MULTI RATES (0NE, TWO AND THREE) OF PRODUCTION INVENTORY MODELS FOR DETERIORATING ITEMS WITH COMPARATIVE STUDY

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ABSTRACT
In production inventory system, there are situations, in which it is not possible to have single rate of production throughout the production period. Items are produced at different rates during sub periods so as to meet various constraints that arise due to change in demand pattern, market fluctuations, etc., In this paper, a production inventory model with deteriorative items in which multi-rates (one, two and three) of production are considered and it is possible that production started at one rate and after some time it may be switched over to another rate. Such a situation is desirable in the sense that by starting at a low rate of production, a large quantum stock of manufacturing items at the initial stage is avoided, leading to reduction in the holding cost. A suitable mathematical model is developed and the optimal production lot size which minimizes the total cost is derived. The global optimal solution is derived and an illustrative example is provided and numerically verified. The validation of result in this model was coded in Microsoft Visual Basic 6.0

KEYWORDS: Production, cycle time, demand, one rate, two rates, three rates of production and optimality.

MSC: 90B05

RESUMEN
En un sistema de inventario de producción, hay situaciones, en las cuales no es posible tener una simple tasa de producción a través del periodo de producción. Los ítems son producidos, con diferentes tasas durante sub periodos para satisfacer varias restricciones, que aparecen durante el cambio del patrón de demanda, fluctuaciones del mercado, etc. En este paper un modelo, para los inventarios de producción con deteriorables ítems en los que hay multi-tasas (una , dos y tres) de producción, es considerado; y es posible que la producción comience a una tasa y después de algún tiempo puede cambiar a otra. Tal situación es deseable en el sentido de que se comienza a una baja tasa producción, así una gran cantidad del stock de los manufacturados ítems en la inicial etapa es evitada, llevando a la reducción del costo de mantenimiento. Un modelo matemático es desarrollado y el tamaño optimal del lote de producción, que minimiza el costo total es derivado. La solución global optimal es derivada y un ejemplo ilustrativo se presenta y numéricamente se verifica. La validación del resultado en este un modelo fue instrumentado en Microsoft Visual Basic 6.0

PALABRAS CLAVE: Producción, ciclo de tiempo, demanda, una tasa, dos tasas, tres tasas de producción y optimalidad.

1. INTRODUCTION

The primary operation strategies and goals of most manufacturing firms are to seek a high satisfaction to customer's demands and to become a low-cost producer. To achieve these goals, the company must be able to effectively utilize resources and minimize costs. Harris (1913) introduced EOQ model with minimize total inventory costs (cost of holding inventory and cost of setup) and derive the formula for number of units to be purchased. Perumal and Arivarignan (2002) developed two rates of production inventory models, shortages are not permitted. Cardenas-Barrown (2009) developed production inventory model with corrected some mathematical expressions in the work of Sarkar, B.R., Jamal, A.M.M., Mondal, S. 2008, optimal batch sizing in a multi-stage production system with rework consideration, European Journal of Operational Research, 184(3): 915-929. Bhownuck and Samanta (2011) considered production inventory model for deteriorating items with shortages and developed mathematical model for the production rate is changed to another at a time when the inventory level reaches prefixed level. Aalikar (2014) developed multi-product multi-period production inventory models in which inventory costs are derived under inflation condition and further, the products are delivered in boxes of known number of items and the aim is to find the number of boxes of the products in different periods to minimize the total inventory cost. Sivashankari and Panayappan (2014) developed two rates

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of production inventory models with incorporated a multi-delivery policy for defective items with the purpose of reducing the holding cost. Sivashankari and Panayappan(2015) developed two rates of productions inventory models for deteriorating items with the aim of reducing in holding cost. Sivashankari and Krishnamoorthi (2016) developed three levels of production inventory models for deteriorating items with the aim of reducing in holding cost. Tiwari, S., Cárdenas-Barrón, L.E., Goh, M., Shaikh, A.A., (2018) developed an inventory model for deteriorating items under a two-level partial trade credit with allowable shortages. This paper considers a supplier-retailer-customer supply chain in which (a) for settling the cost of purchasing, the retailer receives a partial trade credit from the supplier and at the same time the retailer offers a separate partial trade credit to the customer, (b) the downstream credit period not only increases demand but also opportunity cost, (c) the deterioration rate is non-decreasing over time and the product is fully deteriorated close to its expiration date, and (d) shortages are allowed. Tiwari, S., Jaggi, C. K., Gupta, M., Cárdenas-Barrón, L.E., (2018) developed a two-echelon supply chain model for deteriorating items in which the retailer's warehouse capacity of display area is limited. Therefore, the retailer stores remaining units in the back room which has unlimited capacity. The demand rate is assumed to be dependent on the retailer's selling price and displayed stock level. Sunil Tiwaria, Leopoldo Eduardo Cárdenas, Barrónb Ali, AkbarShaikh and Mark Gohad (2018)establishes an economic order quantity inventory model for deteriorating items, with allowable shortages and permissible partial delay in payment based on the order quantity. This paper presents theoretical results to determine the optimal replenishment time and the length of time for the stock to draw down completely, and with these time values the optimal ordering and backlogging policies are calculated for the retailer in order to minimize the total inventory cost per unit time Shaikh, A.A., Bhunia, A.K., Cárdenas-Barrón, L.E., Sahoo, L. (2018) considered a fuzzy inventory model for a deteriorating item with permissible delay in payments and the demand depends on selling price and the frequency of the advertisement. Mahata (Expert Syst Appl 39(3):3537–3550, 2012) developed an economic production quantity (EPQ) inventory model for exponentially deteriorating items under permissible delay in payments considering that both demand and production are constant and known. This paper, applying well-known approximation mathematical expressions, derives closed-form formulas for the time at which the production ends, the cycle length and the total cost of inventory system. Moreover, this work presents a comparison of the solutions to the numerical examples by approximation closed-form formulas and Mahata (2012)'s method. The approximated method works properly because the percent of penalty is negligible less than 0.09%. Bhunia, A.K., Shaikh, A.A., Dhaka,V., Pareek, S., Cárdenas-Barrón, L.E., (2018) developed an inventory model for single deteriorated item considering the impact of marketing decisions and the displaced stock level on the demand. Partial backlogged shortages are allowed. Analyzing the storage capacity of the shop and demand parameters, different scenarios have been investigated. For each scenario, the corresponding problem has been formulated as a nonlinear mixed integer optimization problem and solved by real coded genetic algorithm and particle swarm optimization technique. Nita H. Shah and Chetansinh R. Vaghela (2018) developed an economic production quantity (EPQ) model for deteriorating items with both up-stream and down-stream trade credits and associated profit function is maximized with respect to selling price and cycle time using classical optimization. Shaikh, A.A., Cárdenas-Barrón, L.E., Bhunia, A.K., Tiwari, S. (2019) Considered an inventory model for a deteriorating item with variable demand dependent on the selling price and frequency of advertisement of the item under the financial trade credit policy. Shortages are allowed and these are partially backlogged with a variable rate dependent on the duration of waiting time until to the arrival of next order. In this inventory model, the deterioration rate follows a three-parameter Weibull distribution. Shaikh, A.A., Cárdenas-Barrón, L.E., Tiwari, S. (2019) considered a two-warehouse inventory model for non-instantaneous deteriorating items with interval-valued inventory costs and stock-dependent demand under inflationary conditions. The proposed inventory model permits shortages, and the backlogging rate is variable and dependent on the waiting time for the next order, and inventory parameters are interval-valued. The main aim of this research is to obtain the retailer's optimal replenishment policy that minimizes the present worth of total cost per unit time. Sahoo, Bhabani, S. Mohanty and P.K. Tripathi (2019) developed a inventory model with three parameter Weibull distribution. Item of deterioration and cost of holding are in linear function of time Fuzziness. Both crisp and fuzzy models are illustrated to determine the optimal cycle time and optimal inventory cost. Mihir S. Suthar and Kunal T. Shukla (2019) considered for non-instantaneous deteriorating items with price sensitive ramp type demand pattern. Pre deterioration discount is considered to be smallest than the post deterioration discount as per trend. Anima Bag and Tripathy P.K. (2019) developed an inventory model for decaying goods with time and selling price induced quadratic demand to determine optimal cycle time, optimal purchase quantity and minimum total cost of the inventory system. This paper analysis a situation in which the production period is consisting of many sub periods each with difference production rates. We assume that in each sub period the inventory is built up by a constant amount

