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ABOUT THE BOOTSTRAP WEAK CONVERGENCE FOR THE FOSTER-GREER-THORBECKE POVERTY INDEX

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ABSTRACT
We assume the Foster-Greer-Thorbecke (FGT) poverty index as a centered and normalized empirical process indexed by a particular Donsker class or collection of functions and define this poverty index as a bootstrapped empirical process, to show that the weak convergence of the FGT empirical process centered and normalized is a necessary and sufficient condition for the weak convergence of the FGT bootstrap empirical process centered and normalized. Thus, this result reflects that under certain conditions, the consistency in weak convergence of the FGT empirical process considered as a classical estimator of poverty (statistics) and the consistency in weak convergence of the FGT bootstrap empirical process considered as a bootstrap estimator of poverty (bootstrap statistics) are asymptotically equivalents for random samples of incomes statistically large and representative of a statistical universe of households.
KEYWORDS: Foster-Greer-Thorbecke poverty index, Convergence of empirical processes, Donsker classes, Bootstrap empirical processes.
MSC: 91B82, 97K60, 62G30, 62F40.

RESUMEN
Nosotros asumimos el indicador de pobreza de Foster-Greer-Thorbecke (FGT) como un proceso empírico centrado y normalizado indexado por una particular clase o colección de funciones Donsker y definimos este indicador de pobreza como un proceso empírico del tipo bootstrap, para probar que la convergencia débil del proceso empírico FGT centrado y normalizado es una condición necesaria y suficiente para la convergencia débil del proceso empírico bootstrap FGT centrado y normalizado. Así, este resultado refleja que bajo ciertas condiciones, la consistencia en convergencia débil del proceso empírico FGT considerado como un estimador clásico de pobreza (estadístico) y la consistencia en convergencia débil del proceso empírico bootstrap FGT considerado como un estimador bootstrap de pobreza (estadístico bootstrap) son asintóticamente equivalentes para muestras aleatorias de ingresos estadísticamente grandes y representativas de un universo estadístico de hogares.
PALABRAS CLAVE: Indicador de pobreza de Foster-Greer-Thorbecke, Convergencia de procesos empíricos, Clases Donsker, Procesos empíricos bootstrap.

1. INTRODUCTION

The problem of estimating *one-dimensional poverty measures* is theoretically addressed in this paper, developing a *central limit theorem*, in the framework of *bootstrap empirical processes*, and to achieve

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this goal first we introduce some basic aspects. The properties of the *axiomatic method of poverty* introduced by Sen in 1976 [see [14]], provided the basis to study the problem posed as a phenomenon that depends only on income, acquiring this focus a greater *mathematical rigor* within the *economic theory*. In fact, several measures of poverty begin are proposed, all of which are supported in the *Sen's axiomatic definition*.

This type of measures is commonly known as *one-dimensional poverty indices*, because in their construction, only one variable or economic dimension is considered: *the income* [see [17], for a detailed discussion about the axiomatic method and all the one-dimensional poverty indices proposed in the literature].

Formally, let N be a *statistical universe* of individuals (let say households), such that for each one of them it is possible to determine its *level of income* among other features, for any random sample of n individuals withdrawn from this population, a *measure or classic index of poverty* is a function $\mathcal{P}:\mathbb{R}_+^{n+1}\to[0,1]$, where the value of $\mathcal{P}(y,z)$ indicates the *degree or level of poverty* associated with the *vector of incomes* $y=(y_1,...,y_n)\in\mathbb{R}_+^n$ and the *poverty line fixed* $z\in\mathbb{R}_+$, such that any j-th individual of the random sample is considered *poor* if $y_j<z$ [see e.g. [13], and [8]].

In this framework, one of the most important measures is the Foster-Greer-Thorbecke (FGT) poverty index, defined in [5] by

$$FGT(y,z,\alpha)=\frac{1}{n}\sum_{j=1}^{q}\left(\frac{z-y_j}{z}\right)^{\alpha}, \tag{1.1}$$

and emphasizes the *degree of aversion to poverty* represented by a parameter $\alpha\geq 0$, where q the *number of poor individuals for a random sample of size n*. In the particular cases $\alpha=0,1,2$, we have that:
(*i*) $FGT(y,z,0)$ reduces to $H(y,z)=\dfrac{q}{n}$, the *headcount ratio of poor individuals* that reflects the *incidence of poverty*.

$$(ii)\;\; FGT(y,z,1)=\frac{q}{n}\left(1-\frac{\mu}{z}\right)=H(y,z)I(y,z):=HI(y,z),$$

where $\mu=\dfrac{1}{q}\displaystyle\sum_{j=1}^{q}y_j$ is the *average income of the poor individuals in the sample*, $I(y,z)$ is defined as the *income gap ratio*, and consequently $HI(y,z)$ is the *combined income gap ratio* that reflects the *intensity or severity of poverty* as the *product of the proportion of the poor due to the poverty gap*.
(*iii*) $FGT(y,z,2)$ is interpreted as the *depth or inequality among poor*.

The results introduced by Lo and Seck in [11], establish that the FGT poverty index defined in (1.1) understood as an empirical process satisfies a *central limit theorem*. However, in this article we will introduce an important weak convergence relationship between the *FGT empirical process of Lo and Seck* and a very particular *FGT bootstrap empirical process*, defined by us below in the next part or section.

Our theoretical proposal presented here is a particular contribution over the literature: formally states that under certain conditions, *the consistency in weak convergence of the centered and normalized FGT empirical process of Lo and Seck considered as a classical estimator of poverty (statistics), and the consistency in weak convergence of our centered and normalized FGT bootstrap empirical process considered as a bootstrap estimator of poverty (bootstrap statistics), are asympto-tically equivalents for random samples of incomes statistically large and representative of a statistical universe of households*. In fact, our theoretical result goes hand in hand with [1] and [7] among others, where these point out precisely that the classic inference for this type of one-dimensional poverty indices such as the FGT, when only small random samples of incomes are available, can present problems in assuming the convergence of the statistics to the normal distribution; and naturally the same analysis is valid for the corresponding bootstrap statistics.

This paper is organized as follows. In Section 2, we present the problem statement. The Section 3 presents our main result and some consequences. The conclusions are given in Section 4. Finally, the Appendix contains all the tools required for the comprehension of the central proof in our theoretical proposal.

2. THE PROBLEM

Lo and Seck in [11] define the *class of functions* $\mathcal{F}_\Gamma = \{f_\alpha : \alpha \geq 0\}$, where

$$f_\alpha(x) = \left|\frac{z-x}{z}\right|^\alpha \mathcal{I}\{x < z\}.$$

For any i.i.d. collection $\{Y_j\}_{j=1}^n$ with *empirical measure* $\mathbb{P}_n = \frac{1}{n}\sum_{j=1}^n \delta_{Y_j}$, each projection Y_j on $(\mathcal{X}^\mathbb{N}, \mathcal{A}^\mathbb{N}, \mathbb{P}^\mathbb{N})$ represents the *observed level of income for the j-th statistical individual of the random sample of size* n in the probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, such that for each $\omega = (y_1, y_2, ...) \in \mathcal{X}^\mathbb{N}$ fixed as infinite-numerable sequence of sample points (incomes), we have that the *trajectories or realizations* are

$$\mathbb{G}_n(f_\alpha) = \sqrt{n}(\mathbb{P}_n - \mathbb{P})(f_\alpha) = \frac{1}{\sqrt{n}}\sum_{j=1}^n (f_\alpha(Y_j) - \mathbb{P}(f_\alpha)), \tag{2.1}$$

where

$$\begin{aligned}
\mathbb{P}_n(f_\alpha) &= \frac{1}{n}\sum_{j=1}^n f_\alpha(Y_j) = \frac{1}{n}\sum_{j=1}^n \left|\frac{z-Y_j}{z}\right|^\alpha \mathcal{I}\{Y_j < z\} \\
&= \int_\mathcal{X} f_\alpha(y_j) d\mathbb{P}_n(y_j) \\
&= \int_0^z \left|\frac{z-y_j}{z}\right|^\alpha d\mathbb{F}_n(y_j) \\
&= \mathbb{E}_{\mathbb{F}_n}\left[f_\alpha(Y_j)\right], \qquad (2.2)
\end{aligned}$$

and with *mean functions* given by

$$\mathbb{P}(f_\alpha) = \int_\mathcal{X} f_\alpha(y) d\mathbb{P}_Y(y) = \int_0^z \left|\frac{z-y}{z}\right|^\alpha d\mathbb{F}(y) = \mathbb{E}_\mathbb{F}\left[f_\alpha(Y)\right], \tag{2.3}$$

where $\mathbb{F}(z) = \mathbb{P}(Y \leq z)$ and $\mathbb{F}_n(z) = \frac{1}{n}\sum_{j=1}^n \mathcal{I}\{Y_j < z\} = \frac{\sharp\{Y_j < z : 1 \leq j \leq n\}}{n}$ respectively, with $z \in \mathbb{R}_+$ fixed and $q = n\mathbb{F}_n(z)$.

