FINANCIAL MODELLING WITH MULTIVARIATE MIXED FRACTIONAL BROWNIAN MOTION

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ABSTRACT
In this paper we introduce a multivariate mixed fractional Brownian motion model for the study of financial data. We study the problem of replication of multivariate European derivatives under this model. We also show that this model can be used to generate some scale-dependent correlation structures, and in particular, it reproduces a well known empirical fact present in financial data known as the Epps effect.

KEYWORDS: fractional Brownian motion, payoff replication, Epps effect

MSC: 60G22, 91G20

RESUMEN
En este artículo se introduce un modelo Browniano fraccionario mixto multivariado para el estudio de datos financieros. Estudiamos el problema de replicación de derivados financieros bajo este modelo multivariado. Tambien demostramos que este modelo genera estructuras de correlación dependientes de la frecuencia de observación. En particular, este modelo es capaz de reproducir el conocido efecto de Epps, qiue se reporta en ciertos datos financieros.

PALABRAS CLAVE: movimiento Browniano fraccionario, replicación de derivados financieros, efecto de Epps

1. INTRODUCTION

Since the 70s, the paradigm model for the pricing of financial derivatives is the Black-Scholes model. In the Black-Scholes model, the price of the underlying asset is modelled as a Geometric Brownian motion

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \tag{1.1}$$

where W is a Brownian Motion.

Multivariate extensions of this model are well known and have been extensively used in many financial applications, in particular for the pricing of multivariate derivatives (see for example [9], among many other papers on the subject).

Despite the fact that both univariate and multivariate versions of the Black-Scholes models fail to describe many empirical facts observed in financial data, they still are widely used by practitioners

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for their mathematical tractability and ease of interpretation.

A univariate mixed fractional Brownian motion has the form

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t + s B_t^H\right) \tag{1.2}$$

where W is a Brownian motion and B^H is a fractional Brownian motion with Hurst index H. Univariate models like the one in (1.2) have been present in the financial literature for the last twenty years, see for example [10], [5], [1] among others.

The addition of the fractional Brownian process B^H to the classical Black-Scholes model has some practical and conceptual consequences. When the Hurst index $H \neq 1/2$, the increments of the fractional Brownian motion are not independent, so in general the increments of the mixed model S will not be independent. This is consistent with the form of long range dependence observed in financial time series, see [11]. This is an empirical property of real financial data that the Black-Scholes model cannot describe.

On the other hand, it is important to mention that if $H \leq 3/4$ and $H \neq 1/2$, the process S is not a semimartingale. At first sight, this seems to be a serious drawback from a financial modelling point of view, in the light of the classical arbitrage results in [12]. However, during the last two decades there have been several papers ([10], [17], [5], [1], [2] , [13] and [18]) that study and solve financial problems, including the arbitrage issue, without necessarily assuming that the price process is a semimartingale. In particular, in many of these papers the existence of arbitrage is avoided by restricting the allowed portfolio strategies. While some of the mentioned papers focus on univariate non-semimartingale models, there are also some arbitrage results, such as the ones in [20], that also apply in multivariate frameworks.

In this direction we would like to point out a well known fact: some derivatives that depend on a single asset can be replicated in a pathwise manner using Föllmer's non probabilistic Itô Calculus, regardless of whether the price process is a semimartingale or not. As far as we know, the first result of this type was given in [6]. More recently, similar results were obtained in [5]. One important consequence of these papers can be roughly summarized as follows:

In a model of the form $S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma W_t + h_t)$ where W is a Brownian motion, and h has null quadratic variation and is independent of W, the perfect replication of a European payoff can be achieved path-by-path using the same delta-hedging as in the Black-Scholes model with volatility σ. In other words, the term h_t will have no effect at all on the price of a derivative that depends on S.

This means that, at least for pricing purposes, the univariate mixed model in (1.2), is as analytically tractable as the Black-Scholes model, at the same time that is able to describe empirical financial data more accurately. Then, some natural questions that arise in this context are:

1) Can we extend the notion of mixed model to the multivariate case in a non trivial way? i.e. not considering the components separately.
2) Is it possible to replicate multivariate derivatives perfectly using multivariate mixed models?
3) Will the multivariate mixed model be able to describe at least some empirical characteristics of financial data that other mainstream models cannot?