I_i (i=1,2,3) at a different rate of production, after allowing consumption by demand. Section 3 is for mathematical modelling and numerical examples. Section 4, a comparative study is carried out. Finally, the paper summarizes and concludes in section 5.

2. ASSUMPTIONS AND NOTATIONS

Assumptions: The assumption of this inventory model are as follows:
1) The demand rate is known, constant and continuous, 2) Items are produced and added to the inventory, 3) one, two and three rates of productions are considered, 4) The item is a single product; it does not interact with any other inventory items, 5) The production rate is always greater than or equal to the sum of the demand rate and defective items, 6) It is assumed that no repair or replacement of the deteriorative items takes place during a given cycle.

Notations: The Notations of this inventory model are as follows:
1) X_1 - productions during one-level of production in units in time T_1, 2) X_2 - production during two- levels of production in units in time T_2, 3) P_3 - production during three-levels of production at time P_3, 4) Y- constant demand rate in units, 5) I_1, I_2, I_3 - maximum inventory levels during one, two and three rates of productions, 6) T_1, T_2, T_3 - production time during one, two and three rates of productions, 7) S_C - setup cost per set, 8) P_C - production cost per unit, 9) H_C - cost of holding of inventory per unit per unit time, 10) D_C - cost of deteriorative per unit, 11) μ - rate of deteriorative items, 12) T- Optimum cycle time, 13) TC (T) – Total cost at time T

Computational Algorithm:

Step 1: Assign values to the parameters with proper units.

Step 2: To find the two variables T_1 and Q in model 1, T_2 and T in model 2, T_3 and T in Model 3. Therefore, the partial differential equation is used in this paper.

Step 3: For optimality condition, a) $\frac{\partial}{\partial T_2} TC(T) = 0$ & $\frac{\partial^2}{\partial T_2^2} TC(T) > 0$

b) $\frac{\partial}{\partial T} TC(T) = 0$ & $\frac{\partial^2}{\partial T^2} TC(T) > 0$

Step 4: The optimum values T and Q for the given data are calculated from the equations (12, 29 & 50).

Step 5 : The sensitivity analysis is used in three models in which it is programmed and the datas are generated from the visual basic 6.0 software.

3. MATHEMATICAL FORMULATION

3.1. Single Rate of Production inventory model for Deteriorative items

In this model, we have considered a single commodity deterministic continuous production inventory model with a constant demand rate Y. The production of the item is started initially at t=0 at a rate X_1(>Y). Once the inventory level reaches I_1 and the production is stopped and the inventory is depleted at a constant rate Y. When the inventory level reaches zero then the next production cycle starts at the lower rate X_1. The duration of the production at the rate X_1 is (0, T_1). The duration when there is no production but only consumption by demand at a rate Y by (T_1, T). The cycle then repeats itself after time T. The duration of a production cycle T is taken as variable. This model is represented by Figure -1

During the production stage, the inventory of good items increases due to production but decreases due to demand and deterioration items. Thus, the inventory differential equation is

$$\frac{dI(t)}{dt} + \mu I(t) = X_1 - Y \; ; 0 \le t \le \; T_1 \tag{1}$$

The inventory differential equation during the consumption period with no production and subsequently reduction in the inventory level due to deterioration items is given by

$$\frac{dI(t)}{dt} + \mu I(t) = -Y \;\; ; \; T_1 \le t \le T \; (2)$$

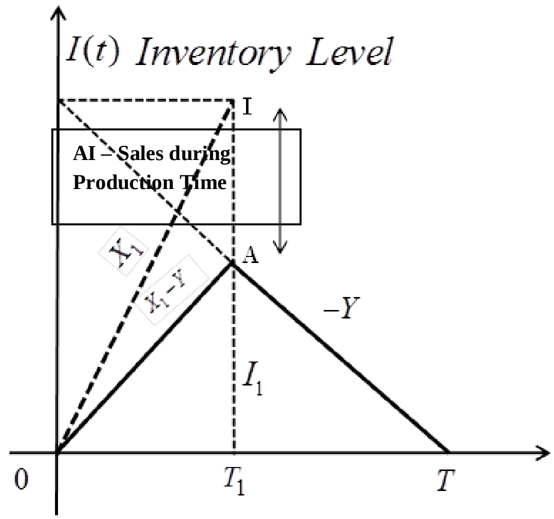

Figure -1 One Rate of Production Inventory Model

With the basic conditions of differential equations: $I(0) = 0$, $I(T_1) = I_1$, $I(T) = 0$

During the first cycle, the inventory level $I(t)$: at time t is equal to

From the equation (1): $I(t) = \frac{X_1 - Y}{\mu}\left[1 - e^{-\theta t}\right]; \; 0 \le t \le T_1$ (3)

From the equation (2): $I(t) = \frac{Y}{\mu}\left(e^{\theta(T-t)} - 1\right); T_1 \le t \le T$ (4)

We know that, $I(T_1) = I(T_1)$ from the equations (3) and (4) , $\frac{X_1 - Y}{\mu}\left[1 - e^{-\theta T_1}\right] = \frac{Y}{\mu}\left[e^{\theta(T-T_1)} - 1\right]$.But in this model, we have considered T_1 as follows,

$(X_1 - Y)T_1 = Y(T - T_1)$, Therefore, $T_1 = \frac{D}{P_1}T$ (5)

The maximum inventory I_1 is as follows: $I(T_1) = I_1 \Rightarrow \frac{X_1 - Y}{\mu}\left(1 - e^{-\theta T_1}\right) = I_1$

Therefore, $I_1 = (X_1 - Y)T_1$ (6)

Total cost: The total cost comprise of the sum of the production cost, ordering cost, holding cost, deteriorating cost. They are grouped together after evaluating the above cost individually.

