INFERENCE FOR INVERTED EXPONENTIATED PARETO DISTRIBUTION BASED ON UPPER RECORD VALUES

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ABSTRACT
Inverted version of exponentiated distributions are found useful as lifetime models. In this paper, we develop maximum likelihood estimators and Bayes estimators of unknown parameters of the inverted exponentiated Pareto distribution based on upper record values. We derive Bayes estimators of the parameters using importance sampling under symmetric and asymmetric loss functions. Credible interval estimation for the parameters is also done. A prediction for future upper record values is proposed based upon non-Bayesian as well as Bayesian approach. Bayes predictive interval estimation is carried out for future upper record value. We assess the validity of the proposed method by using real data and compare the proposed estimators based on mean squared error through a Monte Carlo simulation. The results may be of interest especially when only record values are stored.

KEY WORDS: Maximum Likelihood, Bayes Estimation, Gamma Prior, Credible Interval, Prediction, Simulation

MSC: 62F10,62F15

RESUMEN
La versión invertida de distribuciones exponenciadas ha aparecido como útil para modelos de los tiempos de vida. En este paper, desarrollamos estimadores máximo verosímiles y Bayesianos para los parámetros desconocidos de la distribución Pareto exponenciada invertida basada en los valores mayores de los records. Derivamos estimadores de Bayes de los parámetros usando muestreo de importancia bajo funciones de pérdida simétricas y asimétricas. La estimación creíble de los parámetros también se desarrollò. Una predicción de valores futuros del valor superior de los records es propuesta basándose en los enfoques no-Bayesianos y Bayesianos. La estimación Bayes del intervalo predictivo se desarrolló para valores futuros del valor superior de los records. La validez del método propuesto es validado usando data real y comparando el error cuadrático medio de los estimadores propuestos a través de simulación de Monte Carlo. Los resultados pueden ser de interés especialmente cuando solo valores de los records se almacenan.

PALABRAS CLAVE: Máxima Verosimilitud, Estimación Bayes, Gamma Prior, Intervalo Creíble, Predictivo, Simulación

1. INTRODUCTION

The concept of record values was introduced by Chandler (1952). Two types of record values lower record values and upper record values are observed from the data . Observations of the shortest time for 100m runs or swimming are called lower records values and observation of long jump , high jump, total rainfall of the year greater than the existing respective records, streets-strength life turning data are called upper record values .

Let $X_1, X_2, \dots\dots, X_n$ be a sequence of independent identically distributed random variables with cumulative distribution function (cdf) F(x).Then the observation X_j is called an upper record value if $X_j > X_i$ for every i<j with X_1 as first upper record value. The time indicates for which upper record values occur are given by the record times$\{U_{(n)}, n \geq 1\}$,where

$U_{(n)} = \max\{j \mid j > U(n-1), X_j > X_{U(n-1)}\}$,n>1 with U(1)=1.

A sequence of upper record values are denoted as $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$. A number of research workers have worked on the problems related to the study of records .

Arnold et al (1998) considered a likelihood function for estimating unknown parameter based on record values. Bask and Balakrishnan (2003) obtained a predictive likelihood function for future record values .The

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inferential aspects for different distribution based on records have been studied by many authors. Ahmadi and Arghami (2003) considered comparison of the fisher information in record values. Chaturvedi and Malhotra (2017) made inference for three parameter Burr distribution based on records.
The Bayesian approach becomes more useful for small sample size if sufficient prior information is available. Jaheen (2003) derived Bayesian estimation for the parameters of the Gompertz distribution based on upper record values. Madi and Raqab (2004) provided Bayesian interference to predict the future upper record values based on the observed upper record values from the Pareto distribution.
The Pareto distribution named after Viltredo Pareto (1897) is used as a distribution of income. Now a day it also becomes widely useful in actuarial and life testing experiments as a life time model. Davis and Feldestein (1979) introduced Pareto distribution as a model for life time data .
A generalized form of Pareto distribution known as exponentiated Pareto distribution have also been used by many authors. Mohmoud et. al (2015) and Kumar (2013) have obtained estimators of parameter of exponentiated Pareto distribution under progressively type-II censored data. Nassar et. al (2018) have introduced a new generalization of the exponentiated Pareto distribution and discussed its application. Many authors have used record values for estimating the parameters of exponentiated Pareto distribution. Yoon et. al (2017) have studies lower record values from exponentiated Pareto distribution. They have used classical as well as Bayesian methods of estimation
Inverted version of exponentiated distributions are also found useful as lifetime models. Hassan and Mohamed (2019) have proposed inverted exponentiated Lomax distribution as a new lifetime model. They have used censored data to make interference on the parameters of the model. Not much work based on record values for inverted version of exponentiated distributions are available. Furthermore, we observed that problems of estimation and prediction under both classical and Bayesian approaches for inverted exponentiated Pareto distribution based on upper record values have not been considered. With this motivation, we considered classical and Bayesian inference for inverted exponentiated Pareto (IEP) distribution based on upper record values.
We first consider maximum likelihood estimation of the parameters of IEP distribution based on the upper record values in Section 2. Furthermore, we consider the Bayesian approach in Sections 3 to obtain Bayes estimates and HPD credible intervals of the parameters of the proposed distribution based on the upper record values. The problem of prediction of future upper records is discussed in Section 4. Real data analysis and simulation study are presented in Section 5. Finally, the paper ends with a conclusion in Section 6.
The probability density function (pdf) and cumulative distribution function (cdf) of IEP distribution are given by respectively as,

$$f(x;\theta,\lambda)=\frac{\theta\lambda}{x^2}\left[1-\left(1+\frac{1}{x}\right)^{-\lambda}\right]^{\theta-1}\left(1+\frac{1}{x}\right)^{-(\lambda+1)},\ x>0,\theta>,\lambda>. \tag{1}$$

and

$$F(x;\theta,\lambda)=1-\left[1-(1+\frac{1}{x})^{-\lambda}\right]^{\theta} \tag{2}$$