In this paper, we will respond affirmatively to these three questions. In particular, regarding the third question, we will see that multivariate mixed models can reproduce a well documented stylized fact known as the Epps effect. We are not aware of any other continuous-time model capable of reproducing the Epps effect.

Most of the papers that use this kind of path by path approach to payoff replication deal exclusively with univariate processes and payoffs that depend on a single asset. In this paper we will explore the problem of multi asset payoff replication under the multivariate mixed fractional Brownian model, using a path by path approach. The main technical tool that we will need to use is Föllmer's non probabilistic version of Itô's formula.

Under certain conditions, it will be possible to replicate payoffs that depend on several assets. The replication result itself is a straightforward generalization of the univariate case, so from a mathematical point of view, its originality is somehow limited. However, such a replication result in the multivariate case, has important consequences from the point of view of dependence modelling. As far as we know, these consequences have not been explored in detail.

The structure of the paper is as follows. In section 2, Föllmer's non probabilistic Itô's formula and related results about the quadratic variation are presented. In Section 3, we first introduce the notion of multivariate fractional Brownian motion, and use it to define the bivariate mixed model. Then we study the problem of payoff replication under a bivariate mixed fractional Brownian model. In Section 4 we show that the bivariate mixed fractional Brownian model introduced in Section 3 can reproduce the Epps effect. Section 5 discusses possible extensions of the mixed model.

2. FÖLLMER'S NON PROBABILISTIC ITÔ'S FORMULA

In order to make this paper as self contained as possible, in this section we will briefly summarize some known results on a non probabilistic version of Itô's formula introduced by Föllmer in [16]. In order to clarify the notation, multidimensional objects will be written in boldface type.

Let $T > 0$ be a fixed real number and consider a sequence of partitions of $[0,T]$, $\tau \equiv \{\tau_n\}_{n=1,2,..}$ satisfying that $\tau_n = \left\{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{K(n)}^{(n)} = T\right\}$, $\tau_1 \subset \tau_2 \subset \cdots$ and $\text{mesh}(\tau_n) = \max_{t_k^{(n)} \in \tau_n} |t_k^{(n)} - t_{k-1}^{(n)}|$ approaches 0 as n approaches infinity. An example of such a sequence is provided by the dyadic partition of $[0,T]$: $\tau_n = \left\{t_k^{(n)} = Tk2^{-n}, k = 0, 1, \ldots, 2^n\right\}$

A continuous function $y : [0,T] \to \mathbb{R}$ has continuous quadratic variation $[y]$ along the sequence $\{\tau_n\}$ if for every $0 \le t \le T$

$$[y]_t = \lim_{n\to\infty} \sum_{\substack{t_i^{(n)}, t_{i+1}^{(n)} \in \tau_n \\ t_{i+1}^{(n)} \le t}} \left(y\left(t_{i+1}^{(n)}\right) - y\left(t_i^{(n)}\right)\right)^2 \tag{2.1}$$

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be a continuous function on $[0,T]$ with values in $\mathbb{R}^n$. We say that $\mathbf{y}$ has quadratic variation along (τ_n) if this holds for all functions y_i, $y_i + y_j$, $1 \le i, j \le n$. In this case we put

$$[y_i, y_j]_t = \frac{1}{2}\left([y_i + y_j]_t - [y_i]_t - [y_j]_t\right) \tag{2.2}$$

Let $F(t, \mathbf{y}) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function on $[0,T] \times \mathbb{R}^n$ that is continuously differentiable in $(t, \mathbf{y}) \in (0,T) \times \mathbb{R}^n$ and twice continuously differentiable in $\mathbf{y} \in \mathbb{R}^n$

