IMPACT OF DETERIORATION AND COST OF SUBSTITUTION ON INVENTORY DECISIONS FOR COMPLEMENTARY AND SUBSTITUTABLE ITEMS UNDER JOINT REPLENISHMENT

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ABSTRACT
In this paper, impact of deterioration and cost of substitution on optimal inventory decisions for an inventory system of two substitutable items, where one item is composed with two complementary components has been studied. In real word system, substitution frequently occurs in situation of stock-out. Here, the inventory level depletes due to combined impact of demand and deterioration and in situation of stock-out, the unmet demand of one item is fully substituted by another item. In the substitution process, each substituted unit experiences a cost of substitution. The demand and deterioration are taken to be deterministic and constant and joint replenishment policy is also considered. In the proposed model, we formulate the problem in two possible cases: full substitution and no substitution. Further, pseudo-convexity for total inventory cost functions is obtained to get unique optimal solution and the solution procedure is outlined to determine order quantities, which minimize the total inventory cost function. Numerical example and sensitivity analysis are provided to demonstrate the effect of different input parameters on optimal decisions. The results show the substantial improvements in the optimal total cost in the case of full substitution with respect to total optimal cost in the case of no substitution.

KEYWORDS: Complementary and substitutable items, Deterioration, Inventory decisions, Cost of substitution, Joint replenishment.

MSC: 90B05

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1. INTRODUCTION

This paper belongs to the area of inventory system for substitutable and complementary deteriorating items by considering stock-out, two-way, and full substitution. In general, substitution is a process in which one item is substituted by another alternate item to fulfil the customer's demand and items under the process of substitution, are called substitutable items. For example, different brands of milk, different brands of mobiles phone, Coffee and tea, sim card of different companies and different brands of laptops etc. are categorized as substitutable items. In real word system, the phenomenon of stock-out substitution can be experienced frequently and plays vital role in inventory decisions and our daily life because almost all of customers wish to minimize their purchasing time. In current scenario, it is frequently seen that customers buy substitutable

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items instead of going to other shop when they meet with case of stock-out of preferred items. A survey of Anupindi et al. [1] indicates the above phenomenon. Substitution event enhances the efficiency of inventory system as result, inventory system saves the inventory cost. Another advantage of substitution process is to advertise the substitutable items. In the substitution process, one additional cost is induced, called cost of substitution. In real word, there is occurred another real situation which is related to complementary items. Those items which are consumed together, are called complementary items. For example, tyre and tube, mobile phone and its charger, computer hardware and software, mobiles phone and sim card, android mobile phone and its applications are categorized as complementary items. Complementary items experience joint demand and joint purchasing. On being increase in demand of one complementary item, demand of other complementary item also increases i.e. the change of demands of complementary items follows same direction. In the proposed model, on being out of stock of one item due to demand and deterioration, its remaining demand is completely fulfilled by another alternate item. Therefore, there are one possibility of substitution: full substitution (symmetrical substitution). Substitution can be classified into three types of substitution: inventory-based substitution or stock-out based substitution, price-based substitution and assortment-based substitution. Inventory-based substitution occurs when desired item is out of stock, in this substitution its unsatisfied demand may be fulfilled by the substitute items, price-based substitution occurs when the price differences cause the phenomenon of substitution, and in assortment-based substitution, customer prefer those substitute items which are newly added in assortment (Shin et al. [37]). Further, stock-out based substitution can be also categorized as symmetrical and asymmetrical (Kim and Bell [22]), (Rasouli and NakhaiKamalabadi [34]). According to their definition, symmetrical substitution occurs if lost demands of one item are completely met with substitute items and asymmetrical substitution occurs if a fraction of lost demands of one item is met with substitute items. Next concept introduced in this paper is deterioration. Generally, deterioration is defined as spoilage, damage, decay, obsolescence, evaporation and loss of utility of physical goods which results in its reducing usefulness. In many inventory systems, the effect of deterioration cannot be ignored, especially for food industry, fashion industry, chemical industry etc. Nearly 20% of food never reaches to consumers because of decay or spoilage (Sethi and Shruti [36]). Items such as fresh food, vegetables and fashion goods, etc. may be considered as deteriorating items. Thus, this inventory model by considering complementary items, substitutable items and deterioration simultaneously under phenomenon of full substitution is more realistic than other existing models in this direction. In this paper, we study inventory model for two substitutable deteriorating items with full substitution (symmetrical stock-out substitution), where one item is made with two complementary components. For proposed inventory model, the total cost function is derived mathematically for two possible cases and solution procedures are developed. Aim of this model is to determine optimal order quantities which minimize the total inventory cost.
The rest of the paper is organized as follows. Section 2 describes literature review, section 3 involves notations and assumptions, and section 4 describes mathematical formulation and solution. In section 5, solution procedure is suggested to determine optimal total cost and optimal order quantities. In section 6, numerical examples and sensitivity analysis are presented, and finally section 7 refers to the conclusion and future work.

2. LITERATURE REVIEW

Firstly, we discuss the inventory models for deteriorating items and then inventory models for substitutable items. Thereafter, we discuss the inventory model for complementary and substitutable deteriorating items. The fundamental inventory model was studied by Harris [16] and this inventory model was extended by Wilson [41] to obtain formula for the economic order quantity (EOQ). First inventory model for deteriorating items was studied by Whitin [40] and he considered fashion items as deteriorating items. Further, many researchers studied different types of deteriorating inventory models considering realistic phenomenon. The review papers on inventory model for deteriorating items of Goyal and Giri [14], Bakker et al. [2], and Janssen et al. [19] may be referred by the readers. Further, Yong and Wang [42] proposed a production-inventory model for deteriorating items with demand disruption. In real-life, production-inventory systems, demand disruption and deterioration of products cannot be avoided. Pando et al. [33] proposed inventory model for deteriorating items with stock-dependent demand introducing that the holding cost is nonlinear function in both time and stock level. Joint pricing and inventory control for two competing retailers with deteriorating items is studied by Mahmoodi [26]. The first inventory model for substitutable item was developed by McGillivray and Silver [27] by proposing that the substitutable items have equal unit variable cost and shortage penalty. Reader may refer the review paper on inventory models for substitutable items by

Table 1: Taxonomy of past research works in literature.

Research work	Substitutable Item	Direction of substitution	Complementary item	Deterioration	Cost of substitution	Full/Partial substitution
Whitin [40]				✓		
McGillivray and Silvar [37]	✓	Two-way				Partial
Chand et al. [5]	✓	One-way				Full
Drezner et al. [8]	✓	One-way				Full & Partial
Goyal [13]	✓	One-way				Full
Ernst and Kouvelis [10]	✓	Two-way				Full
Gurnani and Drezner [15]	✓	One-way				Full
Hsu. et al. [18]	✓	One-way			✓	Full
Tang and Yin [39]	✓	Two-way				Full
Zhang et al. [44]	✓	One-way				Partial
Liu et al. [24]	✓	One-way				Full
Salameh et al. [35]	✓	One-way				Partial
Yuhong and Shuya [43]			✓			
Krommyda et al. [23]	✓	Two-way				Partial
Giri et al. [12]	✓	Two-way	✓			
Maddah et al. [25]	✓	Two-way				Partial
Benkherouf et al. [3]	✓	One-way				Full
Mishra and shanker [30]	✓	Two-way			✓	Partial
Mishra and shansker [29]	✓	Two-way		✓		Partial
Mishra [28]	✓	Two-way		✓	✓	Partial
Hemmati et al. [17]			✓			
Pan et al. [32]	✓	Two-way				Full
Chen. et al. [7]	✓	Two-way				Partial
Mokhtari [31]	✓	Two-way	✓			Full
Jing and Mu [20]	✓	One-way		✓	✓	Full
Giri et. al. [11]	✓	One-way & two-				Partial
This model	✓	Two-way	✓	✓	✓	Full

Sin et al. [37]. To the best of our knowledge, the research papers on complementary and substitutable items concurrently are very few contributions in literature. Most of research papers on complementary item consist of price decisions and research papers consisting inventory decisions are rarely available in literature. While, this paper consists of inventory decisions and develops an inventory model for two deteriorating items under substitution and completion, by considering partial substitution, cost of substitution, and joint replenishment. Joint replenishment policy is more beneficial in inventory model of two or more than two items because if two or more than two items are ordered jointly then transportation cost, fixed ordering cost can be reduced. Readers may study review paper on joint replenishment by Khouja and Goyal [21]. Summary of literature review related to our article in categories of substitutable items, direction of substitution, complementary items, deterioration, cost of substitution and full or partial substitution are presented in Table 1 as Taxonomy of past research works in literature. Under One-way substitution, Chand et al. [5] studied parts selection model, Drezner et al. [8] developed an EOQ model for two substitutable items considering joint replenishment policy and studied the cases of full substitution, partial substitution and no substitution and investigated that only partial substitution or no substitution may be optimal and full substitution is never optimal. Goyal [13] studied an inventory model for two substitutable products with full substitution. While, Ernst and Kouvelis [10] proposed the effects of selling packaged goods on inventory decisions in which they studied on two individual products and one packaged product and no substitution between individual products but substitution between one of two individual products and packaged product in case of stock-out substitution under two-way and full substitution. Further, Gurnani and Drezner [15] extended the work of Drezner et al. [8] for multiple products with one-way substitution and full substitution. Hsu et al. [18] studied a dynamic lot-size model under one-way item and full substitution where the items are indexed in such a way that a lower-index item may be used to substitute for the demand of a higher-index item while Tang and Yin [39] studied joint ordering and pricing strategies for two substitutable items under two-way and full substitution. Further, considering one-way substitution Zhang et al. [44] studied EOQ