On the other hand, the *classical nonparametric bootstrap empirical measure* introduced by Efron in [3, 4] is defined as

$$\hat{\mathbb{P}}_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{\hat{Y}_j},$$

where for any collection of coordinate projections $Y_1, ..., Y_n$ i.i.d.$\sim$ $\mathbb{P}$, $\hat{Y}_1, ..., \hat{Y}_n$ denotes a *bootstrap sample with replacement of* $\mathbb{P}_n$. Following [12], among others, we can consider a *triangular array of exchangeable random variables* $\mathbf{W} = \{W_{nj} : n \in \mathbb{N} \text{ and } j = 1, ..., n.\}$ on $(\mathcal{W}, \mathcal{D}, \mathbb{P}_W)$, to define a general empirical measure

$$\hat{\mathbb{P}}_n^W = \frac{1}{n}\sum_{j=1}^{n} W_{nj}\delta_{Y_j},$$

which is precisely known as the *exchangeably weighted bootstrap empirical measure*. Observe that the random variables $\mathbf{W}$ can be interpreted as *random weights*, in the sense that each component W_{nj} reflects the number of times that Y_j is selected for the n trials of any *bootstrap sample with replacement*. Moreover, the classical measure $\hat{\mathbb{P}}_n$ defined above is a special case of $\hat{\mathbb{P}}_n^W$ obtained by taking $(W_{n1}, ..., W_{nn})^{'} = \underline{W}_n = \underline{M}_n$, with $\underline{M}_n = (M_{n1}, ..., M_{nn})^{'} \sim Mult_n(n, (1/n, ..., 1/n))$. In what fo-llows, we assume that $\underline{W}_n = (W_{n1}, ..., W_{nn})^{'}$ satisfies the conditions:

A1. For all $n \in \mathbb{N}$, $\underline{W}_n = (W_{n1}, ..., W_{nn})^{'}$ is *exchangeable*. That is, for any permutation $\pi = (\pi(1), ..., \pi(n))$ of (1,...,n), the joint distribution of $\pi(\underline{W}_n) = (W_{n\pi(1)}, ..., W_{n\pi(n)})^{'}$ is the same as that of $\underline{W}_n$.

A2. For all $j = 1, ..., n$, with $n \in \mathbb{N}$, $W_{nj} \geq 0$ and $\displaystyle\sum_{j=1}^{n} W_{nj} = n$.

Now, combining **A2** and the result obtained by [16], page 598, we obtain

$$(\hat{\mathbb{P}}_n^W - \mathbb{P}_n)(f_\alpha) = \frac{1}{n}\sum_{j=1}^{n}(W_{nj} - 1)(\delta_{Y_j} - \mathbb{P})(f_\alpha) = \frac{1}{n}\sum_{j=1}^{n}\xi_{nj}Z_j(f_\alpha)$$

for any $f_\alpha \in \mathcal{F}_\Gamma$, where $\xi_{nj} = W_{nj} - 1$ and $Z_j = \delta_{Y_j} - \mathbb{P}$ respectively, with $Z_j(f_\alpha) = f_\alpha(Y_j) - \mathbb{P}(f_\alpha)$. Therefore

$$\hat{\mathbb{G}}_n^W = \sqrt{n}(\hat{\mathbb{P}}_n^W - \mathbb{P}_n) = \sqrt{n}\bigg(\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}\delta_{Y_j} - \frac{1}{n}\sum_{j=1}^{n}\xi_{nj}\mathbb{P}\bigg),$$

is the *exchangeably weighted bootstrap empirical measure centered and normalized*, and for each $f_\alpha \in \mathcal{F}_\Gamma$ with $\omega = (y_1, y_2, ...) \in \mathcal{X}^{\mathbb{N}}$ fixed

$$\begin{aligned}\hat{\mathbb{G}}_n^W(f_\alpha) &= \sqrt{n}(\hat{\mathbb{P}}_n^W - \mathbb{P}_n)(f_\alpha) \\ &= \sqrt{n}\bigg(\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}f_\alpha(Y_j) - \frac{1}{n}\sum_{j=1}^{n}\xi_{nj}\mathbb{P}(f_\alpha)\bigg). \qquad (2.4)\end{aligned}$$

The expressions (2.1), (2.2) and (2.3) defined above, allow to describe the FGT poverty index defined in (1.1) as the *$\mathcal{F}_\Gamma$-indexed FGT empirical process of Lo and Seck centered and normalized*, with $\mathcal{F}_\Gamma \subset L_p(\mathcal{X}, \mathcal{A}, \mathbb{P})$, the compositions $f_\alpha(Y_j) = f_\alpha \circ Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X} \to \mathbb{R}$, for all $\alpha \geq 0$, $j = 1, 2, ..., n$, and $\mathbb{E}(Y_j^p \mathcal{I}\{Y_j < z\}) < \infty$; that is, an *additional moment condition* (see Remark 7). Analogously, the expression (2.4) represents the trajectories of our *$\mathcal{F}_\Gamma$-indexed FGT bootstrap empirical process centered and normalized*. In [11] it was shown that

$$\mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - \mathbb{P}) \xrightarrow{weakly} \mathbb{G}, \text{ in } \ell^\infty(\mathcal{F}_\Gamma),$$

where the *limit process* $\{\mathbb{G}(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma, \alpha \geq 0\}$ has *covariance function*

$$Cov\left[f_\alpha, f_\beta\right] = \mathbb{E}\left[f_{\alpha+\beta}(Y)\right] - \mathbb{E}\left[f_\alpha(Y)\right]\mathbb{E}\left[f_\beta(Y)\right], \tag{2.5}$$

for every couple $f_\alpha, f_\beta \in \mathcal{F}_\Gamma$. In consequence, this class $\mathcal{F}_\Gamma = \{f_\alpha : \alpha \geq 0\}$ is $\mathbb{P}$-*Donsker*. Now, starting from this result, we can develop a *Donsker-type charac-terization.* For it, we considering the following conditions for the weights ξ_{nj}, where **B1** is a redefinition of **A1**:

B1. For all $n \in \mathbb{N}$, $\underline{\xi}_n = (\xi_{n1}, ..., \xi_{nn})'$ is *exchangeable.*

B2. The norm $L_{2,1}$ of ξ_{n1} is *uniformly bounded.* That is,

$$\|\xi_{n1}\|_{2,1} = \int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}| \geq t)}dt \leq k < \infty,$$

where in fact $\|\cdot\|_{2,1}$ is a norm in the *space of real random variables*, and is *square-integrable* by exercise 10.5.1 (a) in [10], page 198.

B3. ξ_{n1} satisfies the *weak second-moment condition.* That is,

$$\lim_{t\to\infty} \limsup_{n\to\infty} t^2\mathbb{P}_W(|\xi_{n1}| \geq t) = 0.$$

B4. Exists $b > 0$ such that

$$\lim_{n\to\infty} \mathbb{E}[|\xi_{n1}|^2] = b.$$

3. THE MAIN RESULT AND SOME CONSEQUENCES

To presents in detail the central proof, first we enunciate the general Theorem 2.1 in [10], pages 15–16, 110; among others, about the two conditions required for the weak convergence of a given empirical process to a very particular "*stochastic object*".

Theorem 1. *The empirical process of n random variables, $\{Y_n\}_{n\in\mathbb{N}}$ converges weakly to a tight process Y in $\ell^\infty(\mathcal{F})$, if and only if:*

1. *For all finite collection $\{f_1, ..., f_k\} \subset \mathcal{F}$, the multivariate distribution of $\{Y_n(f_1), \ldots, Y_n(f_k)\}'$ converges to that of $\{Y(f_1), \ldots, Y(f_k)\}'$.*
2. *There exists a pseudometric ρ_P for which $\mathcal{F}$ is totally bounded and*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P^*\Big(\sup_{f,g\in\mathcal{F}:\rho_P(f,g)\leq\delta} |Y_n(f) - Y_n(g)| > \varepsilon\Big) = 0,$$

for all $\varepsilon > 0$.

In what follows, as in Theorem 10.1 in [10], pages 175–177; or Theorem 2.9.2 in [15], pages 179–180, we prove that conditions (1) and (2) of the Theorem 1 enunciated above are equivalent for the processes $\{\mathbb{G}_n(f_\alpha) = \sqrt{n}(\mathbb{P}_n - \mathbb{P})(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma, \alpha \geq 0\}$ and $\{\hat{\mathbb{G}}_n^W(f_\alpha) = \sqrt{n}(\hat{\mathbb{P}}_n^W - \mathbb{P}_n)(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma, \alpha \geq 0\}$ defined in (2.1) and (2.4), respectively.

Theorem 2. *Given* $(\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}}) \times (\mathcal{W}, \mathcal{D}, \mathbb{P}_W) \times (\mathcal{Z}, \mathcal{C}, \mathbb{P}_\epsilon)$*, the basic product probability space. Let* $\{Z_j\}_{j=1}^n$ *be i.i.d. empirical processes, where* $Z_j := \delta_{Y_j} - \mathbb{P}$*, with* $Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}$ *i.i.d.* $\sim \mathbb{P}$ *such that* $\mathbb{E}(Y_j^p \mathcal{I}\{Y_j < z\}) < \infty$ *for all* $j = 1, 2, ..., n$*, where* $z \in \mathbb{R}_+$ *is the poverty line fixed. Let* $\{\xi_{nj}\}_{j=1}^n$ *be i.i.d. random weights independent of* $\{Z_j\}_{j=1}^n$ *that satisfies the conditions* **B1-B4***, with mean* $\mathbb{E}[\xi_{nj}] = \mu$ *and* $\xi_{nj} := W_{nj} - 1$*, where* $\{W_{nj}\}_{j=1}^n$ *satisfies the conditions* **A1-A2***. Let* $\mathcal{F}_\Gamma = \{f_\alpha : \alpha \geq 0\} \subset L_p(\mathcal{X}, \mathcal{A}, \mathbb{P})$ *be a collection or class of functions such that*

$$f_\alpha(y_j) = \left| \frac{z - y_j}{z} \right|^\alpha \mathcal{I}\{y_j < z\},$$

for all $y_j \in \mathcal{X}$*, and* $z \in \mathbb{R}_+$*. Then the following conditions are equivalent:*
(i) $\mathcal{F}_\Gamma$ *is* $\mathbb{P}$*-Donsker.*
(ii) $\hat{\mathbb{G}}_n^W$ *converges weakly to a tight process in* $\ell^\infty(\mathcal{F}_\Gamma)$*.*