Define $\nabla F_{\mathbf{y}} \equiv \left(\frac{\partial F}{\partial y_1}, \frac{\partial F}{\partial y_2}, \ldots, \frac{\partial F}{\partial y_n}\right)$. Then we have Itô-Föllmer's formula:

$$
\begin{aligned}
F(t,\mathbf{y}_t) - F(0,\mathbf{y}_0) &= \int_0^t \nabla F_{\mathbf{y}}(s,\mathbf{y}_s)d\mathbf{y}_s + \int_0^t \frac{\partial F}{\partial t}(s,\mathbf{y}_s)ds \\
&+ \frac{1}{2}\sum_{k,m=1}^{n}\int_0^t \frac{\partial^2 F}{\partial y_k \partial y_m}(s,\mathbf{y}_s)d[y_k,y_m]_s
\end{aligned} \tag{2.3}
$$

where

$$
\int_0^t \nabla F_{\mathbf{y}}\left(s,\mathbf{y}_s\right)d\mathbf{y}_s = \lim_{n\to\infty} \sum_{\substack{t_i^{(n)},t_{i+1}^{(n)}\in\tau_n \\ t_{i+1}^{(n)}\le t}} \nabla F_{\mathbf{y}}\left(t_i^{(n)},\mathbf{y}\left(t_i^{(n)}\right)\right)\cdot\left(\mathbf{y}(t_{i+1}^{(n)}) - \mathbf{y}(t_i^{(n)})\right) \tag{2.4}
$$

Remark 1. *In general, it is not true that*

$$
\int_0^t \nabla F_{\mathbf{y}}\left(s,\mathbf{y}_s\right)d\mathbf{y}_s = \sum_{i=1}^{n}\int_0^t \frac{\partial F}{\partial y_i}\left(s,\mathbf{y}_s\right)dy_i(s)
$$

as the integrals in the right hand side may not exist individually (see [19]).

Remark 2. *The notion of integral in expression (2.4) has a clear economic interpretation in a financial context as accumulated gains/losses by trading a financial asset. There exist other notions of integrals that use a fractional Brownian processes as integrator. In particular, the Wick integral has been proposed in financial contexts but there are conceptual concerns about its use for certain financial problems, because it does not have a clear economic interpretation, as shown in [7].*

Also if $Z\colon \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, then $x = Z \circ \mathbf{y}$ is of quadratic variation along (τ_n), with

$$
[x,x]_t = \sum_{i,j}\int_0^t \frac{\partial Z}{\partial y_i}(\mathbf{y}_s)\frac{\partial Z}{\partial y_j}(\mathbf{y}_s)d[y_i,y_j]_s \tag{2.5}
$$

Other properties about the quadratic variation that will be needed later are stated in the following Lemma. These properties are well know both in the classical stochastic framework, as well as in Föllmer's non probabilistic framework, so a proof is not included.

Lemma 3. *If y_1, y_2, b_1 b_2 are continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $[b_1]_t = [b_2]_t = 0$ for $0 \le t \le T$ then:*

- $[y_1, b_1]_t = 0$ *(this is consequence of Cauchy-Schwartz inequality)*
- $[y_1 + b_1]_t = [y_1]_t$
- $[y_1 + b_1, y_2 + b_2]_t = [y_1, y_2]_t$

3. MULTIVARIATE MIXED FRACTIONAL MODELS

In this section we will introduce the multivariate fractional mixed models and study some of their properties, in particular the quadratic variation. First we have to introduce some results on the multivariate fractional Brownian motion following the presentation in [3].

For the remaining of the paper, assume that we have a probability space $(\Omega, \mathcal{A}, P)$ and all the stochastic processes are defined in this probability space. Expected values are taken with respect to the probability measure P.

Let $H = (H_1, H_2, \dots H_m) \in (0,1)^m$. A multivariate fractional Brownian motion $B = (B^{(1)}, B^{(2)}, \dots B^{(m)})$ satisfies that each component $B^{(i)}$ is a univariate Fractional Brownian motion with Hurst exponent H_i. That means that for all i, $B^{(i)}$ is a Gaussian process with $E\left(B^{(i)}\right) = 0$ and covariance function

$$EB_s^{(i)} B_t^{(i)} = \frac{1}{2}\left(|t|^{2H_i} + |s|^{2H_i} - |t-s|^{2H_i}\right)$$