model for two substitutable items with partial substitution, Liu et al. [24] studied two perishable inventory model with full substitution which is inspired by the ABO issue related to the blood bank system and Salameh et al. [35] studied EOQ model for two substitutable items with partial substitution and joint replenishment policy. Salameh et al. [35] extended the work of Drezner et al. [8] by taking partial and two-way substitution. Taking only complementary items, Yuhong and Shuya [43] studied the joint selling of complementary components under brand and retail Competition and Hemmati et al. [17] developed an integrated two-stage model, which consists of one vendor and one buyer for two complementary products under consignment policy and stock-dependent demand. Under two-way substitution Krommyda et al. [23] proposed optimal order quantity model for two substitutable items with stock-dependent demand considering partial substitution, Giri et al. [12] proposed two-echelon supply-chain system, having a competition of selling two substitutable items and one complementary item using common retailer and Maddah et al. [25] extend the work of Salameh et al. [35] and developed an inventory model for multiple substitutable items to obtain optimal order quantities under joint replenishment with partial substitution. While, Benkherouf et al. [3] developed an inventory decision model for finite horizon problem of substitutable items, taking time varying demand under one-way and full substitution. In addition, under two-way and partial substitution Mishra and Shanker [30] proposed an inventory model of two substitutable items to determine optimal order quantities under joint replenishment with cost of substitution, Mishra and Shanker [29] proposed an inventory model of two substitutable deteriorating items under joint replenishment policy to determine optimal ordering quantities and Mishra [28] extended the work of [29], by considering cost of substitution. Further, under two-way substitution Pan et.al [32] developed an inventory replenishment model for two-inventory based substitutable items with full substitution and obtained the optimal replenishment cycle time and ending inventory levels, Chen. et. al. [7] proposed an inventory model for Joint replenishment decision taking shortages, partial demand substitution, and defective items and Mokhtari [31] developed an EOQ model for two-substitutable items, where one item is composed with two complementary components and he considered joint ordering policy and full substitution. Further, Jing and Mu [20] developed a Forecast horizon for dynamic lot sizing model of two perishable products (one of them is fresh and another is frozen) with one-way and full substitution, also considering cost of substitution and Giri et al. [11] developed joint replenishment model for two substitutable items in fixed time horizon with two-way and one-way substitution. Moreover, Taleizadeh et al. [38] studied pricing decisions for two items3, where items may be complementary or substitutable and Edalatpour et al. [9] analysed simultaneous pricing and inventory decisions for complementary and substitutable items with nonlinear holding cost. This article goals to fill the gaps in above direction by considering complementary and substitutable items simultaneously, deterioration, cost of substitution, and full substitution.
This paper is an extension of the work of Mokhtari [31] in two directions: deterioration and cost of substitution, the work of Mishra and Shanker [30] in three directions complementary items, deterioration, and full substitution and the work of Mishra [28] in two directions complementary items and full substitution. To best of our knowledge, research papers on optimal inventory decisions for complementary and substitutable deteriorating items under joint replenishment with cost of substitution, considering two-way and full substitution are not available in literature. So, this inventory model makes models of Mokhtari [31], Mishra and Shanker [30] and Mishra [28] more realistic by introducing these directions of extension.

3. NOTATIONS AND ASSUMPTIONS

In this paper, the following notations and assumptions are used.

3.1. Notations

The following notations are used throughout the paper.

Parameters

D_1, D_2	Demand rates for items 1 and 2.
θ	Deterioration rate of items 1 and 2.
h_1, h_2	Holding cost per unit of time of items 1 and 2.
A_1, A_2	Ordering cost of items 1 (for complementary components $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$) and 2.
a_1, a_2	Usage rates of two complementary components of item 1.
CS_{12}	Unit substitution cost for item 1 when it is substituted by item 2.
CS_{21}	Unit substitution cost for item 2 when it is substituted by item 1.

Intermediate variables

p_1	Time interval during which substitution occurs in situation (i).
p_2	Time interval during which substitution occurs in situation (ii).
t_1, t_2	Time when items 1 and 2 completely depleted.
z	Inventory level of item 2 at time t_1 in situation (i).
z_1, z_2	Inventory level of two complementary components of item 1 at time t_2 in situation (ii).

Decision variables

q_1, q_2	Ordering quantities of two complementary components $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ of item 1.

Q_2 Ordering quantity of item 2.
q_1^*, q_2^*, Q_2^* Optimal ordering quantities in case of full substitution.
$q_{1w}^*, q_{2w}^*, Q_{2w}^*$ Optimal ordering quantities in case of no substitution.

Functions

$I_1(t),\ I_2(t)$ Inventory levels of two complementary components of item 1.
$I_{11}^1(t)$ Inventory level of first complementary component $\boldsymbol{\alpha}_1$ of item 1 when item 1 depleted before item 2.
$I_{12}^1(t)$ Inventory level of second complementary component $\boldsymbol{\alpha}_2$ of item 1 when item 1 depleted before item 2.
$I_2^1(t)$ Inventory level of item 2 when item 1 depleted before item 2.
$I_3^1(t)$ Inventory level of item 2 during full substitution, when item 1 depleted before item 2.
$I_{11}^2(t)$ Inventory level of first complementary component $\boldsymbol{\alpha}_1$ of item 1 when item 2 depleted before item 1.
$I_{12}^2(t)$ Inventory level of second complementary component $\boldsymbol{\alpha}_2$ of item 1 when item 2 depleted before item 1.
$I_2^2(t)$ Inventory level of item 2 when item 2 depleted before item 1.
$I_3^2(t)$ Inventory level of first complementary component $\boldsymbol{\alpha}_1$ of item 1 during full substitution, when item 2 depleted before item 1.
$I_4^2(t)$ Inventory level of second complementary component $\boldsymbol{\alpha}_2$ of item 1 during full substitution, when item 2 depleted before item 1.

Objective functions

- Case of full substitution

TC_1 Total cost per cycle in situation (i).
TC_2 Total cost per cycle in situation (ii).
TCU_1 Total cost per unit time in situation (i).
TCU_2 Total cost per unit time in situation (ii).

- Case of no substitution

TC_W Total cost per cycle.
TCU_W Total cost per unit time.

3.2. Assumptions

The following assumptions are used in mathematical formulation of inventory model.

1. The inventory system contains two substitutable items (similar in quality), where first item is composed with two complementary components.
2. Both items are deteriorating.
3. Joint ordering policy is used.
4. Lead time is zero and replenishment is instantaneous i.e. replenishment rate is infinite.
5. Demand is deterministic and constant.
6. Deterioration rate is deterministic and constant.
7. Substitution is stock-out.
8. Substitution is two-way.
9. Demand of one item can be fully substituted by another item.

Situations (i) and (ii) for case of full substitution are discussed in further section.

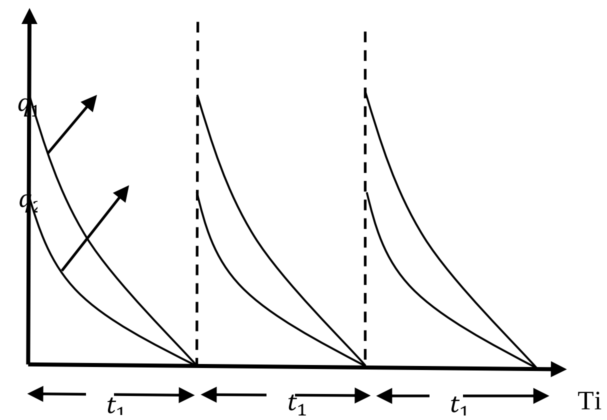

Figure 1: Inventory diagram for two complementary components of item 1.

4. MATHEMATICAL FORMULATION AND SOLUTION

First, we establish the relation between q_1 and q_2. Then, we formulate and find the solution for full substitution and no substitution. It is assumed that item 1 is composed with two complementary components $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ and their consumption rates (usage rates) a_1 and a_2 means that one unit of item 1 is made with a_1 unit of first complementary component $\boldsymbol{\alpha}_1$ and a_2 unit of second complementary component $\boldsymbol{\alpha}_2$. So, demand rates of two complementary components $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are a_1D_1 and a_2D_1 respectively. These components are ordered jointly and replenished instantaneously for the aim of cost saving. Initially, inventory levels of two components $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are q_1 and q_2 respectively whose demand rates are a_1D_1 and a_2D_1. The inventory levels of both complementary components moderately reached to zero on account of deterioration and demand. Inventory diagram for inventory levels of two complementary components of item 1 is represented by figure 1.

Inventory levels of both complementary components of item 1 are governed by the following differential equations.

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -a_1 D_1;\ 0 \le t \le t_1 \qquad (1)$$

With boundary conditions $I_1(0) = q_1$ and $I_1(t_1) = 0$.