2. MAXIMUM LIKELIHOOD ESTIMATOR

The point and confidence interval estimation of the parameters of IEP distribution based on upper record value are obtained using the method of maximum likelihood.
Let $x_{U_{(1)}}, x_{U_{(2)}}, \ldots, x_{U_{(m)}}$ be a sequence of upper record values from IEP distribution. For the sake of simplicity, we assume $x_{U_{(i)}}=x_i$, i= 1, 2,…., m
The likelihood function of observed record $\underline{x}$ is then given by

$$L=L(\theta,\lambda|\underline{x})=\prod_{i=1}^{m}f(x_i,\theta,\lambda)/\prod_{i=1}^{m-1}[1-F(x_i,\theta,\lambda)] \tag{3}$$

It follows, from (1), (2) and (3), that

$$L=\frac{\theta^m\lambda^m}{\prod_1^m x_i^2}\prod_{i=1}^{m}[1-(1+\frac{1}{x_i})^{-\lambda}]^{\theta-1}\frac{\prod_{i=1}^{m}(1+\frac{1}{x_i})^{-(\lambda+1)}}{\prod_{i=1}^{m-1}[1-(1+\frac{1}{x_i})^{-\lambda}]^{\theta}} \tag{4}$$

The natural logarithm of the likelihood function in (4) becomes:

$$\log L = m\log\theta + m\log\lambda - 2\sum_1^m \log x_i + \theta\log\left[1-\left(1+\frac{1}{x_m}\right)^{-\lambda}\right] - \sum_1^m \log[1-\left(1+\frac{1}{x_i}\right)^{-\lambda} - (\lambda+1)\sum_1^m \log(1+\frac{1}{x_i}) \tag{5}$$

Differentiating (5) with respect to θ and λ , we have the likelihood equations for θ and λ as

$$\frac{\partial \log L}{\partial \theta} = \frac{m}{\theta} + A_1(\lambda, x_m) = 0 \tag{6}$$
and
$$\frac{\partial \log L}{\partial \theta} = \frac{m}{\lambda} + \theta A_2(\lambda, x_m) - \sum_{i=1}^{m} A_3(\lambda, x_i) - \sum_{i=1}^{m} A_4(x_i) = 0 \tag{7}$$
where,

$A_1(\lambda, x_m) = \log\,[1-(1+\frac{1}{x_m})^{-\lambda}]$

$A_2(\lambda, x_m) = \frac{(1+\frac{1}{x_m})^{-\lambda}\log(1+\frac{1}{x_m})}{1-(1+\frac{1}{x_m})^{-\lambda}}$

$A_3(\lambda, x_i) = \frac{(1+\frac{1}{x_i})^{-\lambda}\log(1+\frac{1}{x_i})}{1-(1+\frac{1}{x_i})^{-\lambda}}$

$A_4(\lambda, x_i) = \log(1+\frac{1}{x_i})$

From equation (6), we can write
$$\theta = \frac{-m}{A_1(\lambda, x_m)} \tag{8}$$
Substituting θ from (8) in equation (7), we get profile likelihood equation of λ as:
$$\frac{m}{\lambda} - \frac{mA_2(\lambda, x_m)}{A_1(\lambda, x_m)} - \sum_1^m A_3(\lambda, x_i) - \sum_1^m A_4(x_i) = 0$$
Therefore, $\hat{\lambda}$, mle of λ, can be defined as the solution of the non-linear equation
$$h(\lambda) = \lambda \tag{9}$$
where
$$h(\lambda) = \frac{m}{\sum_1^m A_4(x_i) + \sum_1^m A_3(\lambda, x_i) + \frac{mA_2(\lambda, x_m)}{A_1(\lambda, x_m)}} \tag{10}$$
The equation (9) is difficult to solve, so iterative procedure is used. We have use R software to solve the equation.

The asymptotic variance-covariance matrix of MLEs for parameters θ and λ are given by the observed information matrix:
$$I(\theta, \lambda) = E\,[-\frac{\partial^2 \log L}{\partial \theta\, \partial \lambda}] \tag{11}$$
The exact expectations in the above expressions are very difficult to obtain. Therefore, we use the observed asymptotic variance-covariance matrix as:
$$\begin{bmatrix} V(\hat{\theta}) & \mathrm{cov}(\hat{\theta}, \hat{\lambda}) \\ \mathrm{cov}(\hat{\theta}, \hat{\lambda}) & V(\hat{\lambda}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \log L}{\partial \theta^2} & -\frac{\partial^2 \log L}{\partial \theta\, \partial \lambda} \\ -\frac{\partial^2 \log L}{\partial \theta\, \partial \lambda} & -\frac{\partial^2 \log L}{\partial \lambda^2} \end{bmatrix}^{-1} \tag{12}$$
with
$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{m}{\theta^2} \tag{13}$$
$$\frac{\partial^2 \log L}{\partial \theta\, \partial \lambda} = A_2(\lambda, x_m) \tag{14}$$
and
$$\frac{\partial^2 \log L}{\partial \lambda^2} = -\frac{m}{\lambda^2} - \frac{\theta {A_2}^2(\lambda, x_m)}{\left(1+\frac{1}{x_m}\right)^{-\lambda}} + \sum_{i=1}^{m} [\frac{{A_3}^2(\lambda, x_i)}{(1+\frac{1}{x_i})^{-\lambda}} \tag{15}$$
Under regularity conditions, the asymptotic properties of the MLE method ensure that
$$\sqrt{n}(\hat{\Psi} - \Psi) \xrightarrow{d} N_2(0, \quad I^{-1}(\Psi)) \quad \text{as } n \to \infty$$
where $\xrightarrow{d}$ denotes the convergence in distribution.