The cross covariances of the mfBm satisfy the following representation, for all $0 \le i, j \le m$, $i \ne j$:
i) If $H_i + H_j \ne 1$ there exists $(\rho_{i,j}, \eta_{i,j}) \in [-1,1] \times \mathbb{R}$ with $\eta_{i,j} = \eta_{j,i}$ such that

$$EB_s^{(i)} B_t^{(j)} = \frac{1}{2}\Big[\left(\rho_{i,j} + \eta_{i,j}\text{sign}(s)\right)|s|^{H_i+H_j} + \left(\rho_{i,j} - \eta_{i,j}\text{sign}(t)\right)|t|^{H_i+H_j}\Big]$$

ii) If $H_i + H_j = 1$ there exists $(\tilde\rho_{i,j}, \tilde\eta_{i,j}) \in [-1,1] \times \mathbb{R}$ with $\tilde\eta_{i,j} = \tilde\eta_{j,i}$ such that:

$$EB_s^{(i)} B_t^{(j)} = \frac{1}{2}\Big[\tilde\rho_{i,j}\left(|s| + |t| - |s-t|\right) + \tilde\eta_{i,j}\left(t\log|t| - s\log|s| - (t-s)\log|t-s|\right)\Big]$$

It is important to notice that while the univariate fractional Brownian motion is well known, multivariate versions of it have only been studied during the last few years, generally related to applications in fields other than finance. Another recent paper studying the multivariate fractional Brownian motion is [4].

3.1. Bivariate mixed fractional Brownian model

In this subsection we will introduce the multivariate mixed fractional Brownian model. For simplicity we will be limiting our attention to the bivariate case, but all these models and results can be easily extended to the n-dimensional case with $n > 2$.

Let $W = \left(W^{(1)}, W^{(2)}\right)$ be a bivariate Brownian motion such that $dW^{(1)}dW^{(2)} = \rho dt$, that is $W^{(2)} = \rho W^{(1)} + \sqrt{1-\rho^2}\widetilde{W}^{(2)}$, where $W^{(1)}$ and $\widetilde{W}^{(2)}$ are independent univariate standard Brownian Motions. Consider also $B^H = (B^{(1)}, B^{(2)})$, a bivariate fractional Brownian motion, independent of W.

Let $\mathbf{x} \equiv \left(x^{(1)}, x^{(2)}\right)$ be defined by $x_t^{(i)} = x_0^{(i)} \exp\left((u_i - \sigma_i^2/2)t + \sigma_i W_t^{(i)} + s_i B_t^{(i)}\right)$ and define $\mathbf{Y} \equiv \left(\log x^{(1)}, \log x^{(2)}\right)$.

Assume that the Hurst exponent $H = (H_1, H_2)$ is such that $H_i > 1/2$ for $i = 1, 2$. Then, P-a.s. we have that $\left[B^{(i)}\right]_t = 0$ for $i = 1, 2$ and for $0 \le t \le T$. We also know that $\left[W^{(i)}(\omega)\right]_t = t$ and $\left[W^{(1)}(\omega), W^{(2)}(\omega)\right]_t = \rho t$ hold P-a.s. Then we can easily prove the following:

Proposition 4. *The stochastic processes* $\mathbf{x}$ *and* $\mathbf{Y}$ *satisfy the following properties:*

1. $\mathbf{Y}$ *is a (bivariate) Gaussian process. For any* s, t *with* $0 \le s < t \le T$*, the increment of* $\mathbf{Y}$, $\mathbf{Y}(t) - \mathbf{Y}(s)$ *is normally distributed with mean* $\begin{bmatrix} (u_1 - \sigma_1^2/2)(t-s) \\ (u_2 - \sigma_2^2/2)(t-s) \end{bmatrix}$ *and covariance matrix*

$$\begin{bmatrix} \sigma_1^2(t-s) + s_1^2(t-s)^{2H_1} & \sigma_1\sigma_2\rho(t-s) + \rho_{1,2}(t-s)^{H_1+H_2} \\ \sigma_1\sigma_2\rho(t-s) + \rho_{1,2}(t-s)^{H_1+H_2} & \sigma_2^2(t-s) + s_2^2(t-s)^{2H_2} \end{bmatrix} \quad (3.1)$$