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -a_2 D_1;\ 0 \le t \le t_1$$

With boundary conditions $I_2(0) = q_2$ and $I_2(t_1) = 0$. (2)

After solving (1) and (2) we get

$$I_1(t) = \left(q_1 + \frac{a_1 D_1}{\theta}\right) e^{-\theta t} - \frac{a_1 D_1}{\theta};\ 0 \le t \le t_1 \qquad (3)$$

$$I_2(t) = \left(q_2 + \frac{a_2 D_1}{\theta}\right) e^{-\theta t} - \frac{a_2 D_1}{\theta};\ 0 \le t \le t_1 \qquad (4)$$

Now, $I_1(t_1) = 0$ gives as $e^{\theta t_1} = 1 + \frac{\theta q_1}{a_1 D_1}$ (5)

$I_2(t_1) = 0$, gives as $e^{\theta t_1} = 1 + \frac{\theta q_2}{a_2 D_1}$ (6)

From equations (5) and (6), we get

$$q_2 = \left(\frac{a_2}{a_1}\right) q_1 \qquad (7)$$

Which is relation between q_1 and q_2 due to joint replenishment policy.

Now, we developed proposed inventory model for cases; full substitution and no substitution separately. Initially, inventory levels of two complementary components of item 1 are q_1 and q_2 and inventory level of item 2 is Q_2 whose demand rates are $a_1 D_1$, $a_2 D_1$ and D_2 respectively. The inventory levels of both items moderately reached to zero on account of deterioration and demand.

4.1. Case Of Full Substitution

In this case, there are two possible situations;

Situation (i): Item 1 depletes before item 2 i.e. if item 1 is out of stock, as shown in figure 2, then item 1 is completely substituted by the item 2.

Situation (ii): Item 2 depletes before item 1 i.e. if item 2 is out of stock, as shown in figure 3, then item 2 is completely substituted by the item 1.

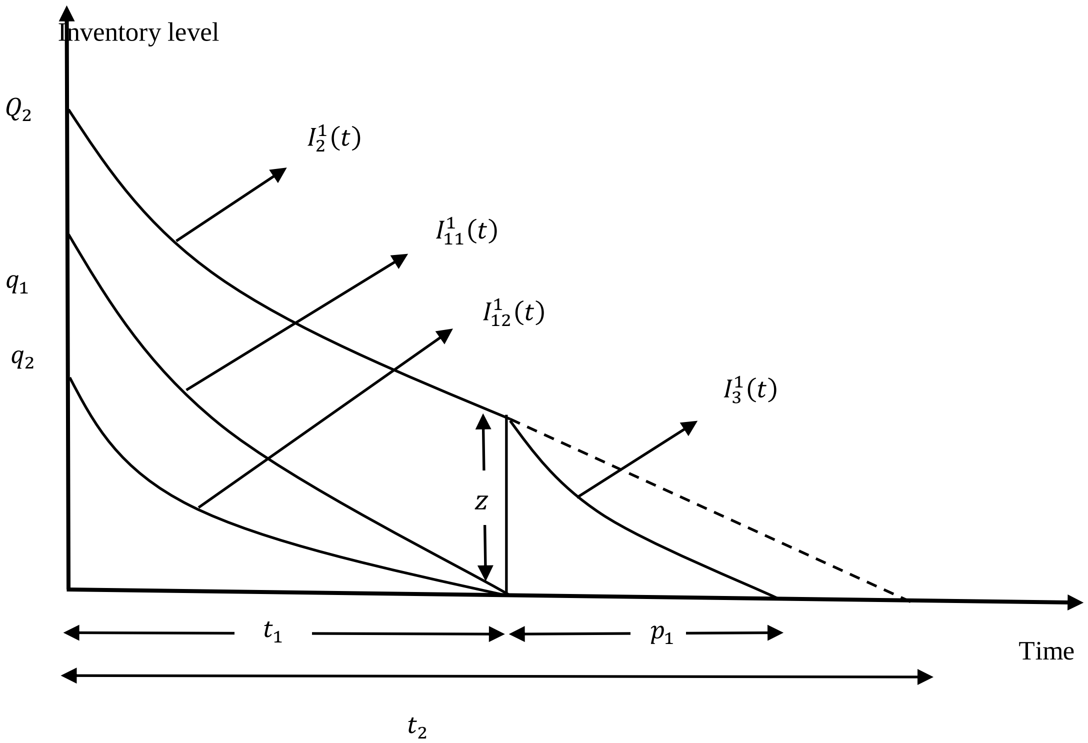

Figure 2: Inventory diagram in situation (i) $(t_1 \le t_2)$.

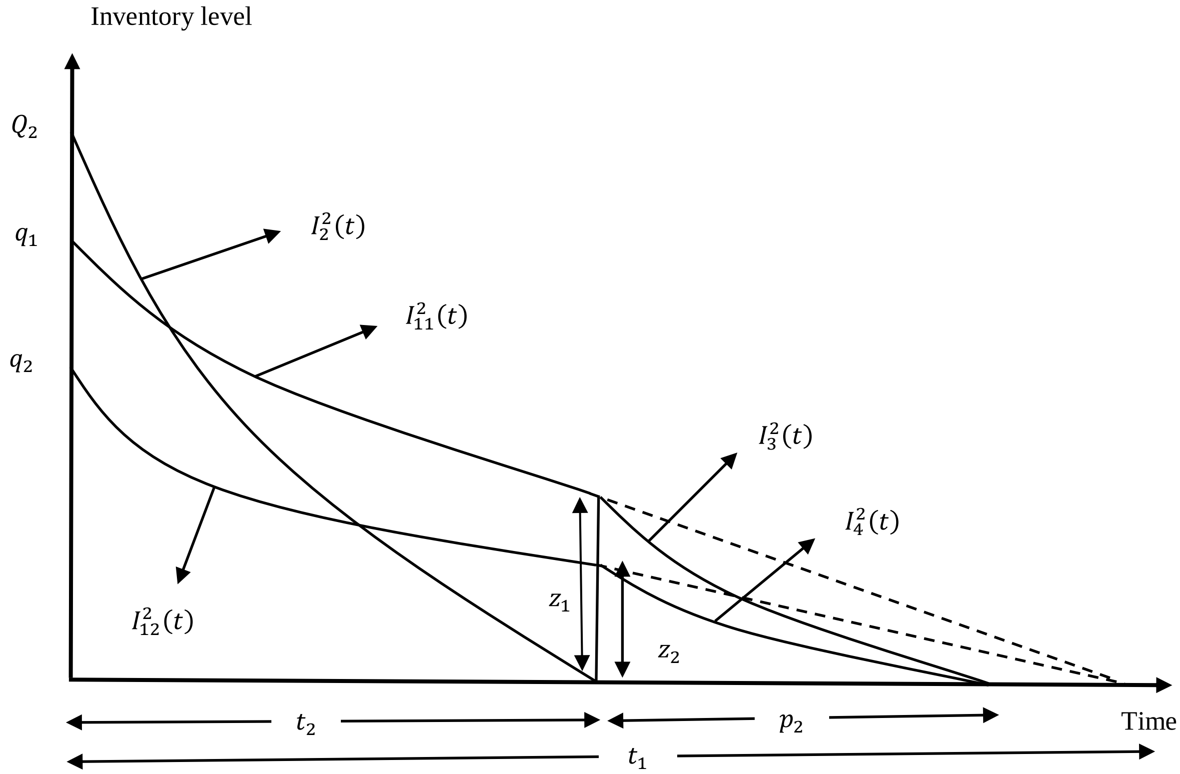

Figure 3: Inventory diagram in situation (ii) $(t_1 \geq t_2)$.

To obtain the total cost per unit in two possible situations, we are describing below.

Situation (i): Item 1 depletes before item 2 $(t_1 \leq t_2)$.

In this situation ($t_1 \leq t_2$ as shown in figure 2), item 1 is completely consumed within inventory cycle of item 2. At this instant, with aim of preventing shortage, substitution occurs for item 1 by item 2. The unsatisfied demand of item 1 is completely fulfilled by remaining inventory of item 2, with consumption rate D_1. Certainly, inventory of item 2 is consumed with consumption rate $(D_1 + D_2)$ during period of substitution (p_1). Here, total cost per inventory cycle consists of fixed ordering costs, holding costs, and cost of substitution and total cost per unit time is obtained by dividing total cost per inventory cycle by length of inventory cycle. To find various cost components, we obtain inventory levels related to this situation.

