}, i = 1, 2, \dots, 7$ decreases with the increase in the value of h.
- For population III, it is noted that
  - For $h = 2, c_i = \rho_{yx}$ and $d_i = 1$, the proposed estimator $\tau_{2,11}$ is efficient among all estimators.
  - For $h = 3$, $c_i = \beta_2(x)$ and $d_i = C_x$, the estimator $\tau_{2,7}$ is efficient among others.
  - For $h = 4, 5$, $c_i = C_x$ and $d_i = \beta_2(x)$, the estimator $\tau_{2,9}$ is efficient among all other estimators.
  - PRE of the all the cases of proposed estimator first decreases for $h = 2, 3$ and then increases for $h = 4, 5$, respectively.

Overall, our proposed estimators in both the situations i.e. complete information on study variable as well as on auxiliary variables and non-response on both the study and auxiliary variables are efficient in terms of highest PRE. In next section, we performed the simulation study to show the performance of the proposed estimators over considered estimators.

6. SIMULATION STUDY

Situation 1: To support our study for different sample sizes, we use simulation. We have generated a normal population of size $N = 500$ where $X \sim rnorm(N, 0, 0.14)$ and $Y \sim X + rnorm(N, 0, 0.14)$. The result for the simulated population is presented in Table 5.

Table 6: PRE of estimators with respect to $\bar{y}$ for different values of c_i and d_i; $i = 1, 2, \dots, 11$ for situation 1.

	c_i	d_i	PRE($\tau_{1,i}$)	PRE(T_{YK})	PRE(T_{SP})
n=70	1	1	758.2448	634.2543	100.0196
	1	-25.2945	761.3533	648.5968	99.9988
	1	-0.01512	758.3799	634.2763	81.4315
	1	0	758.3779	634.2757	22.0229
	-25.2945	-0.01512	710.0375	634.5651	18.8757
	-0.01512	-25.2945	574.4438	634.2627	100.0090
	-0.01512	1	574.6953	634.2761	*
	1	2.8234	757.9992	634.3180	100.0090
	2.8234	1	734.0808	634.3155	100.0090
	2.8234	-25.2945	735.4641	775.1426	*
	-25.2945	2.8234	710.0509	792.6500	*
n=140	1	1	493.0974	429.2612	*
	1	211.3417	498.6139	*	*
	1	0.0313	493.0719	429.2155	*
	1	0	493.0711	429.2144	*
	211.3417	0.0313	439.1258	442.5474	*
	0.0313	211.3417	319.5427	429.8240	100.0000
	0.0313	1	318.7342	429.2155	100.0000
	1	2.7302	493.1429	429.3852	*
	2.7342	1	454.1853	429.3856	*
	2.7342	211.3417	455.2057	*	*
	211.3417	2.7342	439.1259	*	*
n	1	1	650.8223	598.8214	100.0003

	1	-29.0866	649.9758	601.3784	*
	1	-0.0023	650.8046	598.8208	*
	1	0	650.8047	598.8208	*
	-29.0866	-0.0023	663.1646	598.8767	*
	-0.0023	-29.0866	577.5721	598.8206	100.0000
	-0.0023	1	577.5813	598.8208	100.0000
	1	2.8234	650.8501	598.8354	100.0004
	2.8234	1	678.8797	598.8350	*
	2.8234	-29.0866	679.5985	617.5137	*
	-29.0866	2.8234	663.1743	619.6078	*

* The percent relative efficiency is less than 100%.

From table 6, it is seen that our proposed class of estimators $\tau_{1,i}$ performs better than, T_{YK} and T_{SP} for different sample sizes in terms of PRE with respect to $\bar{y}$ in case of complete response on study as well as auxiliary variable.

For sample size $n = 70$, the estimator $\tau_{1,6}$, $\tau_{1,7}$, $\tau_{1,10}$and $\tau_{1,11}$ perform less efficient than T_{YK} (Yadav and Kadilar) estimator and for all other cases it is performing better than the other considered estimators.

For sample size n=140, the estimator $\tau_{1,5}$, $\tau_{1,6}$ and $\tau_{1,7}$ performs less efficient than T_{YK}.

At last, for sample size $n = 280$, $\tau_{1,6}$ and $\tau_{1,7}$performs less efficient than T_{YK}.

Situation 2: For Situation 2, we have generated a normal population of size $N = 800$ where $X \sim \text{rnorm}(N, 0.7, 1.5)$ and $Y = 2 + 4 * X$. The result for the simulated population is presented in Table 6 and Table 7. Also, for simulated population, the Unal and Kadilar estimators ($T_{1,i}$ and $T_{2,i}$) are not performing good.

Table 7: PRE of existing estimators with respect to $\bar{y}^*$ for situation 2.