(i) Production Cost /unit time = $X(t)P_C \frac{P_C}{T} = YP_C$ (7)

(ii) Setup cost per setup $= \frac{S_C}{T} = \frac{Y}{Q} S_C$ (8)

(iii) Holding Cost / unit time : Holding cost is applicable to both stages of the production cycle, as described by

$$HC = \frac{H_C}{T}\left[\int_0^{T_1} I(t)dt + \int_{T_1}^{T} I(t)dt\right] = \frac{H_C}{T}\left[\int_0^{T_1} \frac{X_1 - Y}{\mu}\left(1 - e^{-\mu t}\right)dt + \int_{T_1}^{T} \frac{Y}{\mu}\left(e^{\mu(T-t)} - 1\right)dt\right] =$$

$$\frac{H_C}{T}\left[\frac{X_1 - Y}{\mu^2}\left(\mu t + e^{-\mu t}\right)_0^{T_1} - \frac{Y}{\mu^2}\left(e^{\mu(T-t)} + \mu t\right)_{T_1}^{T}\right] = \frac{H_C}{T}\left[\frac{X_1 - Y}{\mu^2}\left(\mu T_P + e^{-\mu T_1} - 1\right) - \frac{Y}{\varpi^2}\left(1 - e^{\mu(T-t)} + \mu(T - T_1)\right)\right]$$

$$= \frac{H_C}{T}\left[\frac{X_1 - Y}{\mu^2}\left\{\frac{\mu^2 T_1^2}{2}\right\} + \frac{Y}{\mu^2}\left\{\frac{\mu^2 (T - T_1)^2}{2}\right\}\right]$$

$$= \frac{H_C}{T}\left[\frac{(X_1 - Y)T_1^2}{2} + \frac{Y(T - T_1)^2}{2}\right] = \frac{H_C}{T}\left[\frac{XT_1^2}{T} + YT - 2YT_1\right] = \frac{TH_C Y(X_1 - Y)}{2X} \text{ from equation (5)} \quad (9)$$

(iv) Deteriorating Cost/unit time: Deteriorating cost, which is applicable to both stages of the production cycle. Therefore,

$$DC = \frac{D_C}{T}\left[\int_0^{T_1} \mu I_1(t)dt + \int_{T_1}^{T} \mu I_2(t)dt\right] = \frac{D_C}{T}\left[\int_0^{T_1} \theta \frac{X_1 - Y}{\theta}\left(1 - e^{-\mu t}\right)dt + \int_{T_1}^{T} \mu \frac{Y}{\mu}\left(e^{\mu(T-t)} - 1\right)dt\right]$$

Expanding the exponential functions and neglecting second and higher power of θ for small value of θ.

$$= \frac{TY\mu D_C X_1 - Y)}{2X_1} \quad (10)$$

Therefore, Total Cost (TC) = Purchase Cost + Ordering Cost + Holding Cost + Deteriorating Cost

$$= DP_C + \frac{H_C}{T} + \frac{TH_C Y(X_1 - Y)}{2X_1} + \frac{TY\mu D_C (X_1 - Y)}{2X} \quad (11)$$

Optimality conditions

a) $\frac{\partial}{\partial T_2} TC(C) = 0$ and $\frac{\partial^2}{\partial T_2^2} TC(C) = 0$

b) $\frac{\partial}{\partial T} TC(C) = 0$ and $\frac{\partial^2}{\partial T^2} TC(C) = 0$

The total cost equation (11) differentiate w.r.t. T, $\frac{\partial}{\partial T}(TC) = \frac{-S_C}{T^2} + \frac{(H_C + \mu D_C)Y(X_1 - Y)}{2X_1} = 0$ and

$\frac{\partial^2}{\partial T^2} = \frac{2S_C}{T^3} > 0$.

Therefore, $T = \sqrt{\frac{2P_1 S_C}{Y(X_1 - Y)(H_C + \mu D_C)}}$ and $Q = \sqrt{\frac{2DP_1 S_C}{(P_1 - D)(H_C + \theta D_C)}}$ (12)

Numerical Example. Let us consider the cost parameters

X_1 = 4,000 units, Y= 3500 units, H_C =11, P_C = 110, S_C =110, μ = 0.01, D_C =110

Optimum solution: T = 0.2038, Q = 713.50, T_1 = 0.1783, $I_1 = 89.18$,

Production cost = 385000 , Setup cost = 539.58, Holding cost = 490.53 ,
Deteriorating cost = 49.05, and Total cost = 386079.17

Table 1. Rate of Deteriorative items with the Inventory costs in one rate of production inventory model

μ	T	Q	T_1	I_1	Setup cost	Holding Cost	Deteriorative Cost	Total cost

0.01	0.2038	713.50	0.1783	89.18	539.58	490.53	49.05	386079.17
0.02	0.1951	683.13	0.1707	85.39	563.58	469.65	93.93	386127.16
0.03	0.1875	656.33	0.1640	82.04	586.59	451.22	135.36	386173.19
0.04	0.1807	632.45	0.1581	79.05	608.73	434.81	173.92	386217.47
0.05	0.1745	611.01	0.1527	76.37	630.10	420.06	210.03	386260.20
0.06	0.1690	591.60	0.1479	73.95	650.76	406.73	244.03	386301.53
0.07	0.1639	573.94	0.1434	71.74	670.79	394.58	276.21	386341.57
0.08	0.1593	557.77	0.1394	69.72	690.24	383.46	306.77	386380.48
0.09	0.15515	542.89	0.1357	67.86	709.15	373.24	335.91	386418.31
0.10	0.1511	529.15	0.1322	66.14	727.58	363.79	363.79	386455.16

Production cost = 385,000

From table 1, it is concluded that there is positive relationship between increases in rate of deteriorative items with cost of setup, cost of deteriorative and total cost. There is a negative relationship between increases in rates of deteriorative items with cycle time (T): optimum quantity (Q): production time ((T_1)), maximum inventory (I_1): cost of holding inventory.