Proof. Let

$$\mathbb{G}_n(f_\alpha) = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^n f_\alpha(Y_j) - \mathbb{P}(f_\alpha)\right),$$

be an alternative representation of the centered and normalized process defined in (2.1). Then, its correspondent limit process $\mathbb{G}$ indexed by each $f_\alpha \in \mathcal{F}_\Gamma$, $\{\mathbb{G}(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma, \alpha \geq 0\}$, can be defined by

$$\mathbb{G}(f_\alpha) = f_\alpha(Y) - \mathbb{P}(f_\alpha), \tag{3.1}$$

where $\mathbb{P}(f_\alpha) = \mathbb{E}\left[f_\alpha(Y)\right]$. Indeed, $\mathbb{E}\left[\mathbb{G}(f_\alpha)\right] = 0$, and

$$\begin{aligned} Cov\left[\mathbb{G}(f_\alpha), \mathbb{G}(f_\beta)\right] &= \mathbb{E}[\mathbb{G}(f_\alpha), \mathbb{G}(f_\beta)] \\ &= \mathbb{E}\left[(f_\alpha(Y) - \mathbb{E}[f_\alpha(Y)])(f_\beta(Y) - \mathbb{E}[f_\beta(Y)])\right] \\ &= \mathbb{E}[f_\alpha(Y) f_\beta(Y)] - \mathbb{E}[f_\alpha(Y)]\mathbb{E}[f_\beta(Y)] \\ &= \mathbb{P}(f_\alpha f_\beta) - \mathbb{P}(f_\alpha)\mathbb{P}(f_\beta), \end{aligned} \tag{3.2}$$

for every couple $f_\alpha, f_\beta \in \mathcal{F}_\Gamma$. Hence, we have a limit process with zero mean, and a covariance function such that its structure go to the hand with the covariance function (2.5) suggested in [11].

Analogously, let

$$\hat{\mathbb{G}}_n^W(f_\alpha) = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^n \xi_{nj} f_\alpha(Y_j) - \frac{1}{n}\sum_{j=1}^n \xi_{nj}\mathbb{P}(f_\alpha)\right),$$

be our bootstrap process defined in (2.4). Then, its correspondent limit process $\tilde{\mathbb{G}}$ indexed by each function $f_\alpha \in \mathcal{F}_\Gamma$, $\{\tilde{\mathbb{G}}(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma, \alpha \geq 0\}$, can be defined by

$$\tilde{\mathbb{G}}(f_\alpha) = \xi_{n1}[f_\alpha(Y) - \mathbb{P}(f_\alpha)], \tag{3.3}$$

where $\mathbb{P}(f_\alpha) = \mathbb{E}[f_\alpha(Y)]$. Indeed, $\mathbb{E}[\tilde{\mathbb{G}}(f_\alpha)] = 0$, and

$$
\begin{aligned}
Cov\left[\tilde{\mathbb{G}}(f_\alpha), \tilde{\mathbb{G}}(f_\beta)\right] &= \mathbb{E}\left[\tilde{\mathbb{G}}(f_\alpha), \tilde{\mathbb{G}}(f_\beta)\right] \\
&= \mathbb{E}\left[(\xi_{n1} f_\alpha(Y) - \xi_{n1}\mathbb{E}[f_\alpha(Y)])\,(\xi_{n1} f_\beta(Y) - \xi_{n1}\mathbb{E}[f_\beta(Y)])\right] \\
&= \mathbb{E}[\xi_{n1}^2]\mathbb{E}[f_\alpha(Y) f_\beta(Y)] - \mathbb{E}[\xi_{n1}^2]\mathbb{E}[f_\alpha(Y)]\mathbb{E}[f_\beta(Y)] \\
&= \mathbb{E}\left[\xi_{n1}^2\right][\mathbb{P}(f_\alpha f_\beta) - \mathbb{P}(f_\alpha)\mathbb{P}(f_\beta)], \qquad (3.4)
\end{aligned}
$$

for every couple $f_\alpha, f_\beta \in \mathcal{F}_\Gamma$, where by condition **B4** the second moment of ξ_{n1}, $\mathbb{E}[\xi_{n1}^2]$ is finite, and consequently (3.4) is good defined.

Intentionally relaxing the notation, let $\mathcal{F}_k = \{f_1, ..., f_k\} \subset \mathcal{F}_\Gamma$, then

$$\{\mathbb{G}_n(f_1), \mathbb{G}_n(f_2), ..., \mathbb{G}_n(f_k)\}' \xrightarrow{weakly} \mathbb{N}_k(0, \Lambda),$$

where $\Lambda = [\Lambda_{i\ell}]_{i=1,\dots,k;\ \ell=1,\dots,k}$ with

$$\Lambda_{i\ell} = Cov[\mathbb{G}(f_i), \mathbb{G}(f_\ell)] = \mathbb{P}(f_i f_\ell) - \mathbb{P}(f_i)\mathbb{P}(f_\ell), \qquad (3.5)$$

and

$$\left\{\hat{\mathbb{G}}_n^W(f_1), \dots, \hat{\mathbb{G}}_n^W(f_k)\right\}' \xrightarrow{weakly} \mathbb{N}_k(0, \Upsilon),$$

where $\Upsilon = [\Upsilon_{i\ell}]_{i=1,\dots,k;\ \ell=1,\dots,k}$ with

$$\Upsilon_{i\ell} = Cov[\tilde{\mathbb{G}}(f_i), \tilde{\mathbb{G}}(f_\ell)] = \mathbb{E}[\xi_{n1}^2][\mathbb{P}(f_i f_\ell) - \mathbb{P}(f_i)\mathbb{P}(f_\ell)], \qquad (3.6)$$

and the covariance structures (3.5) and (3.6) naturally coincide with the functions (3.2) and (3.4), respectively.

In this instance, we have that the *convergence of all finite-dimensional marginal distributions of* $\mathbb{G}_n$ *and* $\hat{\mathbb{G}}_n^W$ *is equivalent to* $\mathcal{F}_\Gamma$. Thus it suffices to show that the *asymptotic equicontinuity conditions of both processes are equivalent.*

Because the limit process $\mathbb{G}$ defined in (3.1) is tight, we can assume that exists a pseudometric ρ on $\mathcal{F}_\Gamma$ for which the pseudometric space $(\mathcal{F}_\Gamma, \rho)$ is totally bounded and such that $\mathbb{G}$ has a version with almost all its trajectories (sample paths) uniformly continuous for ρ. That is, for every couple $f_\alpha, f_\beta \in \mathcal{F}_\Gamma$

$$|\mathbb{G}(f_\alpha) - \mathbb{G}(f_\beta)| \leq \lambda\rho(f_\alpha, f_\beta) \;\text{ a.s., for some }\; \lambda > 0.$$

Consequently,

$$[\mathbb{G}(f_\alpha) - \mathbb{G}(f_\beta)]^2 \leq [\lambda\rho(f_\alpha, f_\beta)]^2 \;\text{ a.s.}$$

and

$$\left[\mathbb{E}\left[\mathbb{G}(f_\alpha) - \mathbb{G}(f_\beta)\right]^2\right]^{1/2} \leq \lambda\rho(f_\alpha, f_\beta), \qquad (3.7)$$

where

$$\mathbb{E}\left[\mathbb{G}(f_\alpha)-\mathbb{G}(f_\beta)\right]^2 = Var\left[\mathbb{G}(f_\alpha)\right]+Var\left[\mathbb{G}(f_\beta)\right]-2Cov\left[\mathbb{G}(f_\alpha),\mathbb{G}(f_\beta)\right]$$

$$= Var_{\mathbb{P}}\left[f_\alpha(Y))]\right)+Var_{\mathbb{P}}\left[f_\beta(Y)\right]-2Cov_{\mathbb{P}}\left[f_\alpha(Y),f_\beta(Y)\right]$$

$$= \int (f_\alpha-f_\beta)^2 d\mathbb{P}-\left(\int (f_\alpha-f_\beta)d\mathbb{P}\right)^2 \tag{3.8}$$

$$= Var_{\mathbb{P}}\left[f_\alpha(Y)-f_\beta(Y)\right]$$

$$= \rho_{\mathbb{P}}^2(f_\alpha,f_\beta),$$

is a pseudometric for every couple $f_\alpha, f_\beta \in \mathcal{F}_\Gamma$, with (3.8) as in [6], page 853, and [12], page 2054, among others. Therefore, (3.7) implies that if $f_\alpha \in B_\rho(f_i,\varepsilon)$ then $f_\alpha \in B_{\rho_{\mathbb{P}}}(f_i,\lambda\varepsilon)$, where $B_\rho(f_i,\varepsilon)$ and $B_{\rho_{\mathbb{P}}}(f_i,\lambda\varepsilon)$ denote the open balls of center f_i and radius ε and $\lambda\varepsilon$ in $(\mathcal{F}_\Gamma,\rho)$ and $(\mathcal{F}_\Gamma,\rho_{\mathbb{P}})$, respectively. This shows that the pseudometric space $(\mathcal{F}_\Gamma,\rho_{\mathbb{P}})$ is totally bounded, and the same analysis is true for the limit process $\tilde{\mathbb{G}}$ defined in (3.3), considering the pseudometric spaces $(\mathcal{F}_\Gamma,\varrho)$ and $(\mathcal{F}_\Gamma,\varrho_{\mathbb{P}})$, with

$$\varrho_{\mathbb{P}}^2(f_\alpha,f_\beta)=\mathbb{E}\left[\xi_{n1}^2\right]\int (f_\alpha-f_\beta)^2 d\mathbb{P}-\left(\mathbb{E}[\xi_{n1}]\int (f_\alpha-f_\beta)d\mathbb{P}\right)^2, \tag{3.9}$$

for every couple $f_\alpha, f_\beta \in \mathcal{F}_\Gamma$, where the pseudometrics (3.8) and (3.9) are good defined because the elements of the class $\mathcal{F}_\Gamma$ are $\mathbb{P}$-square-integrables.

