REVISTA INVESTIGACIÓN OPERACIONAL VOL. 42, NO. 2, 267-278, 2021

ON THE OSCILLATORY BEHAVIOUR OF SOME FORCED NONLINEAR GENERALIZED DIFFERENTIAL EQUATION

L. M. Lugo*,* Juan E. Nápoles Valdés*,** and Miguel Vivas-Cortez***
*UNNE, FaCENA, Argentina.
**UTN-FRRE, Argentina
***Pontificia Universidad Católica del Ecuador (PUCE), Ecuador.

ABSTRACT
In this paper, we study the oscillatory character of a generalized di-fferential equation of order α with $\alpha \in (0, 1]$. Generalized criteria are obtained, of type Kamenev, which are extensions of several known in the literature both integer and fractional.

KEYWORDS: Generalized ordinary differential operator, oscillation.

MSC: 34L30, 34C15

RESUMEN
En este artículo se estudia el carácter oscilatorio de una ecuación diferencial generalizada de orden α con $\alpha \in (0, 1]$. se botienen criterios generalizados tipo Kamenev que son extensiones de resultados conocidos de la literatura para el caso tanto entero como fraccionario.

PALABRAS CLAVE: operador diferencial ordinario generalizado, oscilación

1. INTRODUCTION

The so-called Fractional Calculus has expanded in multiple directions in recent years, not only from a theoretical point of view, but also from applications.
It is noteworthy that, in general, they are centered on equations with the classical "global" fractional derivatives (see [9] and the references cited there), and qualitative research is almost non-existent using local fractional derivatives (see [20, 21, 22] for attempts in this direction, although the techniques used will be different from those used in this work).
The continuous extensions of various notions of derivatives in recent years, of non-integer and / or variable order, are extensions of the classical differential equations, and have different theoretical and applied fields. (cf. [14], [17], [27] and [28]). More than five decades ago, intensified in recent years, that different researchers have been studying various qualitative aspects of fractional equation solutions (local and global) (see [5] and references cited therein). However, we should note that, in general,

*lmlmb@yahoo.com.ar

little attention is paid to the study of the oscillatory nature of the solutions, being one of the central properties in mathematical modeling.

Although certain differential operators that are called local fractional derivatives have appeared since the 1960s, it is not until in 2014 when a local derivative (conformable) was formalized and in 2018 (non-conformable) with very good properties, what we highlight is that all these operators can be considered as particular cases (including the ordinary classic) of the following definition of Generalized Derivative, as discussed below.

Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in recent years, in such a way that a single definition of "fractional derivative or integral" does not exist, or at least is not unanimously accepted, in [3] suggests and justifies the idea of a fairly complete classification of the known operators in non-integer order Calculus, in addition, in the work [2] some reasons are presented why new operators linked to applications and developments theorists appear every day. These operators, both classic (global) or local, have been obtained by numerous mathematicians, some well known and others have not gone far enough (the Sonin derivative is enough as an example), if to this we add that, for some reason, local differential operators, which we prefer to call generalized, have been ignored and underestimated by numerous researchers, today they have been the source of development of new global operators based on their formulations.

In addition, Chapter 1 of [1] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 LIMITATIONS ... concludes "We can therefore conclude that both the Riemann – Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [34] that, the local fractional operator is not a fractional derivative" (p.24). As we said before, they are new tools that have demonstrated their usefulness and potential in the modeling of different processes and phenomena.

For the second order linear equation:

$$y'' + b(t)y = 0, \tag{1.1}$$

We will focus our work using the new Kamenev oscillation criteria, esta-blished in [11], in which making use of an integral average method, specifically affirms that if

$$\lim_{t\to\infty} sup \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} b(s)ds = +\infty, \tag{1.2}$$

for $n > 2$, then the differential equation (1.1) is oscillatory.

In this paper we will study a forced generalized nonlinear equation using a Riccatti Transformation and then formulate two general oscillation criteria of Kamenev type, when $q(t)$ is allowed to take negative values for sufficiently large values of t. For this we have divided the study into two parts, first a particular case is studied and subsequently we study the general equation itself.

2. PRELIMINARY RESULTS

In this paper we will consider the following generalized differential equation:

$$N_F^\alpha\left[p(t)f(N_F^\alpha x)\right] + q(t,x,N_F^\alpha x)g(x(t)) = r(t,x,N_F^\alpha x), \tag{2.1}$$

where the differential operator N_F^α, $0<\alpha\le 1$, will be defined later and the functions considered satisfy $p\in C([t_0,\infty),(0,\infty))$, $q\in C([t_0,\infty)x\mathbb{R}^2,\mathbb{R})$, $f\in C(\mathbb{R},\mathbb{R})$ such that $yf(y)>0$ for $y\neq 0$, $g\in C(\mathbb{R},\mathbb{R})$, $xg(x)>0$ for $x\neq 0$, with $(N_F^\alpha y)g(N_F^\alpha y)>0$ for $N_F^\alpha y\neq 0$.