Table 2. Effect of Demand and cost parameters on optimal values in one rate of Production inventory model

Parameters		Optimum values							
		T	Q	T_1	I_1	Setup cost	Holding cost	Deteriorative cost	Total Cost
	3800	0.2565	897.81	0.2362	70.87	428.28	389.84	38.98	385857.64
	3900	0.2250	787.68	0.2020	80.79	488.77	444.34	44.43	385977.54
X_1	4000	0.2038	713.50	0.1783	89.18	539.58	490.53	49.05	386079.17
	4100	0.1884	659.43	0.1608	96.50	583.84	530.76	53.07	386167.67
	4200	0.1765	617.91	0.1471	102.99	623.06	566.42	56.64	386246.13
	0.01	0.2038	713.50	0.1783	89.18	539.58	490.53	49.05	386079.17
	0.02	0.1951	683.13	0.1707	85.39	563.58	469.65	93.93	386127.16
μ	0.03	0.1875	656.33	0.1640	82.04	586.59	451.22	135.36	386173.19
	0.04	0.1807	632.45	0.1581	79.05	608.73	434.81	173.92	386217.47
	0.05	0.1745	611.01	0.1527	76.37	630.10	420.06	210.03	386260.20
	90	0.1843	645.39	0.1613	80.67	488.07	443.70	44.47	385976.15
	100	0.1943	680.30	0.1700	85.03	514.47	467.70	46.77	386028.95
S_C	110	0.2038	713.50	0.1783	89.18	539.58	490.53	49.05	386079.17
	120	0.2129	745.23	0.1863	93.15	563.58	512.34	51.23	386127.17
	130	0.2216	775.66	0.1939	96.95	586.59	533.26	53.32	386173.19
	9	0.2231	780.96	0.1952	97.62	492.98	439.29	53.69	385985.96
	10	0.2128	744.95	0.1862	93.11	516.81	465.59	51.29	386033.62
H_C	11	0.2038	713.50	0.1783	89.18	539.58	490.53	49.05	386079.17
	12	0.1959	685.73	0.1714	85.71	561.44	514.29	47.14	386122.88
	13	0.1888	660.96	0.1652	82.62	582.47	537.03	45.44	386164.95
	90	0.2055	719.47	0.1798	89.93	535.11	494.64	40.47	386070.22
	100	0.2047	716.47	0.1791	89.55	537.35	492.57	44.77	386074.70
D_C	110	0.2038	713.50	0.1783	89.18	539.58	490.53	49.05	386079.17
	120	0.2030	710.57	0.1776	88.82	541.81	488.52	53.29	386083.62
	130	0.2021	707.68	0.1769	88.46	544.03	486.53	57.49	386088.06
	90	0.2055	719.47	0.1798	89.93	535.11	494.64	40.47	315000.00 316079.17
	100	0.2047	716.47	0.1791	89.55	537.35	492.57	44.77	350000.00 351079.17
P_C	110	0.2038	713.50	0.1783	89.18	539.58	490.53	49.05	385000.00 386079.17
	120	0.2030	710.57	0.1776	88.82	541.81	488.52	53.29	420000.00 421079.17
	130	0.2021	707.68	0.1769	88.46	544.03	486.53	57.49	455000.00 456079.17

Sensitivity Analysis:

The total cost functions are the real solution in which the model parameters are assumed to be static values. It is reasonable to study the sensitivity i.e. the effect of making changes in the model parameters over a given optimum solution. It is important to find the effects on different system performance measures, such as cost function, inventory system, etc. For this purpose, sensitivity analysis of various system parameters for models of this research are required to be observed, whether the current solutions remain unchanged, or infeasible, etc.

Managerial insights: A sensitivity analysis is performed to study the effects of change in the system parameters, setup cost (S_C), holding cost (H_C), deteriorative cost (D_C): total cost, optimal cycle time (T): optimal quantity (Q): production time (T_1), maximum inventory (I_1). The sensitivity analysis is performed by changing (increasing or decreasing) the parameter taking at a time, keeping the remaining parameters at their original values. The following influences can be obtained from sensitivity analysis based on table 2.

1) there is a positive relationship between increase in rate of production P_1 with maximum inventory (I_1): setup cost, holding cost, deteriorative cost, total cost. There is a negative relationship between increase in rate of production P_1 with optimum cycle time (T): optimum quantity (Q) and production time (T_1).

2) there is a positive relationship between with the increase in rate of deteriorating item (μ), with the cost of setup, Deteriorating cost, and total cost. There is a negative relationship with increase in rate of deteriorating items (μ), with optimal cycle time T, Maximum inventory level I_1, Optimal quantity Q, Holding cost.

3) there is a positive relationship between with the increase in setup cost per unit (S_C) , with optimum quantity (Q*): cycle time (T): production time (T_1),production time (T_1), maximum inventory (I_1), holding cost, deteriorative cost, and total cost.

4) there is a positive relationship between with the increase in holding cost per unit per unit time (H_C) with the setup cost, cost of holding inventory and total cost increases but there is negative relationship between increase in holding cost per unit per unit time with optimal cycle time (T) and optimal lot size (Q): production time (T_1) , maximum inventory (I_1),

5) Similarly, other parameters deteriorative cost per unit (D_C), production cost per unit (P_C), can also be observed from the table 2.

3.2. Two Rates of Productions Inventory Model for Deteriorating Items

In this model, we have considered a single commodity deterministic continuous production inventory model with a constant demand rate Y. The production of the item is started initially at t =0 at a production rate X_1 (> Y). Once the inventory level reaches I_1, the rate of production is switched over to X_2 (> X_1) and the production is stopped when the level of inventory reaches $I_2 > (I_1)$ and the inventory is depleted at a constant rate Y. When the inventory level reaches to zero the next production cycle starts at the lower rate X_1. The duration of production at the rate X_1 is [0, T_1]. The duration of production at the rate X_2 is [T_1, T_2]. There is no production but only consumption by demand at a rate Y during the time [T_2, T] . The cycle then repeats itself after time T. The duration of a production cycle T is taken as variable. This model is represented by figure 2. Let I(t) denote the inventory level of the system at time t. The differential equation describing the system in the interval (0,T) are given by

$$\frac{d}{dt} I(t) + \mu I(t) = X_1 - Y , \quad 0 \le t \le T_1 \tag{13}$$

$$\frac{d}{dt} I(t) + \mu I(t) = X_2 - Y , \quad T_1 \le t \le T_2 \tag{14}$$

$$\frac{d}{dt} I(t) + \mu I(t) = -Y , \quad T_2 \le t \le T \tag{15}$$

with the basic conditions of differential equations are

I(0) =0, $I(T_1) = I_1,\ I(T_2) = I_2,$ I(T) =0 (16)

The solutions of the above differential equations are as follows

From the equation (13): $I(t) = \dfrac{X_1 - Y}{\mu}(1 - e^{-\mu t})$ (17)