On the other hand, making a slight modification in our Lemma 8 of the Appendix, it is clear that

$$\frac{1}{2}\|\xi_{n1}-\mu\|_1\mathbb{E}^*\left[\left\|\frac{1}{\sqrt{n}}\sum_{j=1}^n \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right] \le \mathbb{E}^*\left[\left\|\frac{1}{\sqrt{n}}\sum_{j=1}^n \xi_{nj} Z_j\right\|_{\mathcal{F}_\Gamma}\right]$$

$$\le 2n_0\mathbb{E}^*\left[\|Z_1\|_{\mathcal{F}_\Gamma}\right] \tag{3.10}$$

$$\mathbb{E}\left[\max_{1\le j\le n}\frac{|\xi_{nj}|}{\sqrt{n}}\right]+4\|\xi_{n1}\|_{2,1}$$

$$\max_{n_0<k\le n}\left\{\mathbb{E}^*\left\|\frac{1}{\sqrt{k}}\sum_{j=n_0+1}^k \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right\}$$

for any $1 \le n_0 < n$. If the class $\mathcal{F}_\Gamma$ is $\mathbb{P}$-Donsker, then by Remark 9 it is $\mathbb{P}$-Glivenko-Cantelli too. Therefore, by Lemma 8.13, page 141 in [10], it follows that $\mathbb{E}^*[\|Z_1\|_{\mathcal{F}_\Gamma}]=\mathbb{P}^*(\|f_\alpha(Y_1)-\mathbb{P}(f_\alpha)\|_{\mathcal{F}_\Gamma})<\infty$, which is also true by Lemma 10, under the hypothesis (ii). Making a slight modification in the Lemma 11, is clear that $\mathbb{E}\left[\max_{1\le j\le n}\frac{|\xi_{nj}|}{\sqrt{n}}\right]\to 0$, under the conditions **B2** and **B3**. Combining this with (3.10), it follows

$$\begin{aligned}
&\frac{1}{2}\|\xi_{n1}-\mu\|_1 \limsup_{n\to\infty} \mathbb{E}^*\left[\left\|\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\epsilon_j Z_j\right\|_{\mathcal{F}_\delta}\right] \\
&\le \limsup_{n\to\infty} \mathbb{E}^*\left[\left\|\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\xi_{nj} Z_j\right\|_{\mathcal{F}_\delta}\right] \\
&\le 4\|\xi_{n1}\|_{2,1} \sup_{k>n_0}\left\{\mathbb{E}^*\left[\left\|\frac{1}{\sqrt{k}}\sum_{j=1}^{k}\epsilon_j Z_j\right\|_{\mathcal{F}_\delta}\right]\right\} \qquad (3.11)
\end{aligned}$$

for each n_0, $\delta > 0$, where $\mathcal{F}_\delta = \{f_\alpha - f_\beta : f_\alpha, f_\beta \in \mathcal{F}_\Gamma, \rho_{\mathbb{P}}(f_\alpha, f_\beta) < \delta\}$ and the pseudometric $\rho_{\mathbb{P}}(f_\alpha, f_\beta)$ defined in (3.8) common for both proceses, because without loss of generality, we can put $\mathbb{E}[\xi_{n1}^2] = 1$ and $\mathbb{E}[\xi_{n1}] = 1$ with the intention that the pseudometric $\varrho_{\mathbb{P}}(f_\alpha, f_\beta)$ defined in (3.9) coincides with (3.8), where in fact $(\mathcal{F}_\Gamma, \rho_{\mathbb{P}})$ is a *totally bounded pseudometric space.*

By the Lemma 11.2.12, pages 343–344 in [2], the Rademacher variables $\epsilon_1, \epsilon_2, ..., \epsilon_n$ can be removed in (3.11), at the "cost" of changing the constants, with $\|\xi_{n1}\|_{2,1} < \infty$ by condition **B2** and $\|\xi_{n1}-\mu\|_1 < \infty$ by **B4**. Consequently, for any sequence $\delta_n \downarrow 0$ we conclude that

$$\mathbb{E}^*\left[\left\|E_{n_{\mathbb{P}}}\right\|_{\mathcal{F}_{\delta_n}}\right] \to 0 \Longleftrightarrow \mathbb{E}^*\left[\left\|B_{n_{\mathbb{P}}}\right\|_{\mathcal{F}_{\delta_n}}\right] \to 0, \qquad (3.12)$$

where $E_{n_{\mathbb{P}}} = \frac{\sum_{j=1}^{n}(\delta_{Y_j}-\mathbb{P})}{\sqrt{n}}$ and $B_{n_{\mathbb{P}}} = \frac{\sum_{j=1}^{n}\xi_{nj}(\delta_{Y_j}-\mathbb{P})}{\sqrt{n}}$, respectively. Both sides of (3.12) are the *outer mean* (L_1)*-versions of the asymptotic equicontinuity conditions respect to* $\mathbb{G}_n$ *and* $\hat{\mathbb{G}}_n^W$. By Lemma 2.3.11, page 115 in [15], it follows that these L_1-versions are equivalents to the *outer probability versions.* Indeed, just applying the *Markov's inequality*, we have that if $\mathbb{E}^*\left\|E_{n_{\mathbb{P}}}\right\|_{\mathcal{F}_{\delta_n}} \to 0$, for any sequence $\delta_n \downarrow 0$, then it follows that $\left\|E_{n_{\mathbb{P}}}\right\|_{\mathcal{F}_{\delta_n}} \xrightarrow{\mathbb{P}^*} 0$ for any sequence $\delta_n \downarrow 0$, and this is also true for $B_{n_{\mathbb{P}}}$. Therefore, the *asymptotic tightness* for both processes is equivalent, and this completes the proof. □

We finish the section presenting three results, which are a consequence of our Theorem 2.

Corollary 3. *Under the conditions of the Theorem* 2*, we have*

$$\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(\delta_{Y_j}-\mathbb{P}), \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\xi_{nj}(\delta_{Y_j}-\mathbb{P})\right) \xrightarrow{weakly} (\mathbb{G}, \tilde{\mathbb{G}})$$

in $\ell^\infty(\mathcal{F}_\Gamma) \times \ell^\infty(\mathcal{F}_\Gamma)$*, where* $\mathbb{G}$ *and* $\tilde{\mathbb{G}}$ *are independent* $\mathbb{P}$*-Brownian bridges.*

Proof. Indeed, $(\mathbb{G}_n, \hat{\mathbb{G}}_n^W)$ is *jointly asymptotically tight.* Furthermore, the two coordinates are *uncorrelated*, and the joint marginals converge to *multivariate normal distributions.* So, $(\mathbb{G}_n, \hat{\mathbb{G}}_n^W)$ *converge jointly in distribution* to the vector $(\mathbb{G}, \tilde{\mathbb{G}})$. □

Corollary 4. *The following conditions are equivalent:*
(i) $\mathcal{F}_\Gamma$ *is* $\mathbb{P}$*-Donsker.*
(ii) $\hat{\mathbb{G}}_n = \sqrt{n}(\hat{\mathbb{P}}_n - \mathbb{P}_n)$ *converges weakly to a tight process in* $\ell^\infty(\mathcal{F}_\Gamma)$*.*

Proof. This is a direct consequence of the Theorem 2 proved above. □

Corollary 5. *Finally,*

$$\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(\delta_{Y_j}-\mathbb{P}),\frac{1}{\sqrt{n}}\sum_{j=1}^{n}M_{nj}(\delta_{Y_j}-\mathbb{P})\right)\xrightarrow{weakly}(\mathbb{G},\tilde{\mathbb{G}})$$

in $\ell^\infty(\mathcal{F}_\Gamma)\times\ell^\infty(\mathcal{F}_\Gamma)$*, where* $\mathbb{G}$ *and* $\tilde{\mathbb{G}}$ *are independent* $\mathbb{P}$*-Brownian bridges.*

Proof. This is a direct consequence of the Corollary 4. □

4. CONCLUSIONS

It is evident that our main result above developed describes a *theoretical inferential problem* of the type

$$\sqrt{n}(\theta_n-\theta)\xrightarrow{weakly}\mathcal{B}\sim\mathbb{N}(0,\Lambda)\Leftrightarrow\sqrt{n}(\hat{\theta}_n-\theta_n)\xrightarrow{weakly}\mathcal{B}^{'}\sim\mathbb{N}(0,\Upsilon)$$

in $\ell^\infty(\mathcal{F}_\Gamma)$, where $\theta(f_\alpha):=\mathbb{P}(f_\alpha)$, $\theta_n(f_\alpha):=\mathbb{P}_n(f_\alpha)$ and $\hat{\theta}_n(f_\alpha):=\hat{\mathbb{P}}_n^W(f_\alpha)$, respectively, for any $f_\alpha\in\mathcal{F}_\Gamma$, with $\mathcal{B}(f_\alpha):=\mathbb{G}(f_\alpha)$ and $\mathcal{B}^{'}(f_\alpha):=\tilde{\mathbb{G}}(f_\alpha)$. The Theorem 2 is an *unconditional result*, in the sense that the original data of functions or coordinate projections of incomes $\{Y_j\}_{j=1}^n$ is not fixed in the statement. However, according with [10], in pages 174–175, or [15], in pages 176-177, among others, our unconditional central limit theorem represents the basis for developing *conditional results* that go to the hand with *in probability and outer-almost-sure conditional central limit theorems.*

On the other hand, if we consider, for example the finite class of functions $\mathcal{F}_\Gamma=\{f_\alpha:\alpha=0,1,2\}$, then we could prove that conditioning the data, our FGT bootstrap empirical process converges weakly to a normal distribution with their respective parameters, where $FGT(Y_j,z,0)$ is a $\mathcal{F}_\Gamma$-indexed empirical process centered and normalized that reflects the *incidence of poverty* (the number of poor individuals), $FGT(Y_j,z,1)$ is a process that reflects the *intensity or severity of poverty* (the degree of poverty of the individuals), and $FGT(Y_j,z,2)$ is interpreted as the *depth or inequality among poor* (the income distribution of the poor individuals). Thus, in a further paper we could develop and discuss this and others theoretical results more refined and deeper in this framework.