Remark 1. *Equation* (2.1) *is not a minor case, for example, if $F\equiv 1$, then* (2.1) *contains as a particular case the equation* (1.1) *taking $p\equiv 1$, $f(z)=z$, $q(t,x,N_F^\alpha x)=q(t)$ and $r(t,x,N_F^\alpha x)=0$; if, instead of the above, we consider $q(t,x,N_F^\alpha x)=q(t)$ and $r(t,x,N_F^\alpha x)=r(t,x)$, then* (2.1) *becomes to that studied in [25].*

We will say that a nontrivial solution of (2.1) is called oscillatory if it has an infinite number of roots, that is, if it has infinite zeros. If all solutions of the equation (2.1) are oscillatory, we will say that the equation is oscillatory.
In [24] a generalized fractional derivative was defined in the following way (see also [23] and [38]).

Definition 1. *Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous function. The generalized N-derivative of f of order α is defined by*

$$N_F^\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon F(t,\alpha)) - f(t)}{\varepsilon} \tag{2.2}$$

for all $t>0$, $\alpha\in(0,1)$ being $F(\alpha,t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a),Re(b)>0$. Also we consider $E_{a,b}(t^{-\alpha})_k$ is the k-th term of $E_{a,b}(.)$.
If f is $\alpha-$differentiable in some $(0,\alpha)$, and $\lim_{t\to 0^+} N_F^{(\alpha)}f(t)$ exists, then define $N_F^{(\alpha)}f(0)=\lim_{t\to 0^+}N_F^{(\alpha)}f(t)$, note that if f is differentiable, then $N_F^{(\alpha)}f(t)=F(t,\alpha)f'(t)$ where $f'(t)$ is the ordinary derivative.

The classic Mittag–Leffler function plays an active role in fractional calculus. It is defined by

$$\mathbf{E}_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)},\quad \alpha\in C,\quad Re(\alpha)>0,\quad z\in\mathbb{C},$$

with Γ the classic Gamma function. In 1903 Mittag – Leffler established the original function $E_{\alpha,1}(z)=E_\alpha(z)$ as a one-parameter function, see [18, 19]. It is a direct generalization of the exponential function. Wiman proposed and studied a extension, the bi-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$, (see [35]). In 1971, Prabhakar in [29] introduced a new generalization $E^\gamma_{\alpha,\beta}(z)$ defined as

$$E^\gamma_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\quad \alpha,\beta,\gamma\in C;\quad Re(\alpha),Re(\beta),Re(\gamma)>0,\quad z\in C,$$

where $(\gamma)_k$ is the Pochhammer symbol (see [30]). Gorenflo et al. (see [6, 7]), and Kilbas and Saigo (see [13, 31]) investigated several properties and applications of the original Mittag-Leffler function and its generalizations.
To point out only one particular case of our definition, let us point out that if $F(t,\alpha)=t^\alpha$ we obtain the conformable derivative of [12].
In what follows, the definition of the integral operator related to the diffe-rential operator (2.2) is presented (see [10] and [38]):

Definition 2. *Let $\alpha \in (0,1]$ and $0 \leq u \leq v$. We say that a function $h : [u,v] \to \mathbb{R}$ is α-integrable on $[u,v]$, if the integral:*

$$ {}_N J_u^\alpha h(x) = {}_{N_F} J_u^\alpha h(x) = \int_u^x \frac{h(t)}{F(t,\alpha)} dt \tag{2.3}$$

exists and is finite.

Remark 2. *Taking into account the examples of kernels presented above, it is clear that we will have different integral operators. To name just one case, if $F(t,\alpha) \equiv 1$ we will have the classic Riemann integral.*

The following theorem is similar to a known result of classical calculus (see [10] and [38]).

Theorem 3. *Let f be N-differentiable function in (t_0, ∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have:*

a) If f is differentiable ${}_{N_F} J_{t_0}^\alpha \left(N_F^\alpha f(t)\right) = f(t) - f(t_0)$.

b) $N_F^\alpha \left({}_{N_F} J_{t_0}^\alpha f(t)\right) = f(t)$.

As in the classic case, we have the well-known Integration by Parts property.