Hence, 100(1-α)% confidence interval for θ and λ are given respectively as follows:
$$\hat{\theta} \pm z_{\alpha/2}\sqrt{V(\hat{\theta})} \qquad \text{and} \qquad \hat{\lambda} \pm z_{\alpha/2}\sqrt{V(\hat{\lambda})} \tag{16}$$
where $z_{\alpha/2}$ is the $[100(1-{}^{\alpha}/_{2})]^{th}$ percentile of standard normal distribution.

3. BAYES ESTIMATION

In this section we derive Bayes estimate of the Parameters of IEP Distribution based on upper record values under symmetric and asymmetric loss functions namely (i) Squared Error loss function(SELF) and (ii) Linex Loss function(LLF) respectively.
We assume gamma (a_1,b_1) and gamma (a_2,b_2) Priors for parameters θ and λ with respective pdf as,

$$\pi_1(\theta) = \frac{e^{-a_1\theta}\theta^{b_1-1}a_1{}^{b_1}}{\Gamma b_1}, \quad \theta>0 \; ; a_1, b_1>0 \tag{17}$$

and

$$\pi_2(\lambda) = \frac{e^{-a_2\lambda}\lambda^{b_2-1}a_2{}^{b_2}}{\Gamma b_2}, \quad \lambda>0 \; ; \; a_2, b_2>0 \tag{18}$$

Here all the prior parameters $a_1, b_1, a_2,$ and b_2 are assumed to be known within their range.
The Joint Posterior distribution of (θ , λ) is given by,

$$\pi\,(\theta, \lambda \,|\, \underline{x}) = \frac{L\,\pi_1(\theta)\pi_2(\lambda)}{\int_\theta \int_\lambda L\,\pi_1(\theta)\pi_2(\lambda)\,d\lambda\,d\theta} \tag{19}$$

Using (4), (17) and (18) the joint posterior distribution of θ and λ reduces to,

$$\pi\,(\theta, \lambda \,|\, \underline{x}) = \frac{\theta^{m+b_1-1}\,e^{-\theta D_1}\,\lambda^{m+b_2-1}\,e^{-\lambda D_2}e^{-D_3}}{D_4} \tag{20}$$

where,

$$D_1 = a_1\text{-}A_1(\lambda_1, x_m)$$
$$D_2 = a_2 + \sum_1^m A_4(x_i)$$
$$D_3 = \sum_1^m \log[1-\left(1+\frac{1}{x_i}\right)^{-\lambda}]$$
$$D_4 = \frac{\Gamma(m+b_1)}{D_1^{m+b_1}}\int_0^\infty \lambda^{m+b_2-1}\,e^{-\lambda D_2}e^{-D_3}d\lambda$$

The marginal posterior distribution of θ and λ can be obtained respectively as,

$$\pi_1(\theta \,|\, \underline{x}) = \int_\lambda \pi\,(\theta, \lambda \,|\, \underline{x})\;d\lambda$$
$$= \frac{\theta^{m+b_1-1}}{D_4}\int_0^\infty \lambda^{m+b_2-1}\,e^{-\lambda D_2 - D_3 - \theta D_1}\,d\lambda,\; 0<\theta<\infty \tag{21}$$

and

$$\pi_2(\lambda|\underline{x}) = \int_\theta \pi\,(\theta, \lambda \,|\, \underline{x})\;d\theta$$
$$= \frac{\Gamma(m+b_1)}{D_4\,D_1{}^{m+b_1}}\;\lambda^{m+b_2-1}\,e^{-\lambda D_2 - D_3},\;\; 0<\lambda<\infty \tag{22}$$

The SELF is a symmetric loss function that assigns equal losses to under and over estimation. The Bayes estimate under SELF is given by the posterior expectation,
That is $\hat{\theta} = E_\pi(\theta)$. (23)
where π is the marginal posterior distribution of θ.
However such a restriction may not be practical. In reliability estimation over estimation is more serious than under estimation. In such situation we consider an asymmetric loss function. There are many types of asymmetric loss functions, but in this paper we consider LINEX asymmetric loss function, introduced by Varian(1975). The LLF is expressed as,
$L(\delta) \propto \exp(\upsilon\delta) - \upsilon\delta - 1,\; \upsilon\neq 0$ where $\delta = \hat{\theta} - \theta$
The sign and magnitude of the parameter υ represents the direction and degree of symmetry, respectively. When υ is positive, than over estimation is more serious than under estimation. The opposite is true when υ is negative, for υ =1 the LLF becomes quite asymmetric about zero. For υ tends to zero the LLF reduces to squared error loss function.
The Bayes estimate of θ under LLF is given by Zellner (1986) as,