2. *The trajectories* $\mathbf{x}(\omega)$, $\mathbf{Y}(\omega)$ *are of quadratic variation along the dyadic partitions* P*-a.s.*
3. $\left[Y^{(i)}\right]_t = \sigma_i^2 t$ P*-a.s. for* $i = 1, 2$
4. $\left[Y^{(1)}, Y^{(2)}\right]_t = \rho\sigma_1\sigma_2 t$ P*-a.s.*

5. $\left[x^{(i)}\right]_t = \int_0^t \sigma_i^2 x_s^{(i)} ds$ *P-a.s. for* $i = 1, 2$

6. $\left[x^{(1)}, x^{(2)}\right]_t = \int_0^t \rho\sigma_1\sigma_2 x_s^{(1)} x_s^{(2)} ds$ *P-a.s.*

7. $\lim_{t\to s^+} Corr\left(Y_t^{(1)} - Y_s^{(1)}, Y_t^{(2)} - Y_s^{(2)}\right) = \rho$

Proof. Statement 1 follows from the definitions of $\mathbf{x}$ and $\mathbf{Y}$. Statements 2-6 follow from expression 2.5 and Lemma 3. Statement 7 follows from expression (3.1) and the fact that H_1, H_2 are both greater than 1/2. □

Remark 5. *Statement 1 describes the dependence structure between the increments of* $Y^{(1)}$ *and* $Y^{(2)}$*. We can see in the covariance expression given in (3.1) that this dependence structure is a combination between the dependence structures of the Brownian motions* $W^{(i)}$*'s and the fractional Brownian motions* $B^{(i)}$*'s. However, under the assumption that* H_1*,* H_2 *are both greater than 1/2, we get that when* $\Delta = t - s$ *is close to 0, the terms in (3.1) corresponding to the fractional Brownian motions are negligible with respect to the terms describing the Brownian motions dependence. That is what statement 7 reflects, meaning that at high frequencies, the dependence structure between the fractional Brownian motions will not be noticeable. Also, it is important to notice that the quadratic variation expressions in statements 3-6 remain the same as in the classical bivariate Black-Scholes framework.*

The existing similarities between the multivariate mixed model and the usual multivariate Black-Scholes model, mentioned in Remark 5, can be exploited for the pricing of multivariate derivatives.

3.2. Replication of bivariate derivatives

In this subsection we will study the pricing problem for a derivative with European pay-off $G\left(x_T^{(1)}, x_T^{(2)}\right)$ at maturity time T. An example of such a pay-off is $G(s_1, s_2) = (s_1 - s_2 - K)^+$, corresponding to a spread option.

Consider a market with two risky assets, so that their prices $x^{(1)}$ and $x^{(2)}$ evolve jointly according to the bivariate mixed fractional model described in subsection 3.1.. We also assume that there is a non risky bond that evolves with interest rate r, which for simplicity will be set to $r = 0$.

It is worth noticing that the components $x^{(i)}$ of the mixed model are not necessarily semimartingales, see [10], therefore the risk neutral approach for pricing does not work for mixed models. However, as in the univariate case (see [6], [5], [1]) it is possible to replicate perfectly the payoff G, as if we were working with the bivariate Black-Scholes model, using a self financing portfolio strategy containing the two risky assets and the non-risky bond.

Consider the following PDE.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 V}{\partial x_1^2} + \rho\sigma_1\sigma_2 x_1 x_2 \frac{\partial^2 V}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_1^2 x_2^2 \frac{\partial^2 V}{\partial x_2^2} = 0 \tag{3.2}$$

We have the following:

Theorem 6. *Let* $V : [0, T] \times \mathbb{R}^2$ *be the solution of (3.2) subject to the terminal condition* $V(T, \mathbf{x}) = G(\mathbf{x})$*. If the prices of the underlying assets evolve according to the mixed model* $\mathbf{x}$ *from subsection 3.1., then there exists a portfolio strategy of initial value* $V(0, \mathbf{x}_0)$ *that replicates the pay-off* G *at maturity time* T*.*

Proof. This proof follows along the same lines of the classical Black Scholes derivation of the price using the parabolic PDE. Let $V(t, \mathbf{x})$ be the solution of (3.2) subject to the terminal condition $V(T, \mathbf{x}) = G(\mathbf{x})$.