Theorem 4. *(Integration by parts) Let u and v be N-differentiable function in (t_0, ∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have:*

$$ {}_{N_F} J_{t_0}^\alpha \left((uN_F^\alpha v)(t)\right) = [uv(t) - uv(t_0)] - {}_{N_F} J_{t_0}^\alpha \left((vN_F^\alpha u)(t)\right). \tag{2.4}$$

Proof. From Theorem 3, part d) of [24] we have $N_F^\alpha(uv)(t) = uN_F^\alpha(v)(t) + vN_F^\alpha(u)(t)$. Integrating in both members of t_0 a t, and using Proposition 1 of [10] we obtain

$$\begin{aligned} {}_{N_F} J_{t_0}^\alpha \left(N_F^\alpha(uv)(t)\right) &= {}_{N_F} J_{t_0}^\alpha \left((uN_F^\alpha(v))(t)\right) + {}_{N_F} J_{t_0}^\alpha \left((vN_F^\alpha(u))(t)\right) \\ (uv)(t) - (uv)(t_0) &= {}_{N_F} J_{t_0}^\alpha \left((uN_F^\alpha(v))(t)\right) + {}_{N_F} J_{t_0}^\alpha \left((vN_F^\alpha(u))(t)\right). \end{aligned}$$

After ordering, the required equality is obtained without difficulty. $\square$

The following definition is a natural extension of the classic case and contains as a particular case Definition 1 of [16].

Definition 3. *Given a real valued function $f : \mathbb{R}^n \to \mathbb{R}$ and $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ a point whose ith component is positive. Then the non conformable partial N-derivative of f of order α in the point $\overrightarrow{a} = (a_1, \ldots, a_n)$ is defined by*

$$ N_{F,x_i}^\alpha f(\overrightarrow{a}) = \lim_{\varepsilon \to 0} \frac{f(a_1, .., a_i + \varepsilon F(a_i, \alpha), \ldots, a_n) - f(a_1, \ldots, a_n))}{\varepsilon} \tag{2.5}$$

if it exists, is denoted $N_{F,x_i}^\alpha f(\overrightarrow{a})$, and called the ith generalized partial derivative of f of the order $\alpha \in (0,1]$ at $\overrightarrow{a}$.

3. MAIN RESULTS

In this section we will prove results for the equation (2.1) which are a genera-lization of several reported in the literature. For this, we will consider a general function $H(t, t_0)$ defined as follows.

Let $H : D = \{(t, s) : t \geq s \geq t_0\} \to \mathbb{R}$ satisfying $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ for $t > s \geq t_0$, having a continuous and nonpositive partial derivative on $D_0 = \{(t, s) : t > s \geq t_0\}$ with respect to the variable s. Moreover, let $h_1, h_2 : D_0 \to \mathbb{R}$ be a continuous functions for which we have

$$-N^{\alpha}_{F,t}H(t,s) = h_1(t,s)\sqrt{H(t,s)},$$
$$-N^{\alpha}_{F,s}H(t,s) = h_2(t,s)\sqrt{H(t,s)},$$

for all $(t, s) \in D_0$. Thus we can state the following result.

Theorem 5. *Under the assumptions:*

i) There is a differentiable function $u : \mathbb{R} \to \mathbb{R}$ satisfying $xu(x) > 0$ for $x \neq 0$, $N^{\alpha}_{F}u(x) \geq c > 0$,
ii) $\frac{q(t,x,N^{\alpha}_{F}x)g(x)}{u(x)} \geq a\,(t)$, for $x \neq 0$,
iii) $\frac{r(t,x,N^{\alpha}_{F}x)}{u(x)} \leq b\,(t)$, for $x \neq 0$,
iv) There are functions γ and δ such that

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_0)} \quad {}_{N}J^{\alpha}_{t_0}\left[\gamma(t)\left(H(t,s)\Phi(t) - \frac{h_1^2(t,s)a(t)\gamma(t)}{4c}\right)\right](t) = \infty, \tag{3.1}$$

where

$$\gamma(s) = exp\left(-2c_N J^{\alpha}_{t_0}(\delta)(s)\right) \tag{3.2}$$

and

$$\Phi(t) = cp(t)\delta^2(t) + a(t) - b(t) - N^{\alpha}_{F}\left(p(t)f\left(N^{\alpha}_{F}x(t)\right)\delta(t)\right), \tag{3.3}$$

then every solution of (2.1) *is oscillatory.*

Proof. Suppose the contrary that there exists a non-oscillatory solution of the equation (2.1) and (3.1) is satisfied, and consider, without loss of generality, that $x(t) > 0$ for $t \in [\tau_0, \infty)$ for some $\tau_0 \geq t_0$. Let's define the function

$$w(t) = \gamma(t)\left[\frac{p(t)f(N^{\alpha}_{F}x)g(x)}{u(x)} + p(t)f(N^{\alpha}_{F}x)\delta(t)\right], \tag{3.4}$$