A TOOLS

Note 6. *Lemma 8 is similar to the Lemmas 2.9.1, pages 177–179 in [15]; or 2.2, pages 595–596 in [16]. Lemma 10 is a new brand result proposed by the authors. Lemma 11 is an alternative proof to Lemma 4.7, page 2071 in [12], considering now the random weights* ξ_{nj}*. For more details, see [9], and for a theoretical discussion about exchangeably weighted bootstraps and other classical types of bootstrap, see e.g. [6, 12, 16].*

Remark 7. *The* FGT *empirical process centered and normalized defined in (2.1) through their trajectories can alternatively be denoted by*

$$FGT(Y_j,z,\alpha)=\sqrt{n}\left[\frac{1}{n}\sum_{j=1}^{q}\left(\frac{z-Y_j}{z}\right)^\alpha-\mathbb{P}(f_\alpha)\right],\tag{A1}$$

where $\alpha \geq 0$, and $z \in \mathbb{R}_+$. Let $\mathbb{P}(f_\alpha) = \mathbb{E}_{\mathbb{F}}(f_\alpha(Y))$, specifically, with $\alpha = 2$ it follows that

$$\begin{aligned}\mathbb{G}_n(f_2) &= \sqrt{n}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{z-Y_j}{z}\right)^2 \mathcal{I}\{Y_j < z\} - \mathbb{E}_{\mathbb{F}}(f_2(Y))\right] \\ &= \sqrt{n}\Bigg(\mathbb{F}_n(z) - \frac{2}{nz}\sum_{j=1}^{n} Y_j\mathcal{I}\{Y_j < z\} \\ &\qquad + \frac{1}{nz^2}\sum_{j=1}^{n} Y_j^2\mathcal{I}\{Y_j < z\} - \mathbb{P}(f_2)\Bigg).\end{aligned} \tag{A2}$$

Applying expected value *in (A2), it follows that*

$$\begin{aligned}\mathbb{E}[\mathbb{G}_n(f_2)] &\leq \sqrt{n}\mathbb{F}(z) - \frac{2}{\sqrt{n}z}\sum_{j=1}^{n}\mathbb{E}(Y_j\mathcal{I}\{Y_j < z\}) \\ &\qquad + \frac{1}{\sqrt{n}z^2}\sum_{j=1}^{n}\mathbb{E}(Y_j^2\mathcal{I}\{Y_j < z\}) - \sqrt{n}\mathbb{P}(f_2).\end{aligned} \tag{A3}$$

Since $n, z < \infty$ and $\mathbb{F}(z) = \mathbb{P}(Y \leq z) \leq 1$, the right-hand side of (A3) is finite if $\mathbb{E}(Y_j^2\mathcal{I}\{Y_j < z\}) < \infty$, for all $j = 1, 2, ..., n$. Hence, for each $\omega \in \mathcal{X}^{\mathbb{N}}$ fixed the trajectories of the empirical process explicitly represented in (A1) are well defined (in mean) for $0 \leq \alpha \leq 2$, if the second moment *of the projections $Y_j\mathcal{I}\{Y_j < z\}$ is finite for all $j = 1, 2, ..., n$. Analogously, it is not difficult to see that for $\alpha = 3$, the condition can be denoted by $\mathbb{E}(Y_j^3\mathcal{I}\{Y_j < z\}) < \infty$, and then, we can extend it for all $\alpha \geq 0$ through their* p-th moments.

Lemma 8. *Given $(\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}}) \times (\mathcal{W}, \mathcal{D}, \mathbb{P}_W) \times (\mathcal{Z}, \mathcal{C}, \mathbb{P}_\epsilon)$, the basic prob. space. Let $\{Z_j\}_{j=1}^n$ be i.i.d. empirical processes such that $\mathbb{E}^*[\|Z_j\|_{\mathcal{F}_\Gamma}] < \infty$ for each $j \leq n$, independent of the i.i.d. Rademacher variables $\{\epsilon_j\}_{j=1}^n$. Then for every i.i.d. sample $\{\xi_{nj}\}_{j=1}^n$ of exchangeable random weights with $\|\xi_{n1}\|_{2,1} < \infty$ and $\mathbb{E}[\xi_{nj}] = \mu$ independent of $\{Z_j\}_{j=1}^n$ and any $1 \leq n_0 < n$,*

$$\begin{aligned}\frac{1}{2}\|\xi_{n1} - \mu\|_1\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right] &\leq \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj} Z_j\right\|_{\mathcal{F}_\Gamma}\right] \\ &\leq 2n_0\mathbb{E}^*[\|Z_1\|_{\mathcal{F}_\Gamma}]\,\mathbb{E}\left[\max_{1\leq j\leq n}\frac{|\xi_{nj}|}{n}\right] \\ &+ 4\left(\frac{\|\xi_{n1}\|_{2,1}}{\sqrt{n}}\right) \\ &\quad \max_{n_0<k\leq n}\left\{\mathbb{E}^*\left\|\frac{1}{\sqrt{k}}\sum_{j=n_0+1}^{k}\epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right\}.\end{aligned} \tag{A4}$$

For symmetrically distributed variables ξ_{nj} around μ, the constants $1/2$, 2 and 4 can all be replaced by 1, and μ in the left-hand side of (A4) *is removed.*

Proof. Respect to the inequality on the left-hand side. If the random weights ξ_{nj} are symmetrically distributed around μ, then the random variables $\epsilon_j|\xi_{nj}|$ possess the same joint distribution as the ξ_{nj}, and by properties of conditional expectation, Jensen's inequality and independence between the terms in the summation, we have

$$\begin{aligned}
\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \xi_{nj} Z_j\right\|_{\mathcal{F}_\Gamma}\right] &= \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j|\xi_{nj}|Z_j\right\|_{\mathcal{F}_\Gamma}\right] \\
&= \mathbb{E}^*\left[\mathbb{E}\left[\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j|\xi_{nj}|Z_j\right\|_{\mathcal{F}_\Gamma}\right]\right] \\
&\geq \mathbb{E}^*\left[\left\|\mathbb{E}\left[\frac{1}{n}\sum_{j=1}^n \epsilon_j|\xi_{nj}|Z_j\right]\right\|_{\mathcal{F}_\Gamma}\right] \\
&= \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j Z_j \mathbb{E}|\xi_{nj}|\right\|_{\mathcal{F}_\Gamma}\right] \\
&= \|\xi_{n1}\|_1 \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right].
\end{aligned}$$

For the general case, let $\{\xi^{'}_{nj}\}_{j=1}^n$ be an independent copy of $\{\xi_{nj}\}_{j=1}^n$. Then applying the same argument of above, we obtain

$$\begin{aligned}
\|\xi_{nj} - \xi^{'}_{nj}\|_1 &= \mathbb{E}[|\xi_{nj} - \xi^{'}_{nj}|] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left|\xi_{nj} - \xi^{'}_{nj}|\right| \xi_{nj}\right]\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[\xi_{nj} - \xi^{'}_{nj}|\xi_{nj}\right]\right] \\
&= \mathbb{E}\left[|\xi_{nj} - \mathbb{E}(\xi^{'}_{nj})|\right] \\
&= \mathbb{E}\left[|\xi_{nj} - \mathbb{E}(\xi_{nj})|\right] \\
&= \|\xi_{nj} - \mu\|_1.
\end{aligned}$$

Since the random variables $(\xi_{nj} - \xi^{'}_{nj})$ are symmetric and have the same joint distribution as the random variables $\epsilon_j|\xi_{nj} - \xi^{'}_{nj}|$, applying the inequality already proved, we obtain

$$\begin{aligned}
\|\xi_{n1} - \mu\|_1 \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right] &\leq \|\xi_{n1} - \xi^{'}_{n1}\|_1 \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right] \\
&\leq \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j|\xi_{nj} - \xi^{'}_{nj}|Z_j\right\|_{\mathcal{F}_\Gamma}\right] \\
&= \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n (\xi_{nj} - \xi^{'}_{nj})Z_j\right\|_{\mathcal{F}_\Gamma}\right] \\
&\leq 2\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^n \xi_{nj} Z_j\right\|_{\mathcal{F}_\Gamma}\right].
\end{aligned}$$