$$\hat{\theta} = \frac{-1}{\upsilon}\log[E_\pi(e^{-\upsilon\theta})], \tag{24}$$

provided expectation under marginal posterior distribution(π) of θ exist.
From equations (21) and (22), it is clear that Bayes estimate under SELF and LLF cannot be observed in closed form. In such situation some other methods are used like Lindley's approximation, importance sampling, numerical integration etc.
Here we consider importance sampling method to obtain Bayes estimate of parameter θ and λ under SELF and LLF. The advantage of this method is that we can easily obtain credible interval for the parameters.
First we rewrite the joint posterior distribution of θ and λ from (20) as,

$$\pi\,(\theta, \lambda \,|\, \underline{x}) \propto \frac{\theta^{m+b_1-1}\,e^{-\theta D_1}\,D_1{}^{m+b_1}}{\Gamma(m+b_1)}\,\frac{\lambda^{m+b_2-1}\,e^{-\lambda D_2}\,D_2{}^{m+b_2}}{\Gamma(m+b_2)}\,\frac{e^{-\sum_1^m \log[1-\left(1+\frac{1}{x_i}\right)^{-\lambda}]}}{D_1{}^{m+b_1}}$$

$$=G_{\theta|\lambda}\,(m+b_1,\ D_1)\ \ G_\lambda\,(m+b_2,\ D_2)\ \ W_1(\theta,\lambda) \tag{25}$$

where $G_{\theta|\lambda}\,(m+b_1,\ D_1)$ is a gamma density of θ given λ. $G_\lambda\,(m+b_2,\ D_2)$ is a gamma density of λ. and $W_1(\theta,\lambda)$ is a function of λ and $\underline{x}$ defined as,

$$W_1(\theta,\lambda)\ =\ \frac{e^{-\sum_1^m \log[1-\left(1+\frac{1}{x_i}\right)^{-\lambda}]}}{{D_1}^{m+b_1}} \tag{26}$$

According to the method given in Raqab and Madi (2005) it is quite easy to get a Bayes estimate of parametric function β = γ(θ, λ), using the importance sampling scheme as follows:

Algorithm 1

Step 1. Based on the observed sample upper record values $\{x_1, \dots, x_m\}$ generate λ_1 from $G_\lambda\,(m+b_2,\ D_2)$ distribution.

Step 2. Generate θ_1 from $G_{\theta|\lambda}\,(m+b_1,\ D_1)$ distribution.

Step 3. Repeat Steps 1 and 2, N times and obtain $(\theta_1,\lambda_1), \dots, (\theta_N,\lambda_N)$.

Step 4. Based on the values of θ and λ in Step 3 compute the values $W_1(\theta_1,\lambda_1), W_1(\theta_2,\lambda_2), \dots, W_1(\theta_N,\lambda_N)$.

Step 5. The expected value of γ(θ, λ) under posterior distribution of (θ, λ) can be approximated,

$$\widehat{E}_h(\gamma(\theta,\lambda)) = \frac{\frac{1}{N}\sum_1^N \gamma(\theta_i,\lambda_i) W_1(\theta_i,\lambda_i)}{\frac{1}{N}\sum_1^N W_1(\theta_i,\lambda_i)} \tag{27}$$

Hence we can obtain Bayes estimate of parameter θ and λ under SELF and LLF using(27) in (23) and (24). Now, we obtain the credible interval of θ using the method described by Kundu and Pradhan (2009). The 100 (1 – α)% HPD credible interval for θ will be the interval,

$$R_j = \left(\hat{\theta}^{\left(\frac{j}{N}\right)},\ \hat{\theta}^{\left(\frac{j+(1-\alpha)N}{N}\right)}\right),\quad j = 1,2, \dots, [\alpha N] \tag{28}$$

Such that it has the smallest width.

$$\text{Where } \hat{\theta}^{(\alpha)} = \begin{cases} \theta_{(1)} & \text{if } \alpha = 0 \\ \theta_{(i)} & \text{if } \sum_{j=1}^{i-1} p_j < \alpha < \sum_{j=1}^{i} p_j \end{cases} \tag{29}$$

$$p_i = \frac{W_1(\theta_i,\lambda_i)}{\sum_1^N W_1(\theta_i,\lambda_i)}\ ,\qquad i=1,2, \dots, N \tag{30}$$

And $\{\theta_{(i)}\}$ and $\{\lambda_{(i)}\}$ are the observed values of $\{\theta_i\}$ and $\{\lambda_i\}$ respectively.

4. PREDICTION OF FUTURE UPPER RECORD VALUES

In this section we address the method of predicting the r^{th} upper record values r > m using Non-Bayesian and Bayesian approaches. Ahmadi and Doostparast (2006) have considered Bayesian prediction for some life time distributions based on record values. Nadar et. al (2013) have predicted future upper record values based on the observed record values in the case of Kumaraswamy distribution.

4.1 Non- Bayesian Approach

Suppose that we observed for first m upper record values from the given distribution having pdf f (x, θ, λ). To predict $y = x_r = r^{th}$ upper record value, r >m, consider the joint predictive likelihood of y and θ, λ as given by Basak and Balakrishnan (2003).

$$L(\,y,\theta,\lambda\,;\,\underline{x}) = \prod_{i=1}^m \frac{K(x_i\,;\theta,\lambda)}{\Gamma(r-m)}\,[K(y;\theta,\lambda) - K(x_m\,;\theta,\lambda)]^{r-m-1} k(y;\theta,\lambda) \tag{31}$$

$$\text{where } K(y;\theta,\lambda) = -\ln[1 - F(y;\theta,\lambda)] \tag{32}$$

$$\text{and}\quad k(x_i\,;\theta,\lambda) = \frac{f(x_i\,;\theta,\lambda)}{1-F(x_i;\theta,\lambda)} \tag{33}$$