so, we have

$$N^{\alpha}_{F}w = -2c\delta\gamma(t)\left[\frac{p(t)f(N^{\alpha}_{F}x)g(x)}{u(x)} + p(t)f(N^{\alpha}_{F}x)\delta(t)\right] -$$
$$-\gamma(t)\left[\frac{N^{\alpha}_{F}u(p(t)f(N^{\alpha}_{F}x)\delta(t))}{u^2(x)}\right] + \gamma(t)\left[(b(t) - a(t)) + N^{\alpha}_{F}(p(t)f(N^{\alpha}_{F}x)\delta(t))\right],$$

from here we have

$$N_F^{\alpha} w \leq -\frac{c}{a(t)\gamma(t)} w^2(t) - \gamma(t)\Phi(t), \tag{3.5}$$

for $t \geq \tau_0$. If, in the previous inequality, we multiply both members by $H(t,s)$ and then integrate with respect to t, from τ to t, with $\tau_0 \leq \tau$ we obtain

$$_N J_\tau^{\alpha}\left[H(t,s) N_F^{\alpha} w\right](t) \leq_N J_\tau^{\alpha}\left[-H(t,s)\frac{c}{a(t)\gamma(t)} w^2 - H(t,s)\gamma(t)\Phi(t)\right](t).$$

From here we obtain

$$_N J_\tau^{\alpha}\left[H(t,s)\gamma(t)\Phi(t)\right](t) \leq -_N J_\tau^{\alpha}\left[H(t,s)\frac{c}{a(t)\gamma(t)} w^2\right](t) -_N J_\tau^{\alpha}\left[H(t,s) N_F^{\alpha} w\right](t). \tag{3.6}$$

Integrating by parts, in the second integral of the right member of the previous inequality we obtain

$$\begin{aligned} & _N J_\tau^{\alpha}\left[H(t,s)\gamma(t)\Phi(t)\right](t) \leq -_N J_\tau^{\alpha}\left[H(t,s)\frac{c}{a(t)\gamma(t)} w^2\right](t) \\ + \quad & H(t,\tau)w(\tau) -_N J_\tau^{\alpha}\left[h_1(t,s)\sqrt{H(t,s)} w\right](t), \end{aligned}$$

from above, after simplifying we obtain

$$_N J_\tau^{\alpha}\left[\gamma(t)\left(H(t,s)\Phi(t) - \frac{h_1(t,s)a(t)\gamma(t)}{4c}\right)\right](t) \leq H(t,\tau)w(\tau),$$

which implies that, for all $\tau_0 \leq t$ we have

$$_N J_{\tau_0}^{\alpha}\left[\gamma(t)\left(H(t,s)\Phi(t) - \frac{h_1^2(t,s)a(t)\gamma(t)}{4c}\right)\right](t) \leq H(t,\tau_0)w(\tau_0) \leq H(t,t_0)\left|w(\tau_0)\right|.$$

Therefore, taking into account $t_0 \leq \tau_0 \leq t$, is obtained

$$\begin{aligned} & _N J_{t_0}^{\alpha}\left[\gamma(t)\left(H(t,s)\Phi(t) - \frac{h_1^2(t,s)a(t)\gamma(t)}{4c}\right)\right](\tau_0) \\ + \quad & _N J_{\tau_0}^{\alpha}\left[\gamma(t)\left(H(t,s)\Phi(t) - \frac{h_1^2(t,s)a(t)\gamma(t)}{4c}\right)\right](t) \\ \leq \quad & H(t,t_0)_N J_{t_0}^{\alpha}\left[\gamma(t)\Phi(t)\right](\tau_0) + H(t,t_0)\left|w(\tau_0)\right| \\ = \quad & H(t,t_0)\left[_N J_{t_0}^{\alpha}\left[\gamma(t)\Phi(t)\right](\tau_0) + \left|w(\tau_0)\right|\right]. \end{aligned}$$

Dividing by $H(t,t_0)$ and taking an upper limit when t tends to ∞ we have

$$\begin{aligned} & \lim_{t\to\infty} sup \frac{1}{H(t,t_0)} \quad _N J_{t_0}^{\alpha}\left[\gamma(t)\left(H(t,s)\Phi(t) - \frac{h_1^2(t,s)a(t)\gamma(t)}{4c}\right)\right](t) \\ \leq \quad & \left[_N J_{t_0}^{\alpha}\left[\gamma(t)\Phi(t)\right](\tau_0) + \left|w(\tau_0)\right|\right] < \infty. \end{aligned}$$

From where a contradiction is obtained with (3.1). In this way the proof of the Theorem is completed.

□

Remark 6. *If we consider* $F \equiv 1$, $g = g(x) = |x|^{p-2}x$, $p > 1$, $f(z) = z$, $q(t, x, N_F^\alpha) = q(t)$ *and* $r \equiv 0$ *we get Theorem 2 of [15].*

Remark 7. *The above result contains Theorem 3.1 of [26], with* $F(x, \alpha) = x^{1-\alpha}$, $F(N_F^\alpha x) = \phi(x)T_\alpha(x)$ *and* $q(t, x, N_F^\alpha x)g(x) = P(t, x, T_\alpha(x))$

If (3.1) is not fulfilled, we can impose additional conditions as obtained in the following result.