In the last step, we use the triangle inequality and the fact that the random weights ξ_{nj} and $\xi^{'}_{nj}$ have the same distribution. Indeed

$$\left\|\frac{1}{n}\sum_{j=1}^{n}(\xi_{nj}-\xi_{nj}^{'})Z_j\right\|_{\mathcal{F}_\Gamma} = \left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}Z_j-\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}^{'}Z_j\right\|_{\mathcal{F}_\Gamma}$$
$$\leq \left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}Z_j\right\|_{\mathcal{F}_\Gamma}+\left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}^{'}Z_j\right\|_{\mathcal{F}_\Gamma},$$

and by this inequality we can write

$$\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}(\xi_{nj}-\xi_{nj}^{'})Z_j\right\|_{\mathcal{F}_\Gamma}\right] \leq 2\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}Z_j\right\|_{\mathcal{F}_\Gamma}\right].$$

To prove the inequality of the right hand-side in (A4), if we assume that the bootstrap weights ξ_{nj} are symmetrically distributed, then

$$\begin{aligned}
\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}Z_j\right\|_{\mathcal{F}_\Gamma}\right] &= \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\epsilon_j|\xi_{nj}|Z_j\right\|_{\mathcal{F}_\Gamma}\right] \\
&= \mathbb{E}^*\left[\left\|\frac{1}{n}\int_0^\infty\left(\sum_{j=1}^{n}\mathcal{I}\{t\leq|\xi_{nj}|\}\epsilon_jZ_j\right)dt\right\|_{\mathcal{F}_\Gamma}\right] \\
&\leq \int_0^\infty\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{\sharp\{j\leq n:|\xi_{nj}|\geq t\}}\epsilon_jZ_j\right\|_{\mathcal{F}_\Gamma}\right]dt \\
&\leq \int_0^\infty\left(\left[\sum_{k=1}^{n}\mathbb{P}_W\left(\sum_{j=1}^{n}\mathcal{I}\{|\xi_{nj}|\geq t\}=k\right)\right]\right. \\
&\qquad\left.\mathbb{E}^*\left\|\frac{1}{n}\sum_{j=1}^{k}\epsilon_jZ_j\right\|_{\mathcal{F}_\Gamma}\right)dt \\
&\leq \left(\int_0^\infty\mathbb{P}_W\left(\sum_{j=1}^{n}\mathcal{I}\{|\xi_{nj}|\geq t\}>0\right)dt\right) \\
&\qquad\left[\max_{k\leq n_0}\left\{\mathbb{E}^*\left\|\frac{1}{n}\sum_{j=1}^{k}\epsilon_jZ_j\right\|_{\mathcal{F}_\Gamma}\right\}\right] \\
&+ \left(\frac{1}{n}\int_0^\infty\sum_{k=n_0+1}^{n}\sqrt{k}\;\mathbb{P}_W\left(\sum_{j=1}^{n}\mathcal{I}\{|\xi_{nj}|\geq t\}=k\right)dt\right) \\
&\qquad\left[\max_{n_0<k\leq n}\left\{\mathbb{E}^*\left\|\frac{1}{\sqrt{k}}\sum_{j=n_0+1}^{k}\epsilon_jZ_j\right\|_{\mathcal{F}_\Gamma}\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_0^\infty \mathbb{P}_W\left\{\max_{1\leq j\leq n}|\xi_{nj}|\geq t\right\}dt\right)\frac{n_0}{n}\mathbb{E}^*[\|Z_1\|_{\mathcal{F}_\Gamma}] \\
&+ \frac{\|\xi_{n1}\|_{2,1}}{\sqrt{n}}\max_{n_0<k\leq n}\left\{\mathbb{E}^*\left\|\frac{1}{\sqrt{k}}\sum_{j=n_0+1}^{k}\epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right\} \\
&= n_0\mathbb{E}^*[\|Z_1\|_{\mathcal{F}_\Gamma}]\mathbb{E}\left[\max_{1\leq j\leq n}\frac{|\xi_{nj}|}{n}\right] \\
&+ \frac{\|\xi_{n1}\|_{2,1}}{\sqrt{n}}\max_{n_0<k\leq n}\left\{\mathbb{E}^*\left\|\frac{1}{\sqrt{k}}\sum_{j=n_0+1}^{k}\epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right\}.
\end{aligned}
$$

With respect to the third term in the last inequality, note that

$$
\begin{aligned}
\sum_{k>n_0}\sqrt{k}\mathbb{P}_W\left\{\sum_{j=1}^{n}\mathcal{I}\{|\xi_{nj}|\geq t\}k\right\} &= \mathbb{E}\left(\sum_{j=1}^{n}\mathcal{I}\{|\xi_{nj}|\geq t\}\right)^{1/2} \\
&\leq \left(\mathbb{E}\sum_{j=1}^{n}\mathcal{I}\{|\xi_{nj}|\geq t\}\right)^{1/2} \\
&= \sqrt{\mathbb{E}\sum_{j=1}^{n}\mathbb{I}\{|\xi_{nj}|\geq t\}} \\
&= \sqrt{\mathbb{E}(\sharp\{j\leq n:|\xi_{nj}|\geq t\})} \\
&= \sqrt{n\mathbb{P}_W(|\xi_{n1}|\geq t)} \\
&= \sqrt{n}\sqrt{\mathbb{P}_W(|\xi_{n1}|\geq t)}.
\end{aligned}
$$

For the general case, note that

$$
\begin{aligned}
\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}[\xi_{nj}-\mathbb{E}(\xi_{nj}^{'})]Z_j\right\|_{\mathcal{F}_\Gamma}\right] &= \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}[\xi_{nj}-\mathbb{E}(\xi_{nj})]Z_j\right\|_{\mathcal{F}_\Gamma}\right] \\
&\leq \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}(\xi_{nj}-\xi_{nj}^{'})Z_j\right\|_{\mathcal{F}_\Gamma}\right].
\end{aligned}
$$

If $\mathbb{E}[\xi_{nj}]=0$ for each $j=1,2,...,n$, obviously

$$
\mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}Z_j\right\|_{\mathcal{F}_\Gamma}\right] \leq \mathbb{E}^*\left[\left\|\frac{1}{n}\sum_{j=1}^{n}(\xi_{nj}-\xi_{nj}^{'})Z_j\right\|_{\mathcal{F}_\Gamma}\right].
$$

And then, we can apply the upper bound for $(\xi_{nj} - \xi_{nj}^{'})$ to obtain

$$\mathbb{E}^* \left[\left\| \frac{1}{n} \sum_{j=1}^{n} (\xi_{nj} - \xi_{nj}^{'}) Z_j \right\|_{\mathcal{F}_\Gamma} \right] \leq n_0 \mathbb{E}^*[\|Z_1\|_{\mathcal{F}_\Gamma}] \mathbb{E} \left[\max_{1 \leq j \leq n} \frac{|\xi_{nj} - \xi_{nj}^{'}|}{n} \right] + \left(\frac{\|\xi_{n1} - \xi_{n1}^{'}\|_{2,1}}{\sqrt{n}} \right) \left(\max_{n_0 < k \leq n} \left\{ \mathbb{E}^* \left\| \frac{1}{\sqrt{k}} \sum_{j=n_0+1}^{k} \epsilon_j Z_j \right\|_{\mathcal{F}_\Gamma} \right\} \right).$$

For the first term in the right-hand side of the inequality, $\|\xi_{nj} - \xi_{nj}^{'}\|_1 \leq 2\|\xi_{nj}\|_1$ by triangle inequality. For the second one, note that for any pair of random variables ξ_{nj} and $\xi_{nj}^{'}$,

$$\mathbb{P}_W(|\xi_{nj} + \xi_{nj}^{'}| \geq t) \leq \mathbb{P}_W(|\xi_{nj}| \geq t/2) + \mathbb{P}_W(|\xi_{nj}^{'}| \geq t/2),$$

and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, therefore

$$\begin{aligned} \|\xi_{n1} + \xi_{n1}^{'}\|_{2,1} &= \int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1} + \xi_{n1}^{'}| \geq t)} dt \\ &\leq \int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}| \geq t/2)} dt + \int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}^{'}| \geq t/2)} dt \\ &= 2 \int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}| \geq t)} dt + 2 \int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}^{'}| \geq t)} dt \\ &= 2\|\xi_{n1}\|_{2,1} + 2\|\xi_{n1}^{'}\|_{2,1}. \end{aligned}$$

Since $\|\xi_{n1} - \xi_{n1}^{'}\|_{2,1} \leq 4\|\xi_{n1}\|_{2,1}$, the right-hand side of (A4) it is true in both cases, and this completes the proof. □

Remark 9. *Any Donsker class is Glivenko-Cantelli too; but the reciprocal is not true. To see this, following e.g. [13], pages 87–88, by the continuity of the uniform norm, particularly we have*

$$\mathbb{G}_n \xrightarrow{weakly} \mathbb{G} \Rightarrow \|\mathbb{G}_n\|_{\mathcal{F}_\Gamma} \xrightarrow{weakly} \|\mathbb{G}\|_{\mathcal{F}_\Gamma}.$$

This implies that $\frac{1}{\sqrt{n}} \|\mathbb{G}_n\|_{\mathcal{F}_\Gamma}$ *converges in law (distribution) to zero, and then converges in probability to the origin, and finally*

$$\frac{1}{\sqrt{n}} \|\mathbb{G}_n\|_{\mathcal{F}_\Gamma} = \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma} \xrightarrow{a.s.^*} 0.$$

In fact, the implication is an application of Slutsky's Lemma, because this result shows that every Donsker class is a Glivenko-Cantelli in probability, and this is also true with "in probability" replaced by "almost surely".