Using (1), (2) in (31) the predictive likelihood function for the IEP distribution is ,

$$L_p(\,y,\theta,\lambda\,;\,\underline{x}) = \frac{1}{\Gamma(r-m)}\,\theta^r\,\lambda^{m+1}\,\frac{1}{\prod_{i=1}^m x_i^2}\prod_{i=1}^m\left[\frac{\left(1+\frac{1}{x_i}\right)^{-(\lambda+1)}}{1-\left(1+\frac{1}{x_i}\right)^{-\lambda}}\right].\left[\ln\left\{\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}\right\}\right]^{r-m-1} \times$$

$$\frac{\left[1-\left(1+\frac{1}{y}\right)^{-\lambda}\right]^{\theta-1}\left(1+\frac{1}{y}\right)^{-(\lambda+1)}}{y^2},\ y > x_m > x_{m-1} > \dots > x_1 > 0 \tag{34}$$

The log likelihood equation to estimate the parameter θ, λ and y are ,

$$\frac{\partial \log L_p}{\partial \theta} = \frac{r}{\theta} + \log\left[1-\left(1+\frac{1}{y}\right)^{-\lambda}\right] = 0$$

$$\frac{\partial \log L_p}{\partial \lambda} = \frac{m+1}{\lambda} - \sum_1^m \left[\frac{\left(1+\frac{1}{x_i}\right)^{-\lambda}\log\left(1+\frac{1}{x_i}\right)}{1-\left(1+\frac{1}{x_i}\right)^{-\lambda}}\right] - \sum_{i=1}^m \log\left(1+\frac{1}{x_i}\right) + (r-m-1)\left[\frac{-\left(\frac{\left(1+\frac{1}{y}\right)^{-\lambda}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}\right)\log\left(1+\frac{1}{y}\right) + \left(\frac{\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}\right)\log\left(1+\frac{1}{x_m}\right)}{\ln\left(1-\left(1+\frac{1}{x_m}\right)^{-\lambda}\right) - \ln\left(1-\left(1+\frac{1}{y}\right)^{-\lambda}\right)}\right]$$

$$+ \frac{(\theta-1)\left\{\left(1+\frac{1}{y}\right)^{-\lambda}\log\left(1+\frac{1}{y}\right)\right\}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}} - \log\left(1+\frac{1}{y}\right) = 0$$

and

$$\frac{\partial \log L_p}{\partial y} = \frac{(r-m-1)\lambda}{y^2}\left(1+\frac{1}{y}\right)^{-\lambda-1}\left[\frac{1}{\ln\left(1-\left(1+\frac{1}{x_m}\right)^{-\lambda}\right) - \ln\left(1-\left(1+\frac{1}{y}\right)^{-\lambda}\right)}\right] - \frac{2}{y} - \frac{(\theta-1)\lambda\left(1+\frac{1}{y}\right)^{-\lambda-1}}{y^2\left[1-\left(1+\frac{1}{y}\right)^{-\lambda}\right]} + \frac{(\lambda+1)}{y^2\left(1+\frac{1}{y}\right)} = 0$$

After simplification these equations are

$$\theta = \frac{-r}{\log\left[1-\left(1+\frac{1}{y}\right)^{-\lambda}\right]} \tag{35}$$

$$g(\lambda) = \frac{m+1}{\lambda} \tag{36}$$

when,

$$g(\lambda) = \sum_1^m \left[\frac{\left(1+\frac{1}{x_i}\right)^{-\lambda}\log\left(1+\frac{1}{x_i}\right)}{1-\left(1+\frac{1}{x_i}\right)^{-\lambda}}\right] - \frac{(\theta-1)\left\{\left(1+\frac{1}{y}\right)^{-\lambda}\log\left(1+\frac{1}{y}\right)\right\}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}} + \sum_{i=1}^m \log\left(1+\frac{1}{x_i}\right) + \log\left(1+\frac{1}{y}\right) - (r-m-1)\left[\frac{\frac{\left(1+\frac{1}{x_m}\right)^{-\lambda}\log\left(1+\frac{1}{x_m}\right)}{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}} - \frac{\left(1+\frac{1}{y}\right)^{-\lambda}\log\left(1+\frac{1}{y}\right)}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}}{\ln\left(1-\left(1+\frac{1}{x_m}\right)^{-\lambda}\right) - \ln\left(1-\left(1+\frac{1}{y}\right)^{-\lambda}\right)}\right]$$

and

$$2y = \frac{\lambda+1}{\left(1+\frac{1}{y}\right)} + \left[\frac{(r-m-1)\,\lambda\left(1+\frac{1}{y}\right)^{-\lambda-1}}{\ln\left(1-\left(1+\frac{1}{x_m}\right)^{-\lambda}\right) - \ln\left(1-\left(1+\frac{1}{y}\right)^{-\lambda}\right)}\right] - \frac{(\theta-1)\lambda\left(1+\frac{1}{y}\right)^{-\lambda-1}}{\left[1-\left(1+\frac{1}{y}\right)^{-\lambda}\right]} \tag{37}$$

On replacing θ from (35) in (36) and (37) we have a system of two equations which can easily be solved by any numerical method.