Theorem 8. *Assume (i) -(iii) of Theorem 5 and let* $\lambda > 1$ *a real number. Suppose that* (3.1) *is not satisfied and that there is a function* g *of class* $C^\alpha[[t_0, +\infty), (0, +\infty)]$ *such that*

$$\lim_{t\to+\infty} sup \frac{1}{t^\lambda} {}_N J_{t_0}^\alpha \left[(t-s)^\lambda \gamma(t)\Phi(t) - \frac{\lambda^2 a(t)\gamma(t)}{4c}(t-s)^{\lambda-2} \right](t) = \infty \tag{3.7}$$

where $\gamma(s)$ *and* $\Phi(s)$ *are as Theorem 5. Then every solution of* (2.1) *is oscillatory.*

Proof. As in the previous proof, we can assume that there is a non-oscillatory solution to the equation (2.1), that satisfies (3.7) and that it is positive in a certain interval $[\tau_0, +\infty)$ with $t_0 \le \tau_0$. Let's take $w(t)$ as in the previous proof. So, if in the (3.5) we multiply both members by $(t-s)^\lambda$ and integrating between τ and t, we obtain

$$\begin{aligned} & {}_N J_\tau^\alpha \left[(t-s)^\lambda N_F^\alpha w \right](t) \\ \le \quad & -{}_N J_\tau^\alpha \left[(t-s)^\lambda \frac{cw^2(t)}{a(t)\gamma(t)} \right](t) - {}_N J_\tau^\alpha \left[(t-s)^\lambda \gamma(t)\Phi(t) \right](t), \end{aligned}$$

Reordering and integrating by parts we obtain

$$\begin{aligned} & {}_N J_\tau^\alpha \left[(t-s)^\lambda \gamma(t)\Phi(t) \right](t) \\ \le \quad & (t-\tau)^\lambda w(\tau) - {}_N J_\tau^\alpha \left[(t-s)^\lambda \frac{cw^2(t)}{a(t)\gamma(t)} \right](t) - \lambda {}_N J_\tau^\alpha \left[(t-s)^{\lambda-1} w \right](t), \end{aligned}$$

From which it is obtained that, for all t such that $t_0 \le t$

$$\lim_{t\to+\infty} sup \frac{1}{t^\lambda} {}_N J_{t_0}^\alpha \left[(t-s)^\lambda \gamma(t)\Phi(t) - \frac{\lambda^2 a(t)\gamma(t)}{4c}(t-s)^{\lambda-2} \right](t) \le w(t_0) < +\infty,$$

which is a contradiction with (3.7). This completes the proof. □

Remark 9. *Under assumptions of Remark 7 we obtain the Theorem 3.2 of [26].*

A variation of the previous result, we can obtain it as follows.

Theorem 10. *Assume that* $i) - iii)$ *of Theorem 5 and* (3.1), (3.2) *and* (3.3) *are not satisfied, and*
iv) $\frac{(N_F^\alpha x)N_F^\alpha u(x)}{f(N_F^\alpha x)} \ge k > 0 \ge k > 0$ *for* $x \ne 0$.
v) We suppose that for sufficiently large $\tau \ge t_0$, $\exists \tau_1, \tau_2, \tau_3$ *with* $\tau \le \tau_2 < \tau_1 < \tau_3$.

vi) Also if there exist $\gamma(t) \in C^{\alpha}[(t_0,\infty),(0,\infty)]$ *such that*

$$\begin{aligned} & \frac{1}{H(\tau_3,\tau_1)} \quad {}_N J^{\alpha}_{\tau_1}\left[H(\tau_3,s)\gamma(t)(a(s)-b(s))\right](t) \\ + & \frac{1}{4cH(\tau_1,\tau_2)} \quad {}_N J^{\alpha}_{\tau_2}\left[H(s,\tau_2)\gamma(t)(a(s)-b(s))\right](\tau_1) \\ > & \frac{1}{4cH(\tau_3,\tau_1)} \quad {}_N J^{\alpha}_{\tau_1}\left[\gamma(s)p(s)A_1^2(\tau_3,s)\right](t) \\ + & \frac{1}{4cH(\tau_1,\tau_2)} \quad {}_N J^{\alpha}_{\tau_2}\left[\gamma(s)p(s)A_2^2(s,\tau_2)\right](\tau_1) \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} A_1(t,s) &= h_1(t,s) - \frac{N_F^{\alpha}\gamma(s)}{\gamma(s)}\sqrt{H(t,s)} \\ A_2(t,s) &= h_2(s,t) - \frac{N_F^{\alpha}\gamma(s)}{\gamma(s)}\sqrt{H(s,t)} \end{aligned}$$