Inventory levels of items 1 and 2 are governed by following differential equations.

$$\frac{dI_{11}^1(t)}{dt} + \theta I_{11}^1(t) = -a_1 D_1;\ 0 \leq t \leq t_1 \tag{8}$$

With boundary conditions $I_{11}^1(0) = q_1$ and $I_{11}^1(t_1) = 0$

$$\frac{dI_{12}^1(t)}{dt} + \theta I_{12}^1(t) = -a_2 D_1;\ 0 \leq t \leq t_1 \tag{9}$$

With boundary conditions $I_{12}^1(0) = q_2$ and $I_{12}^1(t_1) = 0$

$$\frac{dI_2^1(t)}{dt} + \theta I_2^1(t) = -D_2;\ 0 \leq t \leq t_1 \tag{10}$$

With boundary conditions $I_2^1(0) = Q_2$ and $I_2^1(t_1) = z$

$$\frac{dI_3^1(t)}{dt} + \theta I_3^1(t) = -(D_1 + D_2);\ t_1 \leq t \leq t_1 + p_1 \tag{11}$$

With boundary conditions $I_3^1(0) = z$ and $I_3^1(t_1 + p_1) = 0$

After solving (8), (9), (10), and (11), we get inventory levels

$$I_{11}^1(t) = \left(q_1 + \frac{a_1 D_1}{\theta}\right) e^{-\theta t} - \frac{a_1 D_1}{\theta};\ 0 \leq t \leq t_1 \tag{12}$$

$$I_{12}^1(t) = \left(q_2 + \frac{a_2 D_1}{\theta}\right) e^{-\theta t} - \frac{a_2 D_1}{\theta};\ 0 \leq t \leq t_1 \tag{13}$$

$$I_2^1(t) = \left(Q_2 + \frac{D_2}{\theta}\right) e^{-\theta t} - \frac{D_2}{\theta};\ 0 \leq t \leq t_1 \tag{14}$$

$$I_3^1(t) = \left(\frac{D_1 + D_2}{\theta}\right)\left(e^{\theta(t_1 + p_1 - t)} - 1\right);\ t_1 \leq t \leq (t_1 + p_1) \tag{15}$$

In this situation, the total cost of item 1 per cycle $(t_1 + p_1)$ consisting of fixed ordering costs and holding costs of both complementary components **α1** and **α2** is

$TC_{11} = 2A_1 + h_1 \int_0^{t_1} \left(I_{11}^1(t) + I_{12}^1(t)\right) dt$. Using equations (12) and (13), we get

$$TC_{11} = \left[A_1 + \frac{h_1}{\theta^2}\left(\theta q_1 - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right)\right) + A_1 + \frac{h_1}{\theta^2}\left(\theta q_2 - a_2 D_1 \ln\left(\frac{\theta q_2 + a_2 D_1}{a_2 D_1}\right)\right)\right]$$

(16) Which can be simplified as

$$TC_{11} = \left[2A_1 + \frac{h_1}{\theta^2}\begin{pmatrix}\theta(q_1 + q_2) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) \\ -a_2 D_1 \ln\left(\frac{\theta q_2 + a_2 D_1}{a_2 D_1}\right)\end{pmatrix}\right] \tag{17}$$

To calculate the total cost of item 2 per cycle, firstly, we find:

Inventory level of item 2 at time $t_1$in this situation i.e. inventory level of item 2 when item 1 becomes out of stock is $z = \left(\frac{Q_2 a_1 D_1 - D_2 q_1}{\theta q_1 + a_1 D_1}\right)$ (18)

Time when item 1 is completely depleted is $t_1 = \frac{1}{\theta}\ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right)$ (19)

Substitution period is $p_1 = \frac{1}{\theta}\ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)$ (20)

Length of inventory cycle $= t_1 + p_1 = \frac{1}{\theta}\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right)$

(21)

In this situation, the total cost of item 2 per cycle consisting of fixed ordering cost and holding cost is $TC_{12} = A_2 + h_2\left(\int_0^{t_1} I_2^1(t)\, dt + \int_{t_1}^{t_1 + p_1} I_3^1(t)\, dt\right)$

Using equations (14) and (15), we get

$$TC_{12} = \left[A_2 + \frac{h_2}{\theta^2}\begin{pmatrix}\theta Q_2 - D_1 \ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right) \\ - D_2 \ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right)\end{pmatrix}\right] \tag{22}$$

Now, total number of substituted units for item 1 by item 2 per cycle

$= D_1 p_1 = \frac{D_1}{\theta}\ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)$ (23)

Cost of substitution $= CS_{12}\frac{D_1}{\theta}\ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)$ (24)

Total cost per cycle is sum of total cost of item 1 per cycle, total cost of item 2 per cycle and cost of substitution i.e. $TC_1 = TC_{11} + TC_{12} +$ cost of substitution, which gives as

$$TC_1 = \begin{bmatrix}(2A_1 + A_2) + \frac{h_1}{\theta^2}\left(\theta(q_1 + q_2) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) - a_2 D_1 \ln\left(\frac{\theta q_2 + a_2 D_1}{a_2 D_1}\right)\right) \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_1 \ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right) - D_2 \ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right)\right) \\ + CS_{12}\frac{D_1}{\theta}\ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)\end{bmatrix}$$

(25)

Finally, in this situation total cost per unit time, $(TCU_1) = TC_1 / (t_1 + p_1)$

$$\text{So, } TCU_1 = \frac{\theta}{\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right)}\begin{bmatrix}(2A_1 + A_2) + \frac{h_1}{\theta^2}\begin{pmatrix}\theta(q_1 + q_2) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) \\ - a_2 D_1 \ln\left(\frac{\theta q_2 + a_2 D_1}{a_2 D_1}\right)\end{pmatrix} \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_1 \ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)\right) \\ - \frac{h_2 D_2}{\theta^2}\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right) \\ + CS_{12}\frac{D_1}{\theta}\ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)\end{bmatrix} \tag{26}$$

Using, joint replenishment condition, $q_2 = \left(\frac{a_2}{a_1}\right) q_1$ TCU_1 can be written in terms of $q_1$and Q_2 as,

$$TCU_1 = \frac{\theta}{\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right)}\begin{bmatrix}(2A_1 + A_2) + \frac{h_1}{\theta^2}\begin{pmatrix}\theta\left(q_1 + \frac{a_2}{a_1}q_1\right) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) \\ - a_2 D_1 \ln\left(\frac{\theta\frac{a_2}{a_1}q_1 + a_2 D_1}{a_2 D_1}\right)\end{pmatrix} \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_1 \ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)\right) \\ - \frac{h_2 D_2}{\theta^2}\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right) \\ + CS_{12}\frac{D_1}{\theta}\ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1 + D_2)}{(D_1 + D_2)(\theta q_1 + a_1 D_1)}\right)\end{bmatrix}$$

(27)

Now, TCU_1 is a function of decision variables q_1 and Q_2. Hence, condition for phenomenon of this situation i.e. $t_1 \leq t_2$ can be expressed in terms of Q_2 and q_1 as $\frac{q_1}{a_1 D_1} \leq \frac{Q_2}{D_2}$, which will works as constraint of optimization problem for this situation, described in section 5.

Situation (ii): Item 2 depletes before item 1 $(t_1 \geq t_2)$.

In this situation, inventory diagram is shown in figure 3 and using approach analogous to situation (i), we can obtain total inventory cost for this situation.

So, total cost per cycle is sum of total cost of item 1 per cycle, total cost of item 2 per cycle and cost of substitution i.e. $TC_2 = TC_{21} + TC_{22}$ + cost of substitution, which gives as

$$TC_2 = \left[\begin{array}{l} (2A_1 + A_2) + \frac{h_1}{\theta^2}\left(\begin{array}{l} \frac{\theta Q_2\left(\theta(q_1+q_2)+D_1(a_1+a_2)\right)}{(\theta Q_2+D_2)} \\ -D_1(a_1+a_2)\ln\left(\frac{\theta Q_2+D_2}{D_2}\right) \\ +\theta\left(\frac{D_2(q_1+q_2)-Q_2 D_1(a_1+a_2)}{(\theta Q_2+D_2)}\right) \end{array}\right) \\ -\frac{h_1(D_1+D_2)}{\theta^2}\left(\begin{array}{l} a_1 \ln\left(\frac{\theta D_2(Q_2 a_1+q_1)+a_1 D_2(D_1+D_2)}{a_1(D_1+D_2)(\theta Q_2+D_2)}\right) \\ +a_2 \ln\left(\frac{\theta D_2(Q_2 a_2+q_2)+a_2 D_2(D_1+D_2)}{a_2(D_1+D_2)(\theta Q_2+D_2)}\right) \end{array}\right) \\ +\frac{h_2}{\theta^2}\left(\theta Q_2 - D_2\ln\left(\frac{\theta Q_2+D_2}{D_2}\right)\right) \\ +CS_{21}\frac{D_2(a_1+a_2)}{\theta}\ln\left(\frac{\theta D_2(Q_2 a_1+q_1)+a_1 D_2(D_1+D_2)}{a_1(D_1+D_2)(\theta Q_2+D_2)}\right) \end{array}\right] \tag{28}$$

Finally, in this situation, total cost per unit time, $TCU_2 = TC_2/\,(t_2 + p_2)$, which gives as

$$TCU_2 = \frac{\theta}{\ln\left(\frac{\theta(Q_2 a_1+q_1)+a_1(D_1+D_2)}{a_1(D_1+D_2)}\right)}\left[\begin{array}{l} (2A_1 + A_2) + \frac{h_1}{\theta^2}\left(\begin{array}{l} \frac{\theta Q_2\left(\theta(q_1+q_2)+D_1(a_1+a_2)\right)}{(\theta Q_2+D_2)} \\ -D_1(a_1+a_2)\ln\left(\frac{\theta Q_2+D_2}{D_2}\right) \\ +\theta\left(\frac{D_2(q_1+q_2)-Q_2 D_1(a_1+a_2)}{(\theta Q_2+D_2)}\right) \end{array}\right) \\ -\frac{h_1(D_1+D_2)}{\theta^2}\left(\begin{array}{l} a_1 \ln\left(\frac{\theta D_2(Q_2 a_1+q_1)+a_1 D_2(D_1+D_2)}{a_1(D_1+D_2)(\theta Q_2+D_2)}\right) \\ +a_2 \ln\left(\frac{\theta D_2(Q_2 a_2+q_2)+a_2 D_2(D_1+D_2)}{a_2(D_1+D_2)(\theta Q_2+D_2)}\right) \end{array}\right) \\ +\frac{h_2}{\theta^2}\left(\theta Q_2 - D_2\ln\left(\frac{\theta Q_2+D_2}{D_2}\right)\right) \\ +CS_{21}\frac{D_2(a_1+a_2)}{\theta}\ln\left(\frac{\theta D_2(Q_2 a_1+q_1)+a_1 D_2(D_1+D_2)}{a_1(D_1+D_2)(\theta Q_2+D_2)}\right) \end{array}\right] \tag{29}$$