Lemma 10. *Given* $(\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}}) \times (\mathcal{W}, \mathcal{D}, \mathbb{P}_W) \times (\mathcal{Z}, \mathcal{C}, \mathbb{P}_\epsilon)$*, the basic pro-duct prob. space. Let* $\{Z_j\}_{j=1}^n$ *be i.i.d. empirical processes with* $Z_j := \delta_{Y_j} - \mathbb{P}$ *such that* $\mathbb{E}[Z_j(f_\alpha)] = 0$ *and* $Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}$ *i.i.d.* $\sim \mathbb{P}$ *for all* $j = 1, 2, ..., n$ *and* $f_\alpha \in \mathcal{F}_\Gamma$*, independent of the i.i.d. Rademacher variables* $\{\epsilon_j\}_{j=1}^n$*.*

Then for every i.i.d. sample $\{\xi_{nj}\}_{j=1}^n$ of random weights independent of $\{Z_j\}_{j=1}^n$ that satisfies **B1** *and* **B4** *with $\xi_{nj} := W_{nj} - 1$ such that $\{W_{nj}\}_{j=1}^n$ satisfies* **A1-A2***, if $\hat{\mathbb{G}}_n^W$ converges weakly in $\ell^\infty(\mathcal{F}_\Gamma)$ to a tight Gaussian process $\tilde{\mathbb{G}}$, then $\mathbb{E}^*\left[\|Z_1\|_{\mathcal{F}_\Gamma}\right] < \infty$.*

Proof. Let $\{Z_j'\}_{j=1}^n$ be an independent copy of $\{Z_j\}_{j=1}^n$, such that we have $Z_j' = \delta_{Y_j'} - \mathbb{P}$. Then by **A2**,

$$
\begin{aligned}
\hat{\mathbb{G}}_n^W - \hat{\mathbb{G}}_n^{W\prime} &= \frac{1}{\sqrt{n}}\sum_{j=1}^n (W_{nj}-1)(Z_j - Z_j') \\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^n \xi_{nj}(Z_j - Z_j') \\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^n |\xi_{nj}|\operatorname{sign}(\xi_{nj})\epsilon_j(Z_j - Z_j') \ \ \text{(in distribution)} \\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^n |\xi_{nj}|\epsilon_j(Z_j - Z_j'), \qquad \text{(in distribution)}
\end{aligned} \tag{A5}
$$

and by the hypothesis of weak convergence, it follows that the difference between both processes in (A5) converges to a limit Gaussian process $\tilde{\mathbb{G}} - \tilde{\mathbb{G}}'$ in $\ell^\infty(\mathcal{F}_\Gamma)$. By Proposition A.2.3, page 440 in [15], it follows that the uniform norm respect to $\tilde{\mathbb{G}} - \tilde{\mathbb{G}}'$, $\|\tilde{\mathbb{G}} - \tilde{\mathbb{G}}'\|_{\mathcal{F}_\Gamma}$ has moments of all orders. Let $\varepsilon = \mathbb{E}\left[\left\|\tilde{\mathbb{G}} - \tilde{\mathbb{G}}'\right\|_{\mathcal{F}_\Gamma}^2\right] < \infty$, then for all $x > 0$,

$$
\mathbb{P}\left(\left\|\tilde{\mathbb{G}} - \tilde{\mathbb{G}}'\right\|_{\mathcal{F}_\Gamma} \geq x\right) \leq \frac{\varepsilon}{x^2}, \tag{A6}
$$

applying *Markov's inequality.*

Using (A6) and the *portmanteau Theorem* like in [15], Lemma 2.3.9, pages 113–114; or [16], Theorem 4.4, pages 602–605, there exists $N \in \mathbb{N}$, such that for all $n \geq N$,

$$
\begin{aligned}
\mathbb{P}^*\left(\left\|\sum_{j=1}^n |\xi_{nj}|\epsilon_j(Z_j - Z_j')\right\|_{\mathcal{F}_\Gamma} > x\sqrt{n}\right) &\leq 2\mathbb{P}\left(\left\|\tilde{\mathbb{G}} - \tilde{\mathbb{G}}'\right\|_{\mathcal{F}_\Gamma} \geq x\right) \\
&\leq \frac{2\varepsilon}{x^2},
\end{aligned} \tag{A7}
$$

and by *Levy's second inequality* of the Proposition A.1.2, page 431 in [15], we have

$$
\begin{aligned}
&\mathbb{P}^*\left(\left\|\sum_{j=1}^n |\xi_{nj}|\epsilon_j(Z_j - Z_j')\right\|_{\mathcal{F}_\Gamma} > x\sqrt{n}\right) \\
&\geq \frac{1}{2}\mathbb{P}^*\left(\max_{1\leq j\leq n} |\xi_{nj}||\epsilon_j| \left\|Z_j - Z_j'\right\|_{\mathcal{F}_\Gamma} > x\sqrt{n}\right) \\
&= \frac{1}{2}\mathbb{P}^*\left(\max_{1\leq j\leq n} |\xi_{nj}| \left\|Z_j - Z_j'\right\|_{\mathcal{F}_\Gamma} > x\sqrt{n}\right),
\end{aligned}
$$

then combining this result with (A7), it follows

$$
\mathbb{P}^*\left(\max_{1\leq j\leq n} |\xi_{nj}|\|Z_j - Z_j'\|_{\mathcal{F}_\Gamma} > x\sqrt{n}\right) \leq \frac{4\varepsilon}{x^2}. \tag{A8}
$$

Let π be a random permutation of the first n terms. Suppose that $J \in \{1, \ldots, n\}$ satisfies

$$\max_{1\leq j\leq n} \frac{1}{\sqrt{n}} \left\| Z_j - Z_j^{'} \right\|_{\mathcal{F}_\Gamma} = \frac{1}{\sqrt{n}} \left\| Z_J - Z_J^{'} \right\|_{\mathcal{F}_\Gamma}. \tag{A9}$$

Then by the condition **B1** of exchangeability of the random weights ξ_{nj},

$$\max_{1\leq j\leq n} \frac{1}{\sqrt{n}} |\xi_{nj}| \left\| Z_j - Z_j^{'} \right\|_{\mathcal{F}_\Gamma} = \max_{1\leq j\leq n} \frac{1}{\sqrt{n}} |\xi_{n\pi(j)}| \left\| Z_j - Z_j^{'} \right\|_{\mathcal{F}_\Gamma}, \tag{A10}$$

in distribution. π is a random permutation independent of the $\{W_{nj}\}$, $\{Z_j\}$ and $\{Z_j^{'}\}$. Therefore, let $\left\| Z_j - Z_j^{'} \right\|_{\mathcal{F}_\Gamma} = b_j$ and $\left\| Z_J - Z_J^{'} \right\|_{\mathcal{F}_\Gamma} = b_J$, conditioning on the W_{nj}'s, it follows that using (A10) the probability on the left-hand side of (A8) is equal to

$$\begin{aligned}
\mathbb{E}_W \mathbb{P}^* \left(\frac{1}{\sqrt{n}} \max_{1\leq j\leq n} |\xi_{n\pi(j)}| b_j > x \middle| \underline{W} \right) &\geq \mathbb{E}_W \mathbb{P}^* \left(\frac{1}{\sqrt{n}} |\xi_{n\pi(J)}| b_J > x \middle| \underline{W} \right) \\
&= \mathbb{E}_W \left(\frac{1}{n} \sum_{j=1}^n \mathcal{I}\left\{ \frac{1}{\sqrt{n}} |\xi_{nj}| b_J^* > x \right\} \right) \\
&\geq \mathbb{E}_W \left(\frac{1}{n} \sum_{j=1}^n \mathcal{I}\{|\xi_{nj}| > \varepsilon\} \right. \\
&\quad \left. \mathcal{I}\left\{ \frac{1}{\sqrt{n}} b_J^* > \frac{x}{\varepsilon} \right\} \right) \\
&= \left(\mathbb{P}_W(|\xi_{n1}| > \varepsilon) \right. \\
&\quad \left. \mathcal{I}\left\{ \frac{1}{\sqrt{n}} b_J^* > \frac{x}{\varepsilon} \right\} \right).
\end{aligned} \tag{A11}$$

Given

$$\mathbb{P}\left(Y > c\mathbb{E}[Y]\right) \geq \frac{(1-c)^2 \left(\mathbb{E}[Y]\right)^2}{\mathbb{E}\left[Y^2\right]}, \tag{A12}$$

the *Paley-Zygmund argument*, where $Y \geq 0$ and $0 \leq c \leq 1$. Using (A12) with $c\mathbb{E}[Y] = \varepsilon$ and $Y = |\xi_{n1}|$ on the right-hand side of (A11), we have

$$\begin{aligned}
&\mathbb{P}_W\left(|\xi_{n1}| > \varepsilon\right) \left(\mathcal{I}\left\{ \frac{1}{\sqrt{n}} b_J^* > \frac{x}{\varepsilon} \right\} \right) \\
&\geq \left(\frac{\left(1 - \varepsilon/\mathbb{E}[|\xi_{n1}|]\right)^2 \left(\mathbb{E}\left[|\xi_{n1}|\right]\right)^2}{\mathbb{E}\left[|\xi_{n1}|^2\right]} \right) \left(\mathcal{I}\left\{ \frac{1}{\sqrt{n}} b_J^* > \frac{x}{\varepsilon} \right\} \right).
\end{aligned} \tag{A13}$$