4.2 Bayesian Approach

In this section, we consider a prediction of future r^{th} upper record value based on the available first m upper record values using Bayesian approach. Let $y = x_r$, r > m > 1. According to Arnold et al (1998), the Bayes predictive density function of y given $\underline{x} = (x_1, x_2 \ldots, x_m)$ is given by

$$h(y/\underline{x}) = \int_0^\infty \int_0^\infty f(y|x,\theta,\lambda)\pi(\theta,\lambda)\,d\lambda\, d\theta \tag{38}$$

where $f(y|x,\theta,\lambda)$ is the conditional density of $y=x_r$ given x_m, expressed as

$$f(y/\underline{x},\theta,\lambda) = \frac{[K(y;\theta,\lambda)-K(x_m;\theta,\lambda)]^{r-m-1}}{\Gamma(r-m)}\frac{f(y;\theta,\lambda)}{1-F(x_m;\theta,\lambda)},\ x_m<y<\infty \tag{39}$$

From (1), (25), (32) and (39), we have

$$f(y|x,\theta,\lambda) = \frac{\theta^{r-m}\lambda}{\Gamma(r-m)y^2}\frac{\left(1-\left(1+\frac{1}{y}\right)^{-\lambda}\right)^{\theta-1}}{\left(1+\frac{1}{y}\right)^{\lambda+1}}\frac{1}{\left(1-\left(1+\frac{1}{x_m}\right)^{-\lambda}\right)^{\theta}}\left(log\left(\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}\right)\right)^{r-m-1} \tag{40}$$

Using (20) and (40) in (38), the Bayes predictive density can be obtained as

$$h(y/\underline{x}) = \frac{1}{D}\frac{\Gamma(r+b_1)}{\Gamma(r-m)}\left(\frac{1}{y^2(1+{}^1/_y)}\right)\int_0^\infty \frac{\lambda^{m+b_2}e^{-\lambda[D_2+\log\left(1+\frac{1}{y}\right)]}}{[a_1-\log\{1-\left(1+\frac{1}{y}\right)^{-\lambda}\}]^{r+b_1}}\frac{1}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}exp\left(-D_3\left(log\left(\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}\right)\right)^{r-m-1}\right),\ x_m< y<\infty \tag{41}$$

From (41), under SELF, the Bayes point predictor of r^{th} upper record value r>m is obtained as

$\hat{Y} = E[Y/\underline{x}] = \int_{x_m}^{\infty} y\, h(y|\underline{x})\, dy$ (42)

which cannot be simplified in a closed form.

To apply important sampling method, we simplify the joint Bayes predictive distribution of y and λ from (38) as

h (y,λ/$\underline{x}$) = $\int_0^{\infty} f(y/\underline{x}, \theta, \lambda)\pi(\theta, \lambda/\underline{x})\, d\theta$

$$\propto \frac{\lambda}{y^2}\frac{[1+\frac{1}{y}]^{-(1+\lambda)}}{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}\lambda^{m+b_2-1}e^{-\lambda D_2}\left(\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-(1+\frac{1}{y})^{-\lambda}}\right)\frac{e^{-D_3[\log\left\{\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}\right\}]^{r-m-1}}}{[a_1-\log(1-(1+\frac{1}{y})^{-\lambda}]^{r+b_1}}$$

= $f(y/y>x_m)G_{\lambda}(m+b_2,D_2)W_2(\lambda, y)$ (43)

where f (y | y>x_m) is the left truncated density function of X with X>x_m at point x=y, which can be obtained from (1) as

$f_x(y/y>x_m) = \frac{\lambda}{y^2}\frac{[1+\frac{1}{y}]^{-(1+\lambda)}}{1-(1+\frac{1}{x_m})^{-\lambda}}$ (44)

$G_{\lambda}(m+b_2,D_2)$ is the gamma density of parameter λ and $W_2(\lambda,y)$ is a function based on λ and y defined as

$$W_2(\lambda, y) = \left(\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-(1+\frac{1}{y})^{-\lambda}}\right)\frac{e^{-D_3[\log\left\{\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-\left(1+\frac{1}{y}\right)^{-\lambda}}\right\}]^{r-m-1}}}{[a_1-\log(1-(1+\frac{1}{y})^{-\lambda}]^{r+b_1}} \quad (45)$$

The importance sampling method can be applied in the following way to obtain the Bayes predictive estimate of the r^{th} upper record value, r> m.

Algorithm 2

Step 1: Generate λ_1 from G_{λ} (m+b_2,D_2) distribution.

Step 2: Generate y_1 from left truncated IEP distribution for given λ_1.

Step 3: Repeat steps 1 and 2 N times and obtain (λ_1, y_1) , (λ_2, y_2) , …, (λ_N, y_N).

Step 4: Based on the values of λ and y obtained in Step 3, compute the values $W_2(\lambda_1, y_1)$, $W_2(\lambda_2, y_2)$, ……., $W_2(\lambda_N, y_N)$

Then under SELF, the Bayes predictive estimator of r^{th} upper record value is estimated as

$\hat{Y}_{Self} = \hat{E}(y|\underline{x}) = \frac{\frac{1}{N}\sum_1^N y_i W_2(\lambda_i,y_i)}{\frac{1}{N}\sum_1^N W_2(\lambda_i,y_i)}$ (46)

Under LLF, the Bayes estimator of r^{th} upper record value is given by:

$\hat{Y} = -\frac{1}{\nu}\ln\left[\hat{E}_{y|\underline{x}}(e^{-\nu y})\right]$ (47)

where $\hat{E}_{y|\underline{x}}(e^{-\nu y})$ can be obtained as

$\hat{E}_{y|\underline{x}}(e^{-\nu y}) = \frac{\frac{1}{N}\sum_1^N e^{-\nu y_i} W_2(\lambda_i,y_i)}{\frac{1}{N}\sum_1^N W_2(\lambda_i,y_i)}$ (48)