From the equation (14): $I(t) = \dfrac{X_2 - Y}{\mu}(1 - e^{-\mu t})$ (18)

From the equation (15): $I(t) = \dfrac{Y}{\mu}(e^{\mu(T-t)} - 1)$ (19)

From the equations (16) and (17): $I(T_1) = \dfrac{X_1 - Y}{\mu}(1 - e^{-\mu T_1})$, that is, $I_1 = \dfrac{X_1 - Y}{\mu}(1 - e^{-\mu T_1})$

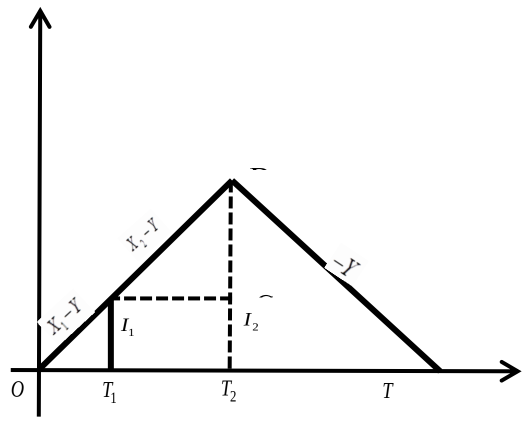

Figure 2 **Production Inventory model with two levels of productions**

On simplification, $I_1 = (X_1 - Y)T_1$ (20)

From the equations (16) and (18): $I_2 = (X_2 - Y)T_2$ (21)

From the triangular inequality OA T_1 and ABC

$$\frac{X_1 - Y}{X_2 - Y} = \frac{T_1 - 0}{T_2 - T_1}, \text{ therefore, } T_1 = \frac{(X_1 - Y)T_2}{X_1 + X_2 - 2Y} \quad (22)$$

Total cost: The total cost comprise of the sum of the production cost, setup cost , holding cost and deteriorative cost. They are grouped together after evaluating the above cost individually.

1. Setup cost = $\dfrac{S_C}{T}$ (23)

2. Production cost = YP_C (24)

3. Holding cost (HC) = $\dfrac{H_C}{T}\left[\int_0^{T_1} I(t)dt + \int_{T_1}^{T_2} I(t)dt + \int_{T_2}^{T} I(t)dt\right] = \dfrac{H_C}{T}\left[\int_0^{T_1} \dfrac{X_1 - Y}{\mu}\left(1 - e^{-\mu t}\right)dt + \int_{T_1}^{T_2} \dfrac{X_2 - Y}{\mu}\left(1 - e^{-\mu t}\right)dt + \int_{T_2}^{T} \dfrac{Y}{\mu}\left(e^{\mu(T-t)} - 1\right)dt\right]$

$$= \frac{H_C}{T}\left[\begin{array}{l} \dfrac{X_1 - Y}{\mu^2}\left(\mu T_1 + e^{-\mu T_1} - 1\right) + \dfrac{X_2 - Y}{\mu^2}\left(\mu T_2 + e^{-\mu T_2} - \mu T_1 - e^{-\mu T_1}\right) \\ - \dfrac{Y}{\mu^2}\left(1 + \mu T - e^{\mu(T-T_2)} - \mu T_2\right) \end{array}\right] = \frac{H_C}{T}\left[\begin{array}{l} \dfrac{X_1 - Y}{\mu^2}\left(\dfrac{\mu^2 T_1^2}{2}\right) + \dfrac{X_2 - Y}{\mu^2}\left(\mu(T_2 - T_1) + e^{-\mu T_2} - e^{-\mu T_1}\right) \\ - \dfrac{Y}{\mu^2}\left(\mu(T - T_2) + 1 - e^{\mu(T-T_2)}\right) \end{array}\right]$$

$$= \frac{H_C}{T}\left[\frac{X_1 - Y}{2}T_1^2 + \frac{X_2 - Y}{2}\left(T_2^2 - T_1^2\right) + \frac{Y}{2}\left(T - T_2\right)^2\right] = \frac{H_C}{2T}\left[(X_1 - Y)T_1^2 + (X_2 - Y)\left(T_2^2 - T_1^2\right) + Y\left(T - T_2\right)^2\right]$$

Substituting the value of T_1 from the equation (22) in the above equation and after some mathematical simplifications

$$HC = \frac{H_C}{2T(X_1+X_2-2Y)^2}\begin{bmatrix}(X_1-Y)^3T_2^2\\ +(X_2-Y)\left((X_1+X_2-2Y)^2-(X_1-Y)^2\right)T_2^2\\ +Y(X_1+X_2-2Y)^2(T-T_2)^2\end{bmatrix} \quad (25)$$

4. Deteriorative cost = $$\frac{\mu D_C}{2T(X_1+P_2-2Y)^2}\begin{bmatrix}(X_1-Y)^3T_2^2\\ +(X_2-Y)\left((X_1+X_2-2Y)^2-(X_1-Y)^2\right)T_2^2\\ +Y(X_1+X_2-2Y)^2(T-T_2)^2\end{bmatrix} \quad (26)$$

Total cost (TC) = Setup cost + Production cost + Holding cost + Deteriorating cost

$$TC(T) = \frac{S_C}{T} + YP_C + \frac{H_C+\mu D_C}{2T(X_1+X_2-2Y)^2}\begin{bmatrix}(X_1-Y)^3T_2^2\\ +(X_2-Y)\left((X_1+X_2-2Y)^2-(X_1-Y)^2\right)T_2^2\\ +Y(X_1+X_2-2Y)^2(T-T_2)^2\end{bmatrix} \quad (27)$$

Optimality conditions

1. $\frac{\partial}{\partial T_2}TC(C)=0$ and $\frac{\partial^2}{\partial T_2^2}TC(C)=0$
2. $\frac{\partial}{\partial T}TC(C)=0$ and $\frac{\partial^2}{\partial T^2}TC(C)=0$

Equation (27) partially differentiate w.r.t. T_2

$$\frac{\partial}{\partial T_2}TC(T) = \begin{bmatrix}(X_1-Y)^3T_2+(X_2-Y)\left[(X_1+X_2-2Y)^2-(X_1-Y)^2\right]T_2\\ +Y(X_1+X_2-2Y)^2(T-T_2)(-)\end{bmatrix}=0$$

On simplification

$$T_2 = \frac{Y(X_1+X_2-2Y)^2T}{(X_1-Y)^3+(X_2-Y)\left[(X_1+X_2-2Y)^2-(X_1-Y)^2\right]+Y(X_1+X_2-2Y)^2} \quad (28)$$