Using joint replenishment condition, $q_2 = \left(\frac{a_2}{a_1}\right) q_1$, TCU_2 can be written in terms of q_1 and Q_2 as

$$TCU_2 = \frac{\theta}{\ln\left(\frac{\theta(Q_2 a_1+q_1)+a_1(D_1+D_2)}{a_1(D_1+D_2)}\right)}\left[\begin{array}{l} (2A_1 + A_2) + \frac{h_1}{\theta^2}\left(\begin{array}{l} \frac{\theta Q_2\left(\theta\left(q_1+\frac{a_2}{a_1}q_1\right)+D_1(a_1+a_2)\right)}{(\theta Q_2+D_2)} \\ -D_1(a_1+a_2)\ln\left(\frac{\theta Q_2+D_2}{D_2}\right) \\ +\theta\left(\frac{D_2\left(q_1+\frac{a_2}{a_1}q_1\right)-Q_2 D_1(a_1+a_2)}{(\theta Q_2+D_2)}\right) \end{array}\right) \\ -\frac{h_1(D_1+D_2)}{\theta^2}\left(\begin{array}{l} a_1 \ln\left(\frac{\theta D_2(Q_2 a_1+q_1)+a_1 D_2(D_1+D_2)}{a_1(D_1+D_2)(\theta Q_2+D_2)}\right) \\ +a_2 \ln\left(\frac{\theta D_2\left(Q_2 a_2+\frac{a_2}{a_1}q_1\right)+a_2 D_2(D_1+D_2)}{a_2(D_1+D_2)(\theta Q_2+D_2)}\right) \end{array}\right) \\ +\frac{h_2}{\theta^2}\left(\theta Q_2 - D_2\ln\left(\frac{\theta Q_2+D_2}{D_2}\right)\right) \\ +CS_{21}\frac{D_2(a_1+a_2)}{\theta}\ln\left(\frac{\theta D_2(Q_2 a_1+q_1)+a_1 D_2(D_1+D_2)}{a_1(D_1+D_2)(\theta Q_2+D_2)}\right) \end{array}\right] \tag{30}$$

Now, TCU_2 is a function of decision variables q_1 and Q_2. Hence, condition for phenomenon of this situation i.e. $t_1 \geq t_2$ can be expressed in terms of q_1 and Q_2 as $\frac{q_1}{a_1 D_1} \geq \frac{Q_2}{D_2}$, which will works as constraint of optimization problem for this situation, described in section 5.

4.2. Case of no Substitution

Items 1 and 2 deplete simultaneously $(t_1 = t_2)$ i.e. both items become out of stock at the same time as shown in figure 4.

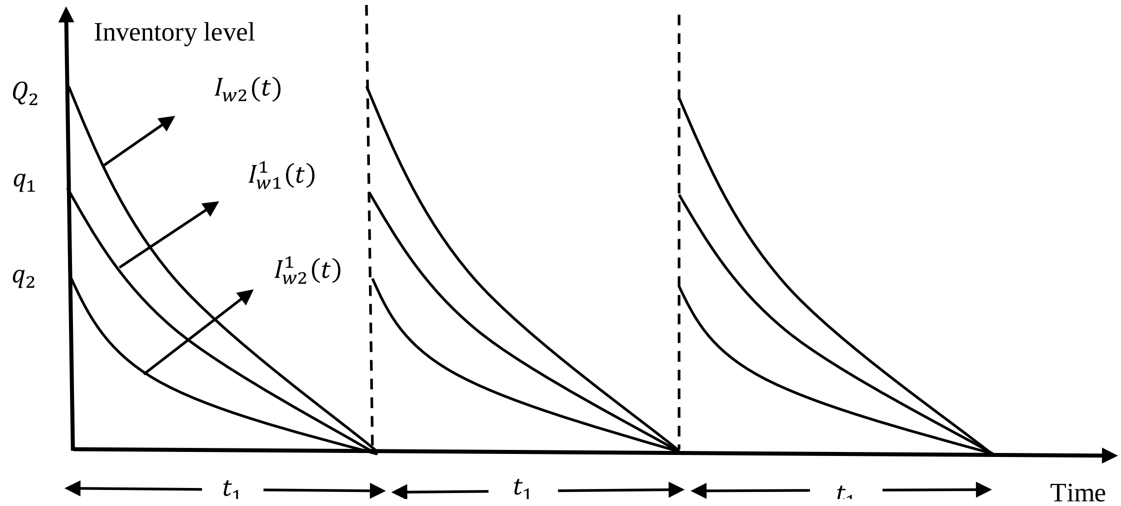

Figure 4: Inventory diagram in case of no substitution.

Total cost per cycle with no substitution under joint replenishment condition consisting of fixed ordering costs and holding costs is

$$TC_W = \begin{bmatrix} (2A_1 + A_2) + \frac{h_1}{\theta^2}\begin{pmatrix} \theta(q_1 + q_2) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) \\ - a_2 D_1 \ln\left(\frac{\theta q_2 + a_2 D_1}{a_2 D_1}\right) \end{pmatrix} \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_2 \ln\left(\frac{\theta Q_2 + D_2}{D_2}\right)\right) \end{bmatrix} \tag{31}$$

Thus, total cost per unit time with no substitution under joint replenishment is given by

$$TCU_W = \frac{\theta}{\ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right)} \begin{bmatrix} (2A_1 + A_2) + \frac{h_1}{\theta^2}\begin{pmatrix} \theta(q_1 + q_2) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) \\ - a_2 D_1 \ln\left(\frac{\theta q_2 + a_2 D_1}{a_2 D_1}\right) \end{pmatrix} \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_2 \ln\left(\frac{\theta Q_2 + D_2}{D_2}\right)\right) \end{bmatrix} \tag{32}$$

Using condition $q_2 = \left(\frac{a_2}{a_1}\right) q_1$, TC_W can be written in terms of q_1 and Q_2 as

$$TCU_W = \frac{\theta}{\ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right)} \begin{bmatrix} (2A_1 + A_2) + \frac{h_1}{\theta^2}\begin{pmatrix} \theta\left(q_1 + \frac{a_2}{a_1} q_1\right) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) \\ - a_2 D_1 \ln\left(\frac{\theta \frac{a_2}{a_1} q_1 + a_2 D_1}{a_1 D_1}\right) \end{pmatrix} \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_2 \ln\left(\frac{\theta Q_2 + D_2}{D_2}\right)\right) \end{bmatrix} \tag{33}$$

Now, TCU_W is a function of decision variables q_1 and Q_2. Hence condition for phenomenon of this situation i.e. $t_1 = t_2$ can expressed in terms of q_1 and Q_2 as $\frac{q_1}{a_1 D_1} = \frac{Q_2}{D_2}$, which will works as constraint of optimization problem for this situation, described in section 5.

Consequently, condition $\frac{q_1}{a_1 D_1} = \frac{Q_2}{D_2}$ can be also treated as condition of joint replenishment for the case of no substitution.

5. SOLUTION PROCEDURE

In this section, firstly, we prove pseudo-convexity for total cost functions in situations (i) and (ii) for full substitution as result of which, the total inventory costs function attains a unique optimal solution. Theorems for pseudo-convexity stated as:

Theorem 1-The total cost function (TCU_1) is pseudo-convex if $h_2 = CS_{12}\theta$.

Proof – See Appendix 1(a).

Theorem 2-The total cost function (TCU_2) is pseudo-convex if $h_1 = CS_{21}\theta$.

Proof – See Appendix 1(b).

Optimal order quantities and optimal total cost will be determined by using the following algorithm.

Algorithm to determine optimal order quantities

Step I Initialize the values of parameters of inventory system

Step II- Solve the nonlinear constrained optimization problem for situation (i) and (ii) of full substitution as follows:

OP₁ – Find (q_1, Q_2) such that min (TCU_1) subject to $\frac{q_1}{a_1 D_1} \leq \frac{Q_2}{D_2}$, $q_1, Q_2 \geq 0$

OP₂ – Find (q_1, Q_2) such that min (TCU_2) subject to $\frac{q_1}{a_1 D_1} \geq \frac{Q_2}{D_2}$, $q_1, Q_2 \geq 0$

Step III- To find optimal total cost (TCU^*), we use $TCU^* =$ Min (minTCU_1, minTCU_2). Optimal ordering quantities corresponding to $TCU^*$are q_1^* and Q_2^*, and value of q_2^* is calculated by $q_2^* = \left(\frac{a_2}{a_1}\right) q_1^*$.