		$h = 2$	$h = 3$	$h = 4$	$h = 5$
n=100	T_R^*	193.1721	193.8012	193.8012	193.8802
	T_{BT}^*	193.9013	194.0094	194.0454	194.0635
	T_{reg}^*	193.7669	193.9420	194.0005	194.0297
	T_R^{**}	203.6920	204.2232	204.4010	204.4901
	T_{BT}^{**}	196.3180	196.4994	196.5600	196.5903
	T_{reg}^{**}	195.7195	195.8308	195.8679	195.8865
n=150	T_R^*	198.6888	199.3419	199.5607	199.6703
	T_{BT}^*	199.8067	199.9033	199.9355	199.9516
	T_{reg}^*	199.5282	199.7637	199.8424	199.8818
	T_R^{**}	209.4933	210.2250	210.4702	210.5931
	T_{BT}^{**}	202.1545	202.3976	202.4788	202.5194
	T_{reg}^{**}	199.8660	199.9326	199.9949	200.0110
n=200	T_R^*	197.1999	197.8506	198.0690	198.1784
	T_{BT}^*	198.2488	198.3780	198.4211	198.4427
	T_{reg}^*	197.9963	198.2514	198.3366	198.3793
	T_R^{**}	207.9179	208.6500	208.8954	209.0184
	T_{BT}^{**}	200.6009	200.8645	200.9526	200.9967
	T_{reg}^{**}	198.5659	198.6957	198.7390	198.7607

Table 8: PRE of proposed estimator with respect to $\bar{y}^*$ for situation 2.

			$\tau_{2,i}$			
	c_i	d_i	h=2	h=3	h=4	h=5
n=100	1	1	1247.0240	2290.9890	3335.9590	4382.0260
	1	1.7587	1238.9200	2281.7090	3325.7430	4371.0820
	1	0.1767	1251.0050	2293.2610	3336.6210	4381.2500
	1	0	1251.1730	2292.6450	3335.2980	4379.3170
	1.7587	0.1767	1269.1010	2313.3170	3357.5300	4402.5780
	0.1767	1.7587	1305.2950	2439.5390	3582.1350	4733.9320
	0.1767	1	1301.2950	2428.3640	3562.8740	4705.0920
	1	2.2574	1231.4940	2272.2390	3314.5910	4358.6100
	2.2574	1	1152.1890	2147.5440	3192.6720	4292.3220

	2.2574	1.7587	1079.4610	2171.8780	3522.8940	5237.2980
	1.7587	2.2574	1117.3800	2129.2280	3225.7040	4417.9950
n=150	1	1	924.8368	1641.6210	2358.7740	3076.4070
	1	1.9572	914.2705	1629.1180	2344.6000	3060.7690
	1	0.0065	929.8934	1645.2530	2360.9230	3077.1460
	1	0	929.9055	1645.2430	2360.8920	3077.0960
	1.9572	0.0065	951.0008	1674.0340	2395.4140	3116.5520
	0.0065	1.9572	967.0626	1761.7270	2563.9680	3374.1680
	0.0065	1	966.8954	1764.3620	2563.2300	3373.0580
	1	2.2330	910.2954	1624.0530	2338.5730	3053.9030
	2.2330	1	842.8369	1507.2260	2200.1260	2924.4160
	2.2330	1.9572	761.4688	1471.6080	2355.8670	3487.6090
	1.9572	2.2330	784.2607	1462.3820	2230.1230	3106.6190
n=200	1	1	740.3329	1273.8880	1807.5510	2341.4560
	1	2.0164	732.0334	1264.3340	1796.9090	2329.8290
	1	0.0300	743.9847	1276.3880	1808.8140	2341.5170
	1	0	744.0277	1276.3580	1808.7090	2341.3400
	2.0164	0.0300	765.4176	1310.5880	1853.719	2396.3340
	0.0300	2.0164	763.3257	1351.0120	1942.4120	2537.8330
	0.0300	1	762.8483	1349.7460	1940.2290	2534.5970
	1	2.2697	729.3121	1260.9410	1792.9020	2325.2600
	2.2697	1	690.6494	1183.9290	1686.8010	2200.6290
	2.2697	2.0164	615.2350	1096.1000	1642.5820	2269.6250
	2.0164	2.2697	632.4573	1111.7140	1632.1700	2199.6310

From table 7 and 8, it is clear that the proposed class of estimator $\tau_{2,i}$ performs better in all cases as compared to the Cochran's ratio estimator's (T_R^* and T_R^{**}), Singh et.al estimator's (T_{BT}^* and T_{BT}^{**}), Classical regression estimators (T_{reg}^* and T_{reg}^{**}). The Unal and Kadilar estimators ($T_{1,i}$ and $T_{2,i}$) doesn't perform better than the proposed and the existing estimators.

7. CONCLUSION

In this paper, a new class of modified exponential estimator has suggested for the population mean of the study variable by using the supplementary information under two situations i.e. Situation I: when there is full response on the study and the auxiliary variables with known population mean of the auxiliary variable; Situation II: when there is non-response on the study and auxiliary variables with known $\bar{X}$. The relative performance of the proposed estimator is compared with conventional estimators and conditions have been obtained. The suggested estimators perform better than the usual unbiased estimator, ratio and regression estimators among other estimators in both the situations for existing population cases as well as for simulated ones. From the above discussion, we recommend our proposed class of estimator will perform efficient in complete response case and also when non-response present on the study and auxiliary variables.

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