then every solution of (2.1) *is oscillatory.*

Proof. Now consider the w function defined as follows $w(t) = \frac{\gamma(t)p(t)f(N_F^{\alpha}x)}{u(x)}$. After deriving with respect to t, we have

$$\gamma(t)(a(t)-b(t)) \leq -N_F^{\alpha}w(t) - \frac{kw^2(t)}{\gamma(t)p(t)} + \frac{N_F^{\alpha}\gamma(t)w(t)}{\gamma(t)} \tag{3.9}$$

reordering, multiplying by $H(t,s)$ and integrating, we get

$$\begin{aligned} & {}_N J^{\alpha}_{\tau_1}\left[H(t,s)\gamma(s)(a(s)-b(s))\right](t) \leq H(t,\tau_1)w(\tau_1) \\ - & {}_N J^{\alpha}_{\tau_1}\left[h_1(t,s)\sqrt{H(t,s)}w\right](t) - k_N J^{\alpha}_{\tau_1}\left[\frac{H(t,s)}{\gamma(s)p(s)}w^2\right](t) \\ + & {}_N J^{\alpha}_{\tau_1}\left[\frac{N_F^{\alpha}\gamma(s)H(t,s)}{\gamma(s)}w\right](t) \end{aligned}$$

Now using function A_1

$${}_N J^{\alpha}_{\tau_1}\left[H(t,s)\gamma(t)(a(s)-b(s))\right](t) \leq H(t,\tau_1)w(\tau_1) +_N J^{\alpha}_{\tau_1}\left[\frac{A_1(t,s)\gamma(s)p(s)}{4c}\right](t).$$

Dividing the previous inequality by $H(t,\tau_1)$ and taking a limit when $t \to \tau_3^-$ we obtain

$$\begin{aligned} & \frac{1}{H(\tau_3,\tau_1)} \quad {}_N J^{\alpha}_{\tau_1}\left[H(\tau_3,s)\gamma(t)(a(s)-b(s))\right](t) \\ \leq & \quad w(\tau_1) + \frac{1}{4cH(\tau_3,\tau_1)} \quad {}_N J^{\alpha}_{\tau_1}\left[\gamma(s)p(s)A_1^2(\tau_3,s)\right](t). \end{aligned} \tag{3.10}$$

Similarly, from the definition of the function w and using A_2, we obtain (with $t \in [\tau_1, \tau_3)$):

$$ {}_NJ_t^{\alpha}\left[H(t,s)\gamma(t)(a(s)-b(s))\right](\tau_1) \le -H(t,\tau_1)w(\tau_1) + {}_NJ_t^{\alpha}\left[\frac{A_2(t,s)\gamma(s)p(s)}{4c}\right](\tau_1). $$

From where, as before, we get

$$ \begin{aligned} & \frac{1}{4cH(\tau_1,\tau_2)} \quad {}_NJ_{\tau_2}^{\alpha}\left[H(s,\tau_2)\gamma(t)(a(s)-b(s))\right](\tau_1) \\ \le \quad & -w(\tau_1) + \frac{1}{4cH(\tau_1,\tau_2)} \quad {}_NJ_{\tau_2}^{\alpha}\left[\gamma(s)p(s)A_2^2(s,\tau_2)\right](\tau_1). \end{aligned} \tag{3.11} $$

Adding the equations (3.10) and (3.11) we obtain a contradiction with (3.8). This completes the proof. □

Remark 11. *Under assumptions of Remark 7 we obtain the Theorem 3.3 of [26].*

We can extend the above result in the following way, considering the function $H(t,s)$ in a particular case.

Theorem 12. *Under assumptions of Theorem 10 we suppose that the following inequality is fulfilled:*

$$ \begin{aligned} & \frac{1}{(\tau_3-\tau_1)^{\lambda}} \quad {}_NJ_{\tau_1}^{\alpha}\left[(\tau_3-s)^{\lambda}\gamma(t)(a(s)-b(s))\right](\tau_3) \\ & \qquad + \frac{1}{(\tau_1-\tau_2)^{\lambda}} \quad {}_NJ_{\tau_2}^{\alpha}\left[(s-\tau_2)^{\lambda}\gamma(t)(a(s)-b(s))\right](\tau_1) \\ > & \frac{1}{4c\,(\tau_3-\tau_1)^{\lambda}} \quad {}_NJ_{\tau_1}^{\alpha}\left[\gamma(s)p(s)(\tau_3-s)^{\lambda-2}\left(\lambda-\frac{N_F^{\alpha}\gamma(s)}{\gamma(s)}(\tau_3-s)\right)^2\right](\tau_3) \\ + & \frac{1}{4c\,(\tau_2-\tau_2)^{\lambda}} \quad {}_NJ_{\tau_1}^{\alpha}\left[\gamma(s)p(s)(s-\tau_2)^{\lambda-2}\left(\lambda+\frac{N_F^{\alpha}\gamma(s)}{\gamma(s)}(s-\tau_3)\right)^2\right](\tau_1) \end{aligned} \tag{3.12} $$