Step IV- Find (q_1, Q_2) such that min (TCU_W) subject to $\frac{q_1}{a_1 D_1} = \frac{Q_2}{D_2}$, $q_1, Q_2 \geq 0$

Step V- Compare optimal total costs obtained in **Step III** and **Step IV.**

6. NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS

6.1. Numerical Example

In this section, to illustrate the applicability and performance of proposed inventory system, we introduce and describe a numerical example. Here, we provide a numerical example whose initial parameters as defined in Table 2.

Table 2. (Initial parameters)

Parameter	Item 1	Item 2
Demand rate D_1, D_2	500	750
Fixed ordering cost A_1, A_2	250	200
Usages rate a_1, a_2	3	4
Holding cost h_1, h_2	0.81	0.81
Deterioration rate θ	0.9	0.9
Cost of substitution CS_{12}, CS_{21}	0.9	0.9

According to algorithm described in above section, the constraint optimization problems are solved using Maple software. Optimal solution of first optimization problem (OP₁) is $q_1 = 277.77$, $Q_2 = 1563.07$, $TCU_1 = 1791.09$ and optimal solution of second optimization problem (OP₂) is $q_1 = 1044.25$, $Q_2 = 522.12$, TCU_2 =2396.57. From **Step III,** we observe that first optimization problem (OP₁) gives the optimal solution of original optimization problem in case of full substitution. Hence, optimal solution of original problem is $q_1^* = 277.77$, $Q_2^* = 1563.07$, $TCU^* = 1791.09$ and $q_2^* = 370.35$. The optimal solution in case of no substitution is $q_{1w}^* = 1044.25$, $Q_{2w}^* = 522.12$, $TCU_W^* = 2396.57$ and q_{2w}^* =1392.33. By introducing phenomenon of full substitution, total inventory cost diminishes from 2396.57 to 1791.09 that shows **25.26 %** saving. Pseudo-convexity of total cost function TCU_1 is shown by graphically in figure 5, figure 6, and figure 7.

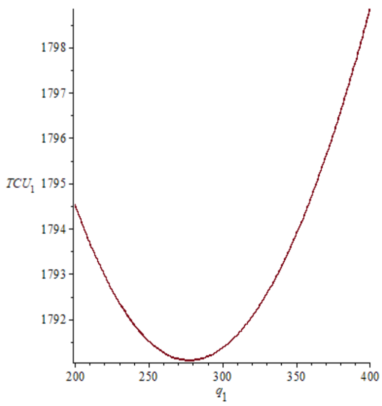

Figure 5: Total cost function (TCU_1) vs. q_1, keeping Q_2 as constant.

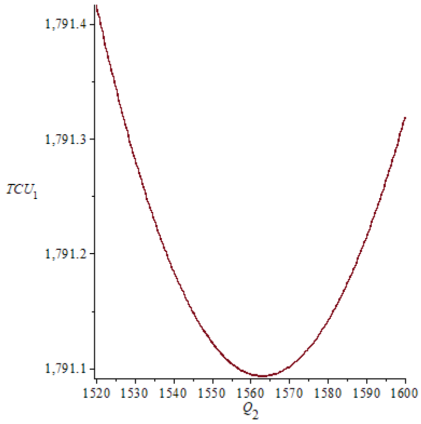

Figure 6: Total cost function (TCU_1) vs. Q_2, keeping $q_1$as constant.

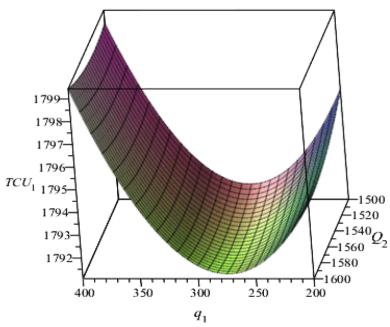

Figure 7: Total cost function (TCU_1) vs. q_1 and Q_2.

6.1. Sensitivity Analysis

Table 3: Sensitivity analysis for optimal solution with respect to various parameters.

Parameter	**Full substitution**				**No substitution**			**% improvement in optimal total cost**
Parameter	Value of para-meter	TCU^*	q_1^*	Q_2^*	TC_W^*	q_{1w}^*	Q_{2w}^*	$\% I_F$
A_1	150	1545.64	277.77	1260.05	2000.00	871.46	435.73	22.72
	200	1672.74	277.77	1416.96	2205.46	960.98	480.49	24.15
	250	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26
	300	1902.59	277.77	1700.74	2576.29	1122.56	561.28	26.15
	350	2008.56	277.77	1831.00	2746.00	1196.82	598.41	26.86
A_2	100	1672.74	277.77	1416.96	2205.46	960.98	480.49	24.15
	150	1732.87	277.77	1491.20	2302.60	1003.31	501.65	24.74
	200	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26
	250	1847.60	277.77	1632.85	2487.71	1083.97	541.98	25.73
	300	1902.59	277.77	1700.74	2576.29	1122.56	561.28	26.15
D_1	300	1530.42	166.66	1500.52	1997.32	778.68	648.90	23.38
	400	1662.16	222.22	1533.16	2206.93	920.99	575.62	24.68
	500	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26
	600	1917.51	333.33	1589.52	2571.04	1154.22	480.92	25.42
	700	2041.65	388.88	1613.16	2733.49	1254.30	447.96	25.31
D_2	550	1693.56	277.77	1442.66	2344.10	1071.83	393.00	27.75
	650	1743.40	277.77	1504.2	2370.49	1057.78	458.37	26.45
	750	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26
	850	1836.87	277.77	1619.60	2422.33	1031.22	584.35	24.17
	950	1880.97	277.77	1674.04	2447.80	1018.64	645.14	23.16
a_1	1	1769.95	138.88	1490.68	2119.91	402.64	603.96	16.51
	2	1782.54	222.22	1534.00	2262.90	744.98	558.74	21.23
	3	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26
	4	1797.27	317.46	1583.94	2522.53	1311.26	491.72	28.75
	5	1801.96	347.22	1599.64	2641.99	1553.2	465.96	31.80
a_2	2	1769.95	416.66	1490.68	2119.91	1207.92	603.96	16.51
	3	1782.54	333.33	1534.00	2262.90	1117.48	588.74	21.23
	4	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26
	5	1797.27	238.09	1583.94	2522.53	983.44	491.72	28.75
	6	1801.96	208.33	1599.64	2641.99	931.92	465.96	31.80
h_1	0.71	1783.89	324.51	1538.60	2281.56	1106.65	553.32	21.81
	0.76	1787.76	299.33	1551.77	2339.84	1074.14	537.07	23.59
	0.81	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26
	0.86	1793.99	259.11	1572.87	2451.86	1016.67	508.33	26.83
	0.91	1796.53	242.80	1581.45	2505.83	991.1	495.55	28.31
h_2	0.71	1717.44	272.17	1694.40	2372.44	1056.76	528.38	27.61
	0.76	1754.9	274.94	1625.33	2384.54	1050.45	525.22	26.41
	0.81	1791.09	277.77	1563.07	2396.57	1044.25	522.12	25.26

Sensitivity analysis is defined as a systematic procedure to study the effect of changes in value of parameters of inventory model on optimum values. In real situations, a substantial impact on optimal values of inventory model is seen on changing the values of parameters of inventory model. In this proposed model, we investigate the impact of values of parameters of this model- ordering costs A_1, A_2, rate of demands D_1, D_2, usages rates a_1, a_2, holding costs h_1, h_2, deterioration rate θ and cost of substitution CS_{12}. Sensitivity analysis for optimal solution with respect to various parameters is given by Table 3.

Table 3 reflects that ordering costs A_1, A_2 have positive impact on optimal total inventory cost in full substitution and no substitution, and percentage improvement in full substitution as well as (shown in figure 8), whereas usages rates a_1, a_2 have equal positive impact (shown in figure 10). Demand rates D_1, D_2 also have positive impact on optimal total inventory cost in full substitution and no substitution and D_2 has negative impact on percentage improvement, whereas demand rate D_1 has positive impact on percentage improvement except at $D_1 = 700$ (slight decrease) (shown in figure 9). Holding cost h_1 has positive impact on optimal total inventory cost in full substitution, no substitution, and percentage improvements and h_2 has same types of effect as $h_1$on optimal total inventory costs in full substitution and no substitution and has negative impact on percentage improvement (shown in figure 11). Deterioration rate θ has positive impact on optimal total inventory cost in full substitution and no substitution, while it has negative impact on percentage improvement (shown in figure 12). Further, cost of substitution CS_{12} has positive impact on optimal total inventory cost in full substitution and has no impact on optimal total inventory cost in no substitution, whereas it has negative impact on percentage improvement (shown in figure 12).

Sensitivity graphs of optimal total inventory costs in full substitution and no substitution, and percentage improvement are shown in below figures.