4.3 Bayesian Predictive Interval for r^{th} Upper Record Value y= x_r, r>m

The Bayesian (1-α)100% predictive interval for Y= X_r r>m is obtained by evaluating

P(L < y = x_r< U)=1-α, r>m

In case of equal tail predictive interval we have P(x_m< y <L) = α/2 = P(y ≥ U), where P(x_m< y < L) = $\int_{xm}^{L} h(y|\underline{x})dy$ and P(y ≥ U) = $\int_{U}^{\infty} h(y|\underline{x})dy$

For given value of α, 0 <α< 1, the P(x_m< y < L) can be obtain using (41) as

P(x_m< y < L)

$$= \frac{\Gamma(r+b_1)}{D\Gamma(r-m)}\int_{xm}^{L}\frac{1}{y^2(1+1/y)}\int_0^{\infty}\frac{\lambda^{m+b_2}e^{-\lambda(D_2+\log\left(1+\frac{1}{y}\right))}}{(a_1-\log(1-\left(1+\frac{1}{y}\right)^{-\lambda})^{r+b_1}}\frac{1}{1-(1+\frac{1}{y})^{-\lambda}}e^{-D_3\left(\log\left(\frac{1-\left(1+\frac{1}{x_m}\right)^{-\lambda}}{1-(1+\frac{1}{y})^{-\lambda}}\right)\right)^{r-m-1}} d\lambda dy$$

which can be solved for L by using any method of numerical integration. Similarly we can obtain the value of upper limit U .

5. SIMULATION STUDY AND DATA ANALYSIS

In this section we assess the performance of the estimators using simulation study. A real data set is also considered to exemplify the results obtained in the paper.

5.1. Simulation Study

We perform a simulation study to examine the behavior of the estimators. Behavior of the estimators of the unknown parameters is measured by their mean square error (MSE_s) and length of the confidence intervals. In this section, a simulation study is conducted to observe the behavior of the estimators proposed by different methods of estimation and prediction. *m(* = 5, 8 and 13) number of record values are simulated from the IEP distribution with θ=5 and λ=2. The simulated data are used to compute MLEs, Bayes estimates, MSEs and interval estimates.
We mention that the simulated results may be influenced by the generated records, however the results based on a large number of repetitions may represent the same phenomenon. In this study, the results are based on 1000 repetitions and are reported in Table 1, 2, and 3. Average estimates and means square error (MSE) values are taken into account to compare the performance of proposed estimators. A Simulation is carried out using R software.
Table 1 to Table 3 report Bayes estimates obtained under SELF and LLF corresponding to the values of ν as -0.1(0.04)0.06, maximum likelihood estimates, MSEs and confidence intervals. The hyper-parameter values are fixed as a_1= 3, a_2= 10, b_1= 20 and b_2= 15 so that the prior means remain close to the true parameter values.

Table 1. Results for Bayes estimate and MLE for m =5

Method	ν	$\hat{\theta}$	$\hat{\sigma}$	MSE($\hat{\theta}$)	MSE($\hat{\sigma}$)	CI for θ	CI for σ
LLF	-0.1	5.04003	2.00021	0.07542	0.02039	(4.42906, 5.48352) 1.05445	(1.65821, 2.24881) 0.59059
	-0.06	5.01412	2.00635	0.06019	0.01705	(4.43304, 5.40843) 0.97539	(1.71416, 2.23049) 0.51633
	-0.02	4.97562	1.98832	0.07986	0.01680	(4.20033, 5.35577) 1.15544	(1.66615, 2.24386) 0.57771
	0.02	4.94578	1.96818	0.07466	0.02396	(4.43011, 5.47868) 1.04857	(1.65963, 2.24867) 0.58904
	0.06	4.98048	2.00669	0.08541	0.01414	(4.33676, 5.44091) 1.10414	(1.72051, 2.19597) 0.47546
SELF		5.01775	1.97584	0.07441	0.01852	(4.37816, 5.50362) 1.12546	(1.69276,2.20633) 0.51357
MLE		9.01526	2.54473	43.33210	2.32997	(4.60145, 10.42907) 5.82762	(2.24404, 2.84542) 3.98220

Table 2. Results for Bayes estimate and MLE for m = 8

Method	ν	$\hat{\theta}$	$\hat{\sigma}$	MSE($\hat{\theta}$)	MSE($\hat{\sigma}$)	CI for θ	CI for σ
LLF	-0.1	4.89849	1.99189	0.13096	0.02165	(4.17987, 5.51791) 1.33804	(1.65505, 2.24975) 0.59471
	-0.06	4.94672	2.00274	0.08665	0.01987	(4.25516, 5.44591) 1.19075	(1.67133, 2.21430) 0.54297
	-0.02	4.90114	2.00197	0.12235	0.01722	(4.19608, 5.45982) 1.26374	(1.71385, 2.22894) 0.51509
	0.02	4.90626	1.97895	0.11149	0.01902	(4.18397, 5.45574) 1.27178	(1.60492, 2.20793) 0.60301
	0.06	4.92202	2.00256	0.09487	0.01841	(4.28271, 5.45910) 1.17639	(1.72158, 2.28653) 0.56494
SELF		4.90183	1.98816	0.10838	0.02288	(4.31952, 5.54689) 1.22737	(1.67081, 2.22838) 0.55757
MLE		4.97507	2.14362	8.05752	0.90945	(4.41590, 5.53423) 1.11833	(1.95576, 2.33148) 0.37571