Combining (A8), (A11) and (A13), this implies that

$$\begin{aligned}
&\left(\frac{\left(1 - \varepsilon/\mathbb{E}[|\xi_{n1}|]\right)^2 \left(\mathbb{E}\left[|\xi_{n1}|\right]\right)^2}{\mathbb{E}\left[|\xi_{n1}|^2\right]} \right) \mathcal{I}\left\{ \frac{1}{\sqrt{n}} \|Z_J - Z_J^{'}\|_{\mathcal{F}_\Gamma}^* > \frac{x}{\varepsilon} \right\} \\
&\leq \frac{4\varepsilon}{x^2},
\end{aligned} \tag{A14}$$

and by (A9), respect to (A14) we have that

$$\left(\frac{\left(1-\varepsilon/\mathbb{E}[|\xi_{n1}|]\right)^2 \left(\mathbb{E}\left[|\xi_{n1}|\right]\right)^2}{\mathbb{E}\left[|\xi_{n1}|^2\right]}\right) \mathcal{I}\left\{\max_{1\leq j\leq n}\frac{1}{\sqrt{n}}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma}^* > \frac{x}{\varepsilon}\right\}$$

$$\leq \frac{4\varepsilon}{x^2}. \tag{A15}$$

By the property $\mathbb{E}^*[\mathcal{I}_B] = \mathbb{P}^*(B)$, in (A15) it follows

$$\left(\frac{\left(1-\varepsilon/\mathbb{E}[|\xi_{n1}|]\right)^2 \left(\mathbb{E}\left[|\xi_{n1}|\right]\right)^2}{\mathbb{E}\left[|\xi_{n1}|^2\right]}\right) x^2\,\mathbb{P}^*\left(\max_{1\leq j\leq n}\frac{1}{\sqrt{n}}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma} > \frac{x}{\varepsilon}\right)$$

$$\leq 4\varepsilon. \tag{A16}$$

By **B4**, and the fact that the convergence in quadratic mean of $|\xi_{n1}|$ to $b > 0$ implies the convergence in mean to this constant, then the first term of (A16) converges to a positive constant. Respect to the second term,

$$\lim_{x\to\infty} x^2\mathbb{P}^*\left(\max_{1\leq j\leq n}\frac{1}{\sqrt{n}}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma} > \frac{x}{\varepsilon}\right) = 0.$$

That is, $\mathbb{P}^*\left(\max_{1\leq j\leq n}\frac{1}{\sqrt{n}}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma} > \frac{x}{\varepsilon}\right) = o(x^{-2})$, as $x \to \infty$. Now, let $x^{'} = \frac{x\sqrt{n}}{\varepsilon}$, since $\mathbb{P}^*\left(\max_{1\leq j\leq n}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma} > x^{'}\right)$ has a superior tail of order $o(x^{-2})$, then all its moments of order $0 < r < 2$ respect to $\max_{1\leq j\leq n}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma}$ exist; i.e., $\mathbb{E}^*\left[\max_{1\leq j\leq n}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma}^r\right] < \infty$, for $0 < r < 2$. In particular, we have $\mathbb{E}^*\left[\max_{1\leq j\leq n}\left\|Z_j - Z_j^{'}\right\|_{\mathcal{F}_\Gamma}\right] < \infty$, and by (A9) it follows that

$$\mathbb{E}^*\left[\left\|Z_1 - Z_1^{'}\right\|_{\mathcal{F}_\Gamma}\right] < \infty. \tag{A17}$$

Finally, by the convexity of the norm $\|\cdot\|_{\mathcal{F}_\Gamma}$ together with $\mathbb{E}[Z_1^{'}] = 0$, respect to (A17) we have

$$\mathbb{E}^*\left[\|Z_1\|_{\mathcal{F}_\Gamma}\right] = \mathbb{P}^*\left(\|f_\alpha(Y_1) - \mathbb{P}(f_\alpha)\|_{\mathcal{F}_\Gamma}\right) < \infty,$$

applying the *Jensen's inequality*. □

Lemma 11. *Let* $\xi = \{|\xi_{nj}| : j = 1, 2, ..., n, n = 1, 2, ...\}$ *be a triangular array of non-negative and exchangeable random variables, defined on the probability space* $(\mathcal{W}, \mathcal{D}, \mathbb{P}_W)$. *If* ξ *satisfies conditions* **B2** *and* **B3**, *this implies that the sequence* $\{|\xi_{n1}|\}_{n\in\mathbb{N}}$ *is uniformly square-integrable; that is,*

$$\lim_{t\to\infty}\limsup_{n\to\infty}\mathbb{E}(|\xi_{n1}|^2\mathcal{I}\{|\xi_{n1}| \geq t\}) = 0. \tag{A18}$$

Furthermore, **B2** *and* **B3** *also imply that*

$$\mathbb{E}\left[\max_{1\leq j\leq n}\frac{|\xi_{nj}|}{n}\right] \to 0. \tag{A19}$$

Proof. Let

$$
\begin{aligned}
\mathbb{E}(|\xi_{n1}|^2 \cdot \mathbb{I}\{|\xi_{n1}| \geq t\}) &= \int_0^\infty \mathbb{P}_W(|\xi_{n1}|^2 \cdot \mathbb{I}\{|\xi_{n1}| \geq t\} > x)dx \\
&= \int_0^\infty \mathbb{P}_W(|\xi_{n1}| \cdot \mathbb{I}\{|\xi_{n1}| \geq t\} > \sqrt{x})dx \\
&= \int_0^\infty \mathbb{P}_W(|\xi_{n1}| \cdot \mathbb{I}\{|\xi_{n1}| \geq t\} > u)2udu \\
&= \int_0^t \mathbb{P}_W(|\xi_{n1}| \cdot \mathbb{I}\{|\xi_{n1}| \geq t\} > u)2udu \\
&+ \int_t^\infty \mathbb{P}_W(|\xi_{n1}| \cdot \mathbb{I}\{|\xi_{n1}| \geq t\} > u)2udu. \qquad \text{(A20)}
\end{aligned}
$$

Since

$$
\mathbb{P}_W(|\xi_{n1}|\mathcal{I}\{|\xi_{n1}| \geq t\} > u) = \begin{cases} \mathbb{P}_W(|\xi_{n1}| \geq t) & \text{si } 0 < u \leq t, \\ \mathbb{P}_W(|\xi_{n1}| \geq u) & \text{si } u > t, \end{cases}
$$

where $\mathbb{P}_W(|\xi_{n1}|\mathcal{I}\{|\xi_{n1}| \geq t\} > u) = \mathbb{P}_W(|\xi_{n1}| \geq t \vee u)$, such that $t \vee u$ denotes the maximum between t and u. Respect to (A20) we obtain

$$
\begin{aligned}
\mathbb{E}(|\xi_{n1}|^2\mathcal{I}\{|\xi_{n1}| \geq t\}) &= \int_0^t \mathbb{P}_W(|\xi_{n1}| \geq t)2udu + \int_t^\infty \mathbb{P}_W(|\xi_{n1}| \geq u)2udu \\
&= t^2\mathbb{P}_W(|\xi_{n1}| \geq t) \\
&+ 2\int_t^\infty u\sqrt{\mathbb{P}_W(|\xi_{n1}| \geq u)}\sqrt{\mathbb{P}_W(|\xi_{n1}| \geq u)}du. \qquad \text{(A21)}
\end{aligned}
$$

By **B3**, let $\varepsilon > 0$. We can choose T large enough so that $t^2\mathbb{P}_W(|\xi_{n1}| \geq t) \leq \frac{\varepsilon^2}{4}$ for each $n \in \mathbb{N}$ when $t \geq T$; that is,

$$
\mathbb{P}_W(|\xi_{n1}| \geq t) \leq \frac{\varepsilon^2}{4t^2}.
$$

Furthermore, for $u > t$,

$$
\mathbb{P}_W(|\xi_{n1}| \geq u) \quad \leq \tfrac{\varepsilon^2}{4u^2} \Rightarrow \sqrt{\mathbb{P}_W(|\xi_{n1}| \geq u)} \quad \leq \frac{\varepsilon}{2u}.
$$

Hence, in (A21) it follows that

$$
\begin{aligned}
\mathbb{E}(|\xi_{n1}|^2\mathcal{I}\{|\xi_{n1}| \geq t\}) &\leq \frac{\varepsilon^2}{4} + \varepsilon\int_t^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}| \geq u)}du \\
&\leq \frac{\varepsilon^2}{4} + \varepsilon\|\xi_{n1}\|_{2,1},
\end{aligned}
$$

and by virtue of **B2**, since ε is arbitrary, the expression (A18) it is satisfied.

To prove (A19), let $\varepsilon > 0$. From (A18) we can choose t large enough so that

$$
\limsup_{n\to\infty} t^2\mathbb{P}_W(|\xi_{n1}| > t) \leq \limsup_{n\to\infty} \mathbb{E}(|\xi_{n1}|^2\mathcal{I}\{|\xi_{n1}| > t\}) \leq \varepsilon^2,
$$

for $t \geq \varepsilon$. Then

$$\mathbb{E}\left[\max_{1\leq j\leq n}|\xi_{nj}|\right] = \int_0^{\varepsilon}\mathbb{P}_W\left(\max_{1\leq j\leq n}|\xi_{nj}|>t\right)dt+\int_{\varepsilon}^{\infty}\mathbb{P}_W\left(\max_{1\leq j\leq n}|\xi_{nj}|>t\right)dt$$
$$\leq \varepsilon+\int_{\varepsilon}^{\infty}t^2\mathbb{P}_W(|\xi_{n1}|>t)t^{-2}dt$$
$$\leq \varepsilon+\varepsilon^2\frac{1}{\varepsilon}=2\varepsilon,$$

and since ε is arbitrary, this finishes the proof of the lemma. □

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