then equation (2.1) *is oscillatory.*

Proof. The proof is tedious but something similar to previous proof. It is enough to take, in the first part, instead of $H(t,s)$, the function $(t-s)^{\lambda}$, in the second part use $(s-t)^{\lambda}$ and follow the idea of the proof. □

Remark 13. *As in the previous remark, we can obtain the Theorem 3.4 of [26].*

Remark 14. *As we indicated before, our results are not contradictory with the ordinary case. Thus, in the case that $F \equiv 1$, $f(z) = g(z) = |z|^{n-2}z$ with $n > 1$, $q \equiv 1$ and $r \equiv 0$, we obtain Theorem 2 of [15]. It is clear that Corollaries 3 and 4 of the aforementioned work are still valid. Obviously in this case, the proof is much simpler.*

Remark 15. *Our results complete those obtained in [33] for the equation*

$$N_F^\alpha(p(t)N_F^\alpha y) + b(t)y = 0,$$

a particular case of (2.1) *with* $g(z) = f(z) = z$, $q \equiv 1$, $r \equiv 0$ *and the kernel* $F(t, \alpha) = t^{1-\alpha}$.

Remark 16. *In the case of the kernel* $F \equiv 1$, *our results are consistent with those obtained in [32], which study a second order nonlinear differential equation of the type* $(p(t)h(x)f(x'))' + q(t)g(x) = r(t, x, x')$, $f(z) = |z|^{n-1}z$ *with* $n > 1$.

4. CONCLUSION

In this work we obtained, through a Riccatti transformation, generalized oscillation criteria that contain as particular cases several reported in the literature, both integer and fractional.
On the other hand, it is noteworthy that in this way we can directly study the oscillatory nature of generalized differential equations, without the need to use a geometric transformation to reduce it to an ordinary differential equation, a matter that in the case of fractional differential equations with global derivatives, classic, it is not possible because there is no Chain Rule.

RECEIVED: APRIL, 2020.
REVISED: NOVEMBER, 2020.

REFERENCES

[1] ATANGANA, A. (2016): **Derivative with a New Parameter Theory, Methods and Applications**, Academic Press.

[2] BALEANU, D. (2019): Comments on: Ortigueira M., Martynyuk V., Fedula M., Machado J.A.T., The failure of certain fractional calculus operators in two physical models, in Fract. Calc. Appl. Anal. 22, **Fract. Calc. Appl. Anal.**, 23: Issue 1, DOI: https://doi.org/10.1515/fca-2020-0012.

[3] BALEANU, D., and A. FERNANDEZ (2019): On Fractional Operators and Their Classifications, **Mathematics** , 7, 830; doi:10.3390/math7090830.

[4] ERDELYI A., MAGNUS F.W., OBERHETTINGER, TRICOMI F.G. (1955): **Higher transcendental functions**, New York, McGraw-Hill, 3.

[5] FENG Q.and F. MENG (2013): Interval oscillation criteria for a class of nonlinear fractional differential equations, **WSEAS Transactions on Mathematics**, 12, 564-571.

[6] GORENFLO R. , A.A. KILBAS, S.V. ROGOSIN (1998): On the generalised Mittag-Leffler type function, **Integral Transforms Spec. Funct.** 7, 215-224.

[7] GORENFLO R, F.MAINARDI (2000): On Mittag-Leffler function in fractional evaluation processes, **J. Comput. Appl.Math.** 118, 283-299.

[8] GUZMAN P. M., G. LANGTON, L. M. LUGO, J. MEDINA, J. E. NÁPOLES VALDÉS (2018): A new definition of a fractional derivative of local type. **J. Math. Anal.**, 9, 88-98.

[9] GUZMAN P. M., L. M. LUGO, J. E. J. E. NÁPOLES VALDÉS (2019): A Note on Stability of Certain Lieńard Fractional Equation, **International Journal of Mathematics and Computer Science**, 14, 301–315

[10] GUZMAN P. M., L. M. LUGO, J. E. J. E. NÁPOLES VALDÉS, M. VIVAS (2020): On a New Generalized Integral Operator and Certain Operating Properties, **Axioms**, 9, 69; doi:10.3390/axioms9020069

[11] KAMENEV I. V. (1978): Integral criterion for oscillations of linear differential equation of second order, **Mat. Zametki** 23 , 249-251 (Russian).