Consider the usual delta-hedging portfolio strategy, that is the self-financing portfolio holding $\phi_i(t, \mathbf{x}_t)$ shares of asset i at time $0 \leq t \leq T$ where $\phi_i(t, \mathbf{x}) = \dfrac{\partial V}{\partial x_i}(t, \mathbf{x})$ for i=1,2.

For any continuous function V on $[0, T] \times \mathbb{R}^2$ that is continuously differentiable in $(t, \mathbf{x}) \in (0, T) \times \mathbb{R}^2$ and twice continuously differentiable in $\mathbf{x} \in \mathbb{R}^2$ we have Itô-Föllmer's formula:

$$dV = \frac{\partial V}{\partial t} dt + \frac{1}{2} \sum_{k,m=1}^{2} \frac{\partial^2 V}{\partial x_k \partial x_m} d[x_k, x_m]_t + \nabla V_{\mathbf{x}} d\mathbf{x} \tag{3.3}$$

We know from Proposition 4 that

$\left[x^{(i)}\right]_t = \int_0^t \sigma_i^2 x_s^{(i)} ds$ a.s. for $i = 1, 2$

$\left[x^{(1)}, x^{(2)}\right]_t = \int_0^t \rho \sigma_1 \sigma_2 x_s^{(1)} x_s^{(2)} ds$ a.s.

Putting these expressions for the quadratic variation and covariation in (3.3) we get

$$dV = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma_1^2 x_1^2 \frac{\partial^2 V}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 x_1 x_2 \frac{\partial^2 V}{\partial x_1 \partial x_2} + \sigma_2^2 x_2^2 \frac{\partial^2 V}{\partial x_1^2}\right)\right] dt + \nabla V_{\mathbf{x}} d\mathbf{x} \tag{3.4}$$

If V satisfies equation (3.2) then we get

$$dV = \nabla V_{\mathbf{x}} d\mathbf{x}$$

which, taking into account that $r = 0$, means that V represents the value of the self-financing portfolio given by ϕ_1 and ϕ_2. Then the value of that portfolio at time $t = T$ will be $V(T, \mathbf{x})$, which is equal to $G(\mathbf{x})$ because of the imposed terminal condition. Then we conclude that the mentioned portfolio replicates the payoff G at maturity.

□

Theorem 6 has important and somehow surprising implications, among them:

1) The replication result above applies on a path by path sense.
2) The replication price of a derivative in the bivariate mixed model, given by $V(0, \mathbf{x_0})$ is the same as in the bivariate Black-Scholes model that is obtained by omitting the fractional Brownian motion terms. In particular, neither s_1, s_2 or the correlation parameter $\rho_{1,2}$ affect at all the derivative price, as they do not appear in the PDE in (3.2).

3) The only correlation parameter that affects the derivative price is ρ, the correlation parameter between the Brownian motions.

These observations, together with Remark 5, suggests that the correlation structure that actually matters for pricing purposes, is the correlation structure that occurs at high frequencies. This kind of analysis is pointless in the Black-Scholes model under which the correlation structure is the same at all frequencies.

4. EPPS EFFECT

In the late 70's Thomas W. Epps [15] , observed that the empirical correlation between the returns of two different stocks decreases as the sampling frequency of data increases (or equivalently when the sampling interval decreases). This empirical fact is know since as the Epps effect. More recent papers that study the Epps effect are [21], [22] and [8].

The likely causes for the Epps effect have been described in the mentioned papers, among others. According to the available literature, the Epps effect is caused by a combination of different factors as:

-Asynchronicity of ticks for different stocks
-Discretization effects
-Human time scale (human reaction time to news).

However, as far as we know, there are no available multivariate continuous-time models for stock prices capable of reproducing and/or explaining the Epps effect. In our opinion, this is an issue that deserves some attention.

Figure 1 represents the theoretical correlation between the increments of the components of a bivariate mixed fractional Brownian motion $Y_t^{(1)} - Y_s^{(1)}$ and $Y_t^{(2)} - Y_s^{(2)}$ as a function of $\Delta = t - s$.