Equation (27) partially differentiate w.r.t. T,

$$\frac{\partial}{\partial T}TC(T) = \left[\frac{-SC_C}{T^2}+\frac{H_C+\mu D_C}{2T^2(X_1+X_2-2Y)}\begin{bmatrix}-(X_1-Y)^3T_2^2\\ -(X_2-Y)\left[(X_1+X_2-2Y)^2-(X_1-Y)^2\right]T_2^2\\ +Y(X_1+X_2-2Y)^2(T^2-T_2^2)\end{bmatrix}\right]=0$$

$$\begin{bmatrix}-(X_1-Y)^3T_2^2\\ -(X_2-Y)\left[(X_1+X_2-2Y)^2-(X_1-Y)^2\right]T_2^2\\ +Y(X_1+X_2-Y)^2(T^2-T_2^2)\end{bmatrix} = \frac{2S_C(X_1+X_2-Y)^2}{H_C+\mu D_C}$$

On some mathematical simplification

$$T^2 = \frac{2S_C\left[(X_1-Y)^3+(X_2-Y)\left\{(X_1+X_2-2Y)^2-(X_1-Y)^2\right\}+Y(X_1+X_2-2Y)^2\right]}{Y(H_C+\mu D_C)\left[(X_1-Y)^3+(X_2-Y)\left\{(X_1+X_2-2Y)^2-(X_1-Y)^2\right\}\right]}$$

Therefore, the optimal cycle time

$$T^* = \sqrt{\frac{2S_C\begin{bmatrix}(X_1-Y)^3+(X_2-Y)\left\{(X_1+X_2-2Y)^2-(X_1-Y)^2\right\}\\ +Y(X_1+X_2-2Y)^2\end{bmatrix}}{Y(H_C+\mu D_C)\left[(X_1-Y)^3+(X_2-YD)\left\{(X_1+X_2-2Y)^2-(X_1-Y)^2\right\}\right]}} \quad (29)$$

For example,

X_1 = 4000, X_2 =5000, Y = 3500, S_C =110, H_C = 11, $\mu = 0.01$, D_C =110, P_C = 110

Solution: Optimum cycle time = 0.1335, Optimal Quantity = 467.52, Production times T_1 = 0.0236 and T_2 = 0.0946, maximum inventory levels I_1 = 11.83 and I_2 = 142.03, setup cost = 823.49, production cost = 385,000, holding cost = 748.62, deteriorative cost = 74.86 and total cost = 386646.98

Note: substituting $P_1 = P_2 = P$ then the above T value from the equation (29) is reduced to the standard production inventory model which is given below T = $\sqrt{\frac{2XS_C}{Y(X-Y)(H_C+\mu D_C)}}$

Table 3. Rate of Deteriorative items with the Inventory costs in two rates of production inventory model

θ	T	Q	T_1	T_2	I_1	I_2	Setup cost	Holding Cost	Deteriorative Cost	Total cost
0.01	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	386646.98
0.02	0.1278	447.61	0.0226	0.0906	11.32	135.98	860.10	716.75	143.35	386720.21
0.03	0.1228	430.05	0.0217	0.0871	10.88	130.65	895.22	688.63	206.59	386790.45
0.04	0.1184	414.41	0.0209	0.0839	10.49	125.89	929.02	663.58	265.42	386858.04
0.05	0.1143	400.36	0.0202	0.0810	10.13	121.62	961.62	641.08	320.54	386923.25
0.06	0.1107	387.64	0.0196	0.0785	9.81	117.76	993.16	620.72	372.43	386986.33
0.07	0.1074	376.07	0.0190	0.0761	9.52	114.25	1023.73	602.19	421.53	387047.46
0.08	0.1044	365.47	0.0185	0.0740	9.25	111.03	1053.41	585.22	468.18	387106.82
0.09	0.1016	355.73	0.0180	0.0720	9.00	108.07	1082.27	569.62	512.65	387164.55
0.10	0.0990	346.72	0.0175	0.0702	8.77	105.33	1110.39	555.19	555.19	387220.78

Production cost = 385,000

From table 3, it is concluded that there is positive relationship between increases in the rate of deteriorative items with cost of setup, cost of deteriorative and total cost. There is a negative relationship between the increases in rate of deteriorative items with optimum cycle time (T): Optimum quantity (Q): production time (T_1, T_2), maximum inventory I_1 and I_2 and cost of holding inventory.

Sensitivity Analysis:

Table 4. Effect of production and demand and cost parameters on optimum values in two rates of production inventory models

Cost Parameters		T	Q	T_1	T_2	I_1	I_2	Setup cost	Holding Cost	Deteriorative Cost	Total cost
X_1	3800	0.1326	464.21	0.0156	0.0934	4.67	140.19	829.35	753.96	95.39	386658.71
	3900	0.1331	465.95	0.0198	0.0941	7.92	141.16	826.26	751.15	75.11	386652.53
	4000	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	386646.98
	4100	0.1339	468.79	0.0272	0.0951	16.31	142.73	821.25	746.59	74.65	386642.51
	4200	0.1342	469.67	0.0304	0.0955	21.27	143.22	819.71	745.19	74.52	386639.42
X_2	4800	0.1410	493.46	0.0289	0.1041	14.46	135.38	780.20	709.27	70.92	386560.40
	4900	0.1370	479.73	0.0261	0.0991	13.05	138.83	802.53	729.57	72.95	386605.07
	5000	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	386646.98
	5100	0.1304	456.06	0.0216	0.0906	10.79	145.02	843.18	766.53	76.65	386686.36
	5200	0.1276	446.78	0.0197	0.0869	9.88	147.82	861.72	783.39	78.34	386723.45
μ	0.01	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	386646.98
	0.02	0.1278	447.61	0.0226	0.0906	11.32	135.98	860.10	716.75	143.35	386720.21
	0.03	0.1228	430.05	0.0217	0.0871	10.88	130.65	895.22	688.63	206.59	386790.45
	0.04	0.1184	414.41	0.0209	0.0839	10.49	125.89	929.02	663.58	265.42	386858.04
	0.05	0.1143	400.36	0.0202	0.0810	10.13	121.62	961.62	641.08	320.54	386923.25
S_C	90	0.1208	422.88	0.0214	0.0856	10.70	128.47	44.87	677.15	67.71	386489.74
	100	0.1273	445.76	0.0225	0.0902	11.28	135.42	785.16	713.78	71.37	386570.33
	110	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	386646.98
	120	0.1395	488.31	0.0247	0.0988	12.36	148.34	860.10	781.91	78.19	386720.21
	130	0.1452	508.25	0.0257	0.1029	12.86	154.40	895.22	813.84	81.38	386790.45
H_C	9	0.1462	511.72	0.0259	0.1036	12.95	155.45	752.36	670.42	81.94	386504.72
	10	0.1394	488.12	0.0247	0.0988	12.35	148.29	788.72	710.56	78.16	386577.45
	11	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	386646.98
	12	0.1283	449.32	0.0227	0.0910	11.37	136.50	856.84	784.89	71.94	386713.68
	13	0.1237	433.09	0.0219	0.0877	10.96	131.57	888.94	819.59	69.35	386777.89
D_C	90	0.1346	471.43	0.0238	0.0954	11.93	143.22	816.65	754.89	61.76	386633.31
	100	0.1341	469.46	0.0237	0.0950	11.88	142.62	820.08	751.74	68.34	386640.16
	110	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	386646.98
	120	0.1330	465.60	0.0235	0.0942	11.78	141.44	826.88	745.55	81.33	386653.77