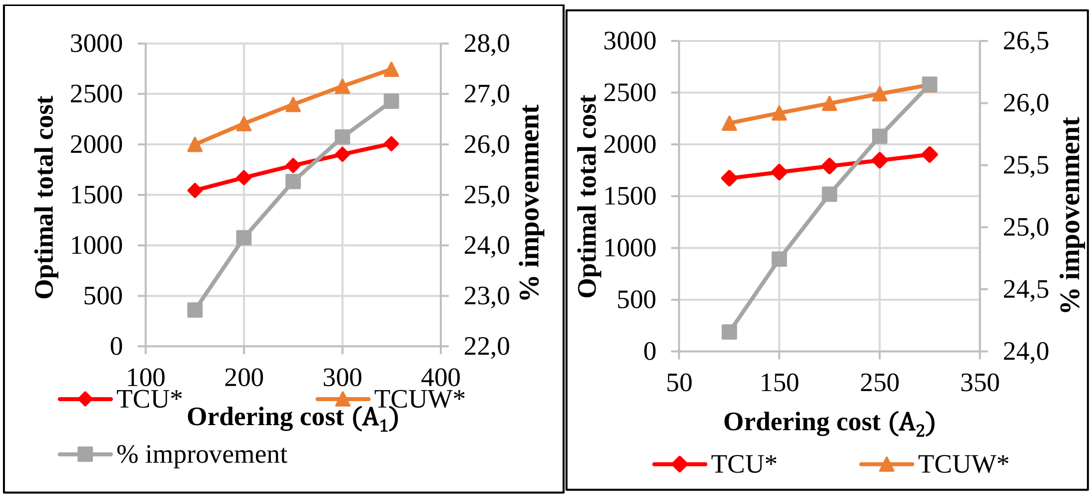

Figure 8: Sensitivity with respect to ordering costs.

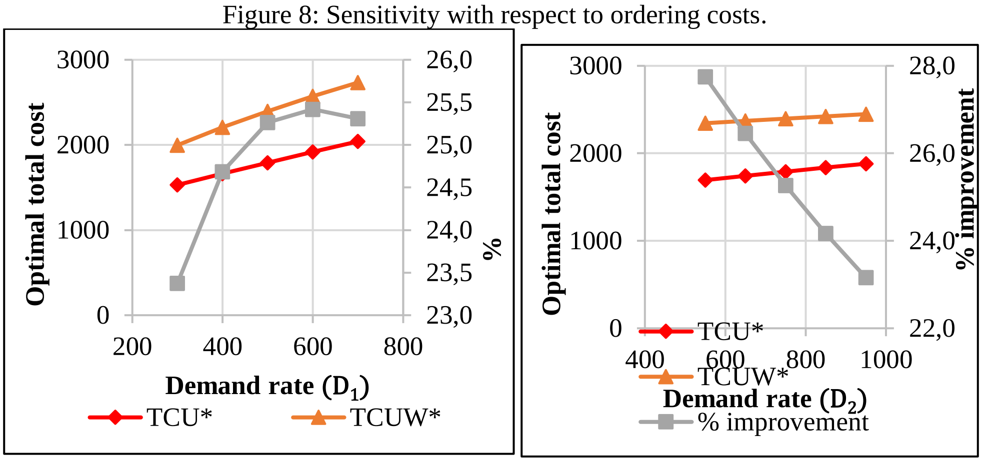

Figure 9: Sensitivity with respect to demand rates.

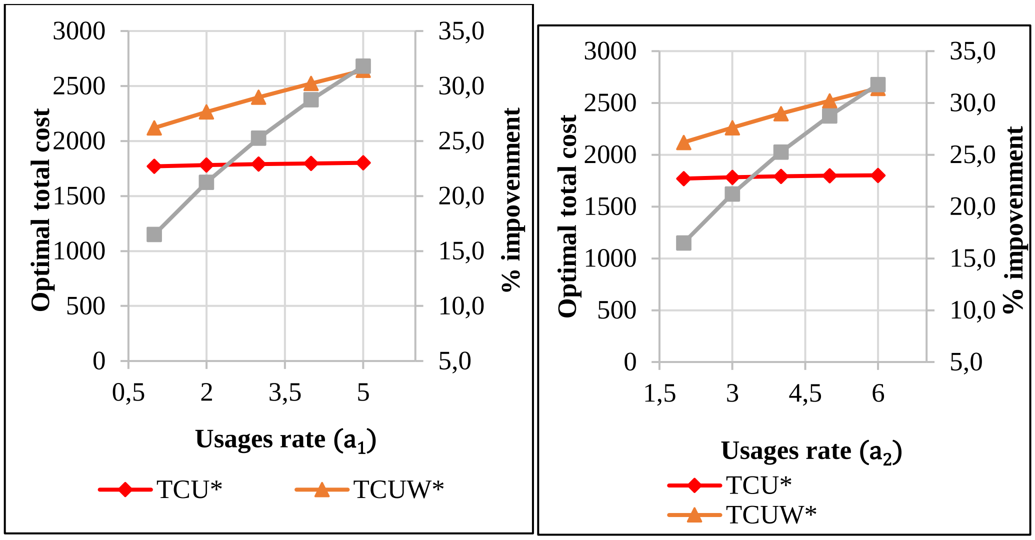

Figure 10: Sensitivity with respect to usages rates of complementary components of item 1.

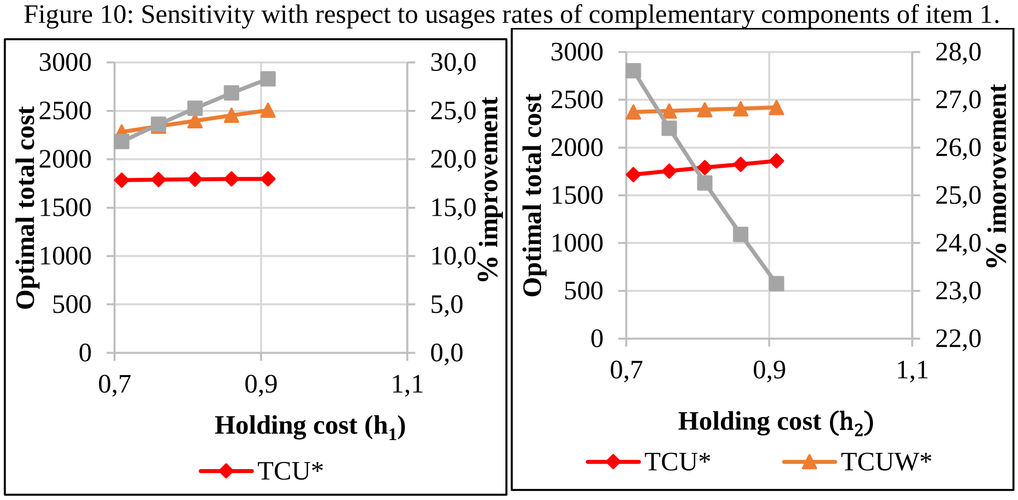

Figure 11: Sensitivity with respect to holding costs.

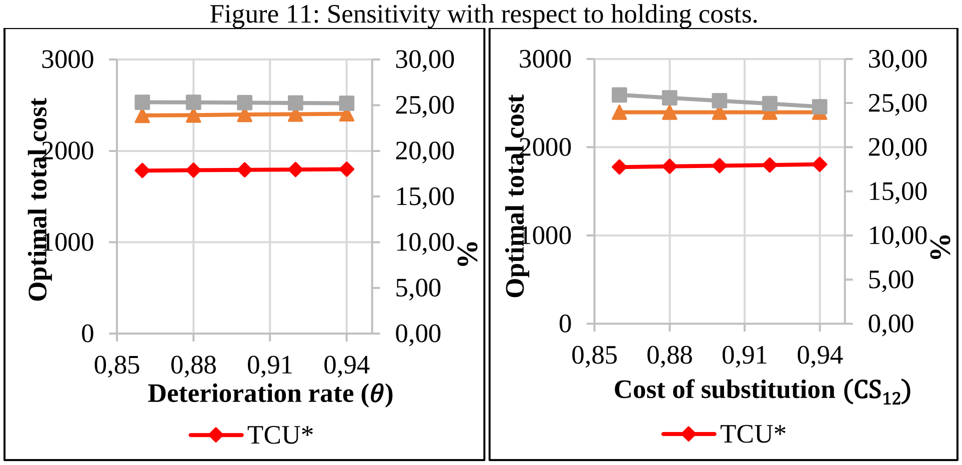

Figure 12: Sensitivity with respect to deterioration rate and cost of substitution.

Further, impact of changes in value of parameters of inventory model on optimum values is given by Table 4.

Table 4: Impact of the changes of values of parameters on optimal total cost with substitution, optimal total cost with no substitution and percentage improvement.

Parameter	Change in parameter	TCU^*	TC_W^*	$\% I_F$

A_1	Increasement	Positive	Positive	Positive
A_2				
D_1	Increasement	Positive	Positive	Positive
D_2		Positive	positive	Negative
a_1	Increasement	Same and positive	Same and positive	Same and positive
a_2				
h_1	Increasement	Positive	Positive	Positive
h_2		Positive	Positive	Negative
θ	Increasement	Positive	Positive	Negative
CS_{12}		Positive	Constant	Negative

6.2. Managerial Implications of the Inventory Model

Now a day's manufacturing companies produce complementary items instead of single item to get full utility. Even, some manufacturing companies produce complementary as well as substitutable items to get best output. So, observing such aspects, with the help of this inventory model, warehouse manager can increase the firm's capability and performance having the inventories of the items of those firms which produce two types of substitutable items where first is composed with two complementary components and second item is assembled with two features as in first item so that substitution is easily possible between both items. This model helps the managers of warehouses to take the decisions for optimal order quantities of items by initializing the input parameters. This inventory model brings a substantial cost saving versus traditional model.