[12] KHALIL, R. ,A. HORANI, M. YOUSEF and M. SABABHEH (2014): A new definition of fractional derivative, **J. Comput. Appl. Math.** 264, 65–70.

[13] KILBAS A.A. , M. SAIGO (1996): On Mittag-Leffler type function, fractional calculus operators and solution of integral equations, **Integral Transforms Spec. Funct.** 4, 355-370.

[14] KILBAS A.A., H. SRIVASTAVA and J. TRUJILLO (2006): Theory and Applications of Fractional Differential Equations, in **Math. Studies**, North-Holland, New York.

[15] . LI H. J, C. C. YEH (1995), An integral criterion for oscillation of nonlinear differential equations, **Math. Japonica** 41, 185-188.

[16] MARTÍNEZ F., J. E. NÁPOLES VALDÉS, Towards a non-confortable fractional calculus of n-variables, **Journal of Mathematics and Applications**, to appear.

[17] MILLER K.S. (1993): **An Introduction to Fractional Calculus and Fractional Differential Equations**, J. Wiley and Sons, New York.

[18] MITTAG–LEFFLER G.M (1903): Sur la nouvelle fonction, **C. R. Acad. Sci.**, Paris, 137, 554–558.

[19] MITTAG–LEFFLER G.M (1904): Sur la reprasentation analytique dâ une branche uniforme d'une fonction monogane, **Acta Math.**, Paris, 29, 101–181.

[20] NÁPOLES J. E., Oscillatory criteria for some non conformable fractional differential equation with damping, **Nonlinearity and Complexity**, to appear.

[21] NÁPOLES J. E.S, On the oscillatory character of some no conformable fractional differential equation, submited.

[22] NÁPOLES J. E., P. M. GUZMAN. (2019): A note on the oscillatory character of some non conformable generalized Liénard system, **Advanced Mathematical Models & Applications**.4, pp.127-133.

[23] Nápoles J. E., P. M. GUZMAN, J. E., L. M. Lugo (2018): Some new results on Nonconformable fractional calculus, **Advances in Dynamical Systems and Applications** ISSN 0973-5321, 13. 167–175.

[24] NÁPOLES J. E., P. M. GUZMAN, J. E., L. M. LUGO, A. KASHURI (2020): The local generalized derivative and Mittag Leffler function, **Sigma J Eng & Nat Sci** 38, 1007-1017.

[25] Nápoles J. E., P. M. GUZMÄN, J. E., L. M. Lugo M. N. Quevedo (2020): On the Oscillatory Nature of Some Generalized Emden-Fowler Equation, **Punjab Univ. J. Math**. Vol. 52, 97-106.

[26] OGUNBANJO A. , P. ARAWOMO (2019), Oscillation of solutions to a generalized forced nonlinear conformable fractional differential equation, **Proyecciones**, 38, 429-445.

[27] OLDHAM K. , J. SPANIER (1974): **The Fractional Calculus. Theory and Applications of Differentiation and Integration of Arbitrary Order**, Academic Press, USA.

[28] PODLUBNY I. (1999): **Fractional Differential Equations**, Academic Press, USA.

[29] PRABHAKAR T.R. (1971): A singular integral equation with a generalized Mittag-Leffler function in the kernel, **Yokohama Math. J.** 19, 7–15.

[30] RAINVILLE E.D.(1960): **Spe cial Functions**, Macmillan, New York.

[31] SAIGO M., A.A. KILBAS (1998): On Mittag-Leffler type function and applications, **Integral Transforms Spec. Funct.** 7, 97-112.

[32] SALHIN et al (2014): **Oscillation theorems for second order nonlinear forced differential equations**, SpringerPlus 3:300.

[33] Shao, Z. Zheng, Kamenev I.V. (2019): Type Oscillatory Criteria for Linear Conformable Fractional Differential Equations, **Discrete Dynamics in Nature and Society Volume**, Article ID 2310185, 10 pages https://doi.org/10.1155/2019/2310185.

[34] UMAROV S. , S. STEINBERG (2009): Variable order differential equations with piecewise constant order-function and diffusion with changing modes, **Z. Anal. Anwend.** 28 , 431-450.

[35] WIMAN A. (1905):Über den fundamentalsatz in der teorie der funktionen $E_\alpha(z)$, **Acta Math.**, 29, 191–201.

[36] WONG J. S. W. (1975): On the generalized Emden-Fowler equation, **SIAM Review**17, 339-360.

[37] WONG J. S. W. (1986): An oscillation criterion for second order nonlinear differential equations, **PAMS** 98, 109-112.

[38] ZHAO D. and M. LUO (2017): General conformable fractional derivative and its physical interpretation, **Calcolo**, D. 54, 903-917.