Table 3. Results for Bayes estimate and MLE for m = 13

Method	ν	$\hat{\theta}$	$\hat{\sigma}$	MSE($\hat{\theta}$)	MSE($\hat{\sigma}$)	CI for θ	CI for σ
LLF	-0.1	4.85974	2.02588	0.15544	0.03975	(4.1526, 5.48891) 1.33631	(1.59786, 2.33230) 0.73445
	-0.06	4.86661	2.02791	0.14508	0.03428	(4.14489, 5.44889) 1.30401	(1.62463, 2.36120) 0.73657
	-0.02	4.85437	2.07318	0.13805	0.02914	(4.11326, 5.56327) 1.45001	(1.66135, 2.33506) 0.67371
	0.02	4.81474	2.03004	0.14863	0.03927	(4.10187, 5.47058) 1.36871	(1.49251, 2.38118) 0.88867
	0.06	4.86711	2.04052	0.11107	0.04192	(4.34500, 5.47359) 1.12859	(1.58910, 2.39896) 0.80987
SELF		4.86087	2.04592	0.15656	0.03221	(4.03052, 5.48677) 1.45625	(1.68120, 2.33259) 0.65139
MLE		4.27429	1.94959	1.26189	0.49983	(4.15665, 5.19921) 1.04256	(1.81033, 2.08886) 0.27854

From the Tables 1 to 3, we observe that number (m) of record values increases, behavior of maximum likelihood estimates improve in terms of smaller MSE values. Furthermore, the confidence interval length for asymptotic confidence intervals decrease. It can be seen that simulated asymptotic confidence intervals contain the estimate as well as true values.

As the number of record values increases, MSEs of Bayes estimators increases and lengths of confidence intervals also increase. A smaller value(-0.06) of parameter (ν) of LINEX loss function provides better estimates of the parameters as well as smaller length confidence intervals compared to the estimators obtained under SELF and MLE. In general, Bayes estimates outperform MLEs and Bayes estimates obtained under LLF outperform Bayes estimates obtained under SELF for suitable values of ν. In case of Bayes estimates under LLF we find that as ν increases, estimates of both the parameters decrease.

5.2. Real data application

In this section a real data set regarding the failure terms of 84 Aircrafts windshield is considered . The windshield is a complete piece of experiment on a large aircraft. Their failures do not result in damage of the aircraft but do result in replacement of the windshield.

This data set is used and analyzed by Murthy et. al (2004). It was shown in Maurya et. al (2019) that IEP distribution fits well to the complete data compared to generalized inverted exponentiated distribution and inverted exponentiated Rayleigh distribution.

We present below only the upper record values obtained from the original data .

0.040, 1.866, 2.385, 3.443,3.467, 3.478, 3.578, 3.595,3.699, 3.779, 3.924, 4.035, 4.121, 4.167, 4.240, 4.255, 4.278, 4.305, 4.376, 4.449.

Based on this data set MLEs and Bayes estimates of the parameters are calculated. We have made prediction for the r-th (= 21, 22, 23, 24) upper record values based on m = 20 observed upper record values along with interval estimates. Bayes estimation under SELF and LLF(ν = 40) are obtained using the hyper-parameter values fixed as a_1= 0.13, a_2= 1.2, b_1= 7 and b_2= 2.2.The results are shown in Tables 4 and 5.

Table. 4 Estimates of the parameters based on m= 20 upper record values for real data.

Parameters	MLE	Bayesian method	
		SELF	LLF(ν = 40)
θ	31.43144	32.83528	13.23990
σ	3.71575	3.35203	1.51685

Table. 5 Prediction and interval estimates of r-th upper record value when m= 20

r	Actual value (Y_r)	Classical method	Bayesian method	
			SELF	LLF(ν = 40)
21	4.485	4.44915	4.77473 (4.54781, 5.09598)	4.56171 (4.47298, 4.78773)
22	4.570	4.65947	4.97084 (4.61859, 5.37907)	4.60594 (4.50666, 4.80463)
23	4.602	4.87357	5.27896 (4.92551, 5.71976)	4.65482 (4.55340, 4.82563)
24	4.663	5.09132	5.54493 (5.17173, 5.95583)	4.70621 (4.61490, 4.84997)

From the Tables 4 and 5 we observe that the Bayes prediction method under LLF outperform in prediction for future records compared to the Bayes prediction made under SELF and by classical method.
The predictive interval for future records obtained by Bayesian method provides shorter length under LLF compared to under SELF.
The length of Bayesian predictive interval for future records increases as r increases in case of SELF where as it decreases in case of LLF when $\nu = 40$.

6. CONCLUSION

In this paper, estimation from inverted exponentiated Pareto distribution based on upper record values has been considered. Both the classical and Bayesian inference of the unknown parameters are provided. It is observed that the MLEs of the unknown parameters cannot be obtained in closed form, hence iterative method is considered. We also consider the Bayes estimates of the unknown parameters based on different loss functions, and it is observed that they cannot be obtained in explicit forms, hence importance sampling has been considered. Prediction for future upper record is also obtained based on classical as well as Bayesian approach. In our study we observe that LINEX loss function provides better estimates of the parameters as well as smaller length confidence intervals compared to the estimators obtained under SELF and MLE. In general, Bayes estimates outperform MLEs and Bayes estimates obtained under LLF outperform Bayes estimates obtained under SELF for suitable values of the parameter of LINEX loss function.

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