7. CONCLUSION AND FUTURE WORK

This paper addresses an inventory decision model for two substitutable deteriorating items, where one item is composed with two complementary components, by taking into account the cost of substitution and considering stock-out substitution, full substitution, two-way substitution, and joint replenishment policy. Two possible cases: full substitution and no substitution are discussed and solution procedures are presented for each situation of all possible cases to compute the optimal order quantities and optimal total cost by considering the impact of deterioration and cost of substitution. This paper computes the optimal order quantities to optimize the total inventory cost. Pseudo-convexity of total cost function has been demonstrated with respect to decision variables for searching of the global optimal decision variables of this inventory model. An analysis of this model reflects that order quantities with substitution save the inventory cost. Numerical and sensitivity analysis are provided to validate the applicability and performance of proposed inventory model.

Further, research is needed to generalize this paper for the multiple products. Also, this inventory system can be generalized for all items consisting of complementary components and partial substitution. Moreover, it can be extended in a different direction introducing stochastic deterioration rate, stochastic demand, stochastic lead time etc.

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APPENDIX 1(A)- SHOWING PSEUDO-CONVEXITY OF TOTAL COST FUNCTION (TCU_1)

Proof of Theorem 1-

The total cost function (TCU_1) per unit time in situation (i) is given by equation (27)

$$TCU_1 = \frac{\theta}{\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1+D_2)}{a_1(D_1+D_2)}\right)} \begin{bmatrix} (2A_1 + A_2) + \frac{h_1}{\theta^2}\begin{pmatrix} \theta\left(q_1 + \frac{a_2}{a_1}q_1\right) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) \\ - a_2 D_1 \ln\left(\frac{\theta\frac{a_2}{a_1}q_1 + a_2 D_1}{a_2 D_1}\right) \end{pmatrix} \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_1 \ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1+D_2)}{(D_1+D_2)(\theta q_1 + a_1 D_1)}\right)\right) \\ - \frac{h_2 D_2}{\theta^2}\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1+D_2)}{a_1(D_1+D_2)}\right) \end{bmatrix}$$

This can be expressed as

$$TCU_1 = TC_1 / \left[\frac{\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1+D_2)}{a_1(D_1+D_2)}\right)}{\theta}\right],$$

Where

$$TC_1 = \begin{bmatrix} (2A_1 + A_2) + \frac{h_1}{\theta^2}\left(\theta\left(q_1 + \frac{a_2}{a_1}q_1\right) - a_1 D_1 \ln\left(\frac{\theta q_1 + a_1 D_1}{a_1 D_1}\right) - a_2 D_1 \ln\left(\frac{\theta\frac{a_2}{a_1}q_1 + a_2 D_1}{a_2 D_1}\right)\right) \\ + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_1 \ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1+D_2)}{(D_1+D_2)(\theta q_1 + a_1 D_1)}\right) - D_2 \ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1+D_2)}{a_1(D_1+D_2)}\right)\right) \\ + CS_{12}\frac{D_1}{\theta}\ln\left(\frac{\theta D_1(Q_2 a_1 + q_1) + a_1 D_1(D_1+D_2)}{(D_1+D_2)(\theta q_1 + a_1 D_1)}\right) \end{bmatrix}$$

Here, we have to show that TCU_1 is pseudo-convex. For this, firstly we show that $TC1$ is convex and use the fact that ratio of positive convex function and positive concave function is pseudo-convex (Cambibi and Martein [4], Chandra [5]).

To show convexity of TC_1, we must prove that its Hessian matrix is positive definite.

The Hessian matrix of cost function TC_1 is given as $H(q_1, Q_2) = \begin{pmatrix} \frac{\partial^2 TC1}{\partial^2 q_1} & \frac{\partial^2 TC1}{\partial q_1 \partial Q_2} \\ \frac{\partial^2 TC1}{\partial Q_2 \partial q_1} & \frac{\partial^2 TC1}{\partial^2 Q_2} \end{pmatrix}$

For positive definiteness of the Hessian matrix $H(q_1, Q_2)$, we prove that $\frac{\partial^2 TC1}{\partial^2 q_1} > 0, \frac{\partial^2 TC1}{\partial^2 Q_2} > 0$

and determinant of the Hessian matrix $|H(q_1, Q_2)| \geq 0$ i.e. $\left(\frac{\partial^2 TC1}{\partial^2 q_1} * \frac{\partial^2 TC1}{\partial^2 Q_2}\right) - \left(\frac{\partial^2 TC1}{\partial q_1 \partial Q_2}\right)^2 > 0$

Now, $\frac{\partial TC1}{\partial q_1} = \left[\begin{array}{c} \frac{a_1 q_1(\theta h_1 q_1 + Q_2 \theta a_2 h_1 + a_2 h_1 D_1 + a_2 h_1 D_2 - h_2 D_2) + \theta a_2 h_1 q_1^2}{a_1(\theta q_1 + a_1(\theta Q_2 + D_1 + D_2))(\theta q_1 + a_1 D_1)} \\ + \frac{a_1^2(h_1 q_1(Q_2 \theta + D_1 + D_2) - D_1(CS_{12} \theta Q_2 + CS_{12} D_2 - Q_2 h_2))}{a_1(\theta q_1 + a_1(\theta Q_2 + D_1 + D_2))(\theta q_1 + a_1 D_1)} \end{array}\right]$

$\frac{\partial TC1}{\partial Q_2} = \frac{h_2 q_1 + a_1(h_2 Q_2 + CS_{12} D_1)}{\theta q_1 + a_1(\theta Q_2 + D_1 + D_2)}$

$\frac{\partial^2 TC1}{\partial^2 Q_2} = \frac{a_1^2(h_2 D_1 + h_2 D_2 - CS_{12} \theta D_1)}{(\theta q_1 + a_1(\theta Q_2 + D_1 + D_2))^2} > 0$ if $h_2 \geq CS_{12}\theta$ (34)

$\frac{\partial^2 TC1}{\partial q_1 \partial Q_2} = \frac{a_1(h_2 D_1 + h_2 D_2 - CS_{12} \theta D_1)}{(\theta q_1 + a_1(\theta Q_2 + D_1 + D_2))^2}$

$\left(\frac{\partial^2 TC1}{\partial^2 q_1} * \frac{\partial^2 TC1}{\partial^2 Q_2}\right) - \left(\frac{\partial^2 TC1}{\partial q_1 \partial Q_2}\right)^2 = \frac{a_1^2 D_1(h_2 D_1 + h_2 D_2 - CS_{12} \theta D_1)(CS_{12}\theta + a_1 h_1 + a_2 h_1 - h_2)}{(\theta q_1 + a_1(\theta Q_2 + D_1 + D_2))^2 (\theta q_1 + a_1 D_1)^2} > 0$ if $h_2 \geq CS_{12}\theta$ and $h_2 \leq CS_{12}\theta$

(35) From (34) and (35),

It is clear that $\frac{\partial^2 TC1}{\partial^2 Q_2} > 0$ and $\left(\frac{\partial^2 TC1}{\partial^2 q_1} * \frac{\partial^2 TC1}{\partial^2 Q_2}\right) - \left(\frac{\partial^2 TC1}{\partial q_1 \partial Q_2}\right)^2 > 0$ simultaneously hold if $h_2 = CS_{12}\theta$ For $h_2 = CS_{12}\theta$,

$$\frac{\partial^2 TC1}{\partial^2 q_1} = \left[\frac{\left(\begin{array}{l} h_1 D_1 a_1^3(\theta Q_2 + D_1 + D_2)^2 + D_1 a_1^2\left(a_2 h_1 D_1^2 + 2 h_1 D_1 \theta(Q_2 a_2 + q_1) + h_1(2 q_1 \theta + Q_2 a_2 \theta + D_2 a_2)(D_2 + \theta Q_2)\right) \\ + D_1 a_1^2(CS_{12}\theta D_1 D_2 + 2 D_1 D_2 h_1 a_2) + q_1^2 \theta^2(CS_{12}\theta D_2 + D_1 a_2 h_1) \\ + D_1 q_1 a_1 \theta(2 a_2 h_1 D_1 + 2 a_2 h_1 D_2 + 2 CS_{12}\theta D_2 + 2 Q_2 a_2 \theta h_1 + q_1 h_1 \theta) \end{array}\right)}{(\theta q_1 + a_1(\theta Q_2 + D_1 + D_2))^2 (\theta q_1 + a_1 D_1)^2}\right]$$

Clearly, $\frac{\partial^2 TC1}{\partial^2 q_1} > 0$ for $h_2 = CS_{12}\theta$

Thus, $\frac{\partial^2 TC1}{\partial^2 Q_2} > 0, \frac{\partial^2 TC1}{\partial^2 q_1} > 0$ and $\left(\frac{\partial^2 TC1}{\partial^2 q_1} * \frac{\partial^2 TC1}{\partial^2 Q_2}\right) - \left(\frac{\partial^2 TC1}{\partial q_1 \partial Q_2}\right)^2 > 0$ if $h_2 = CS_{12}\theta$

So, TC_1 is convex function

It is also clear that $\left[\frac{\ln\left(\frac{\theta(Q_2 a_1 + q_1) + a_1(D_1 + D_2)}{a_1(D_1 + D_2)}\right)}{\theta}\right]$ is positive concave function.

This completes the proof of theorem.

APPENDIX 1(B)- SHOWING PSEUDO-CONVEXITY OF TOTAL COST FUNCTION ($\boldsymbol{TCU_2}$)

Proof of theorem 2- Similar to proof of theorem 1