	130	0.1324	463.70	0.0234	0.0939	11.73	140.87	830.26	742.51	87.75	386660.53
P_C	90	0.1346	471.43	0.0238	0.0954	11.93	143.22	816.65	754.89	61.76	315000.00 316646.98
	100	0.1341	469.46	0.0237	0.0950	11.88	142.62	820.08	751.74	68.34	350000.00 351646.98
	110	0.1335	467.52	0.0236	0.0946	11.83	142.03	823.49	748.62	74.86	385000.00 386646.98
	120	0.1330	465.60	0.0235	0.0942	11.78	141.44	826.88	745.55	81.33	350000.00 351646.98
	130	0.1324	463.70	0.0234	0.0939	11.73	140.87	830.26	742.51	87.75	420000.00 421646.98

Production cost = 385,000

Managerial insights: A sensitivity analysis is performed to study the effects of change in the system parameters, ordering cost (S_C), holding cost (H_C), deteriorative cost (D_C), total cost on optimal values that is optimal cycle time (T): optimal quantity (Q): production time (T_1) and (T_2), maximum inventory (I_1) and (I_2). The sensitivity analysis is performed by changing (increasing or decreasing) the parameter taking at a time, keeping the remaining parameters at their original values. The following influences can be obtained from sensitivity analysis based on table 4.

1) there is a positive relationship between increase in the rate of production (X_1) with optimum cycle time (T): optimum quantity (Q): production times (T_1, T_2), maximum inventory (I_1, I_2). There is a negative relationship between increases in the rate of production (X_1) with setup cost, holding cost, deteriorative cost and total cost.

2) there is a positive relationship between increase in the rate of production (X_2) withcost of setup, cost of holding inventory, cost of deteriorative items, total cost and maximum inventory (I_2). There is a negative relationship between increases in the rate of production (X_2) with cycle time (T): optimum quantity (Q): production time (T_1) and (T_2) and maximum inventory level (I_1).

3) there is a positive relationship between increases in rate of deteriorative items (μ) with cost of cost of setup, cost of deteriorating items, and total cost. There is a negative relationship between increases in rate of deteriorative items (μ) with optimal cycle time T, maximum inventory level I_1 and I_2 production time (T_1) and (T_2) optimal quantity Q, cost of holding inventory.

4) there is a positive relationship between increase in setup cost per unit (H_C) with optimum quantity (Q*): cycle time (T): production time (T_1) and (T_2) maximum inventory I_1 and I_2, cost of setup, cost of holding inventory, cost of deteriorative items and total cost.

5) there is positive relationship between with the increase in cost of holding per unit per unit time (H_C) with cost of setup, cost of holding inventory and total cost. There is a negative relationship between increases in cost of holding per unit per unit time with optimal cycle time (T) and optimal lot size (Q): production time (T_1) and (T_2), maximum inventory I_1 and I_2, deteriorative cost,

6) Similarly, other parameters deteriorative cost per unit (D_C), production cost per unit (P_C), can also be observed from the table 4.

3.3. Three Rates of Productions Inventory Models for Deteriorating Items

In this model, we have considered a single commodity deterministic continuous production inventory model with a constant demand rate Y. The production of the item is started initially at t =0 at a production rate X_1 (>Y). Once the inventory level is reaches Q_1, the rate of production is switched over to X_2 ($> X_1$) and the inventory level is reaches to $I_2 (> I_1)$, the rate of production is switched over to X_3 ($> X_2$) and the production is stopped when the level of inventory reaches $I_3(> I_2)$ and the inventory is depleted at a constant rate D. When the inventory level reaches to zero the next production cycle starts at the lower rate X_1. The duration of production at the rate X_1 is [$0, T_1$], the duration of production at the rate X_2 is [T_1, T_2] and P_3 is [T_2, T_3]There is no production but only consumption by demand at a rate D during the time [T_3, T] . The cycle then repeats itself after time T. The duration of a production cycle T is taken as variable. This model is represented by figure 3. Let I(t) denote the inventory level of the system at time t. The differential equation describing the system in the interval (0,T) are given by

$$\frac{d}{dt}I(t) + \mu I(t) = X_1 - Y, \quad 0 \le t \le T_1 \tag{30}$$

$$\frac{d}{dt}I(t) + \mu I(t) = X_2 - Y, \quad T_1 \le t \le T_2 \tag{31}$$

$$\frac{d}{dt}I(t) + \mu I(t) = X_3 - Y, T_2 \le t \le T_3 \tag{32}$$

$$\frac{d}{dt}I(t) + \mu I(t) = -Y, \quad T_3 \le t \le T \tag{33}$$

with the basic condition of differential equations

I(0) =0, $I(T_1) = I_1$, $I(T_2) = I_2$, $I(T_3) = I_3$ I(T) =0 (34)

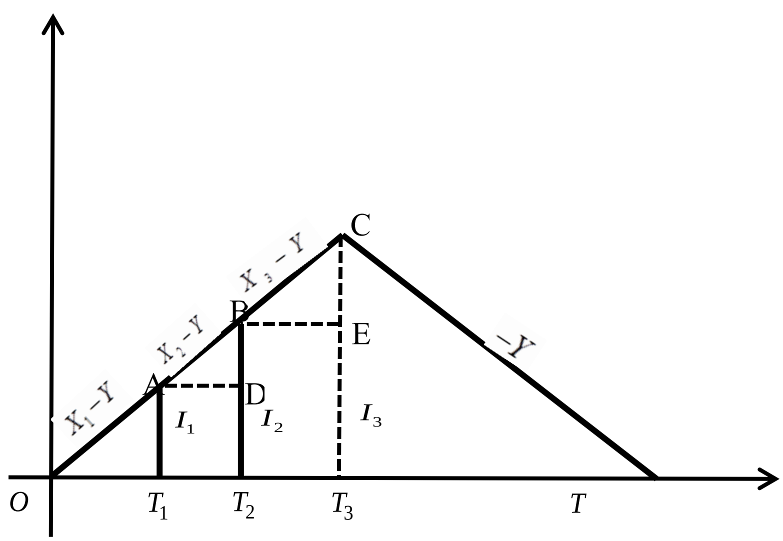

Figure 3 On hand inventory of three levels of production with deteriorative items