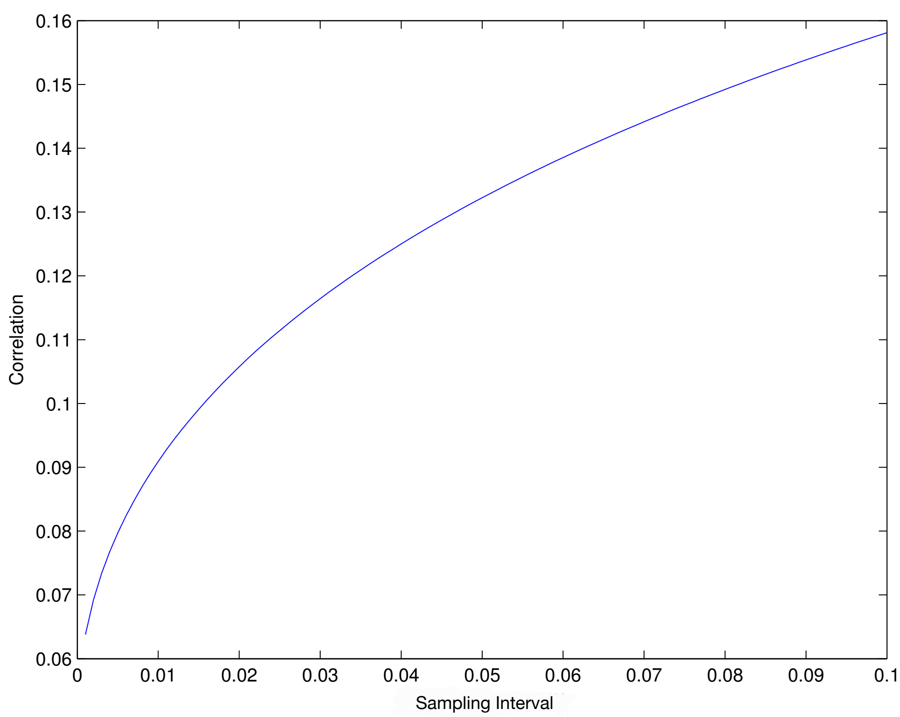

Figure 1: Correlation as a function of the sampling interval $\Delta = t - s$

This correlation can be computed explicitly using expression (3.1). The parameters used to generate Figure 1 are $\sigma_1 = \sigma_2 = s_1 = s_2 = 0.2$, $\rho = 0.05$, $\rho_{1,2} = 0.5$, $H_1 = H_2 = 0.75$. Figure 1 shows that the correlation structure generated by the bivariate mixed fractional Brownian model is consistent with the Epps effect.

This result does not contradict the causes for the Epps effect mentioned above. Actually, what this means is that probably some of the factors causing the Epps effect could be modelled with a fractional Brownian motion.

5. DISCUSSION

The bivariate mixed fractional Brownian model introduced in subsection 3.1. can be generalized in different directions while still remaining analytically tractable, a feature highly appreciated by practitioners.

On one hand, it is straightforward to extend the bivariate model to more than two dimensions using the same equations. The results regarding replication of multivariate derivatives are similar than in the bivariate case.

Another way of getting a model with some additional features would be to introduce some form of dependence between the Brownian motions $W^{(i)}$'s and the fractional Brownian motions $B^{(i)}$'s. In theory, this would be possible by modelling the pair (W, B) jointly as a four-dimensional fractional Brownian motion, in which the first two components have Hurst index equal to 1/2 and the other two are greater that 1/2. In this way, the first two components (corresponding to the Brownian motion) do not have to be independent of the last two components (corresponding to the fractional Brownian motions).

Another variation of the model that gives similar pricing results would be to substitute the fractional Brownian processes $B^{(i)}$'s by other continuous processes of null quadratic variation. One possible substitute for the fractional Brownian process could be a moving average process of the type $h_t = \dfrac{1}{h}\displaystyle\int_{t-h}^{t} Z_s ds$.

These extensions prove that mixed models are very flexible, and from our point of view, it is important to explore whether some of these multivariate mixed models are consistent with empirical financial data. Also, it has been pointed out that there are notable differences between implied and realized correlation, see [14]. These mixed models could explain, at least partially, that discrepancy.

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