The solutions of the above differential equations are as follows

From the equation (30): $I(t) = \frac{X_1 - Y}{\mu}(1 - e^{-\mu t})$ (35)

From the equation (31): $I(t) = \frac{X_2 - Y}{\mu}(1 - e^{-\mu t})$ (36)

From the equation (32): $I(t) = \frac{X_3 - Y}{\mu}(1 - e^{-\mu t})$ (37)

From the equation (33): $I(t) = \frac{Y}{\mu}(e^{\mu(T-t)} - 1)$ (38)

From the equations (34) and (35): $I(T_1) = \frac{X_1 - Y}{\mu}(1 - e^{-\mu T_1})$, that is, $I_1 = \frac{X_1 - Y}{\mu}(1 - e^{-\mu T_1})$

On simplification, $I_1 = (X_1 - Y)T_1$ (39)

From the equations (34) and (37): $I_2 = (X_2 - Y)T_2$ (40)

From the equations (34) and (37): $I_3 = (X_3 - Y)T_2$ (41)

From the triangular inequality OA T_1 and ABC $\frac{X_1 - Y}{X_2 - Y} = \frac{T_1 - 0}{T_2 - T_1}$, therefore, $T_1 = \frac{(X_1 - Y)T_2}{X_1 + X_2 - 2Y}$ (42)

From the triangular inequality OA T_1 and BDE $\frac{X_1 - Y}{X_3 - Y} = \frac{T_1 - 0}{T_3 - T_2}$, therefore, $T_2 = \frac{(X_1 + X_2 - 2Y)T_3}{X_1 + X_2 + X_3 - 3Y}$ (43)

Therefore, $T_1 = \frac{(X_1 - Y)T_3}{X_1 + X_2 + X_3 - 3Y}$

Total cost: Total cost comprise of the sum of the production cost, setup cost, holding cost, deteriorative cost. They are grouped together after evaluating the above cost individually.

1. Setup cost = $\frac{S_C}{T}$ (44)

2. Production cost = YP_C (45)

3. Holding cost (HC) = $\frac{H_C}{T}\left[\int_0^{T_1} I(t)dt + \int_{T_1}^{T_2} I(t)dt + \int_{T_2}^{T_3} I(t)dt + \int_{T_3}^{T} I(t)dt\right] = \frac{H_C}{T}\left[\int_0^{T_1} \frac{X_1 - Y}{\mu}\left(1 - e^{-\mu t}\right)dt + \int_{T_1}^{T_2} \frac{X_2 - Y}{\mu}\left(1 - e^{-\mu t}\right)dt + \int_{T_2}^{T_3} \frac{X_3 - Y}{\mu}\left(1 - e^{-\mu t}\right)dt + \int_{T_2}^{T} \frac{Y}{\mu}\left(e^{\mu(T-t)} - 1\right)dt\right]$

$$= \frac{H_C}{T}\left[\begin{array}{l} \frac{X_1 - Y}{\mu^2}\left(\mu T_1 + e^{-\mu T_1} - 1\right) + \frac{X_2 - Y}{\mu^2}\left(\mu T_2 + e^{-\mu T_2} - \mu T_1 - e^{-\mu T_1}\right) \\ + \frac{X_3 - Y}{\mu^2}\left(\mu T_3 + e^{-\mu T_3} - \mu T_2 - e^{-\mu T_2}\right) - \frac{Y}{\mu^2}\left(1 + \mu T - e^{\mu(T-T_2)} - \mu T_3\right) \end{array}\right] = \frac{H_C}{T}\left[\frac{X_1 - Y}{\mu^2}\left(\frac{\mu^2 T_1^2}{2}\right) + \frac{X_2 - Y}{2\mu^2}\mu^2\left(T_2^2 - T_1^2\right) + \frac{X_3 - Y}{2\mu^2}\left(T_3^2 - T_2^2\right) + \frac{Y}{\mu^2}\left(\mu^2(T - T_3)^2\right)\right]$$

$$= \frac{H_C}{2T}\left[(X_1 - Y)T_1^2 + (X_2 - Y)\left(T_2^2 - T_1^2\right) + \left(X_3 - Y\right)\left(T_3^2 - T_2^2\right) + Y\left(T - T_2\right)^2\right]$$

Substituting the value of T_1 and T_2 from the equations (42, 43) in the above equation and after some mathematical simplifications

$$HC = \frac{H_C}{2T(X_1 + X_2 + X_3 - 3Y)^2}\left[\begin{array}{l} (X_1 - Y)^3 T_3^2 \\ + (X_2 - Y)\left((X_1 + X_2 - 2Y)^2 - (X_1 - Y)^2\right)T_3^2 \\ + (X_3 - Y)\left((X_1 + X_2 + X_3 - 3Y)^2 - (X_1 + X_2 - 2Y)^2\right)T_3^2 \\ + Y(X_1 + X_2 + X_3 - 3Y)^2\left(T - T_3\right)^2 \end{array}\right] = \frac{H_C}{2T(X_1 + X_2 + X_3 - 3Y)^2}\left[\begin{array}{l} \left\{\begin{array}{l} (X_1 - Y)^3 + (X_2 - Y)\left((X_1 + X_2 - 2Y)^2 - (X_1 - Y)^2\right) \\ + (X_3 - Y)\left((X_1 + X_2 + X_3 - 3Y)^2 - (X_1 + X_2 - 2Y)^2\right) \end{array}\right\}T_3^2 \\ + Y(X_1 + X_2 + X_3 - 3Y)^2\left(T - T_3\right)^2 \end{array}\right] \quad (46)$$

4. Deteriorative cost = $\frac{\mu D_C}{2T(X_1 + X_2 + X_3 - 3Y)^2}\left[\begin{array}{l} \left\{\begin{array}{l} (X_1 - Y)^3 + (X_2 - Y)\left((X_1 + X_2 - 2Y)^2 - (X_1 - Y)^2\right) \\ + (X_3 - Y)\left((X_1 + X_2 + X_3 - 3Y)^2 - (X_1 + X_2 - 2Y)^2\right) \end{array}\right\}T_3^2 \\ + Y(X_1 + X_2 + X_3 - 3Y)^2\left(T - T_3\right)^2 \end{array}\right]$ (47)

Total cost (TC) = Setup cost + Production cost + Holding cost + Deteriorating cost

TC(T) = $\frac{S_C}{T} + YP_C + \frac{H_C + \theta D_C}{2T(X_1 + X_2 + X_3 - 3Y)^2}\left[\begin{array}{l} \left\{\begin{array}{l} (X_1 - Y)^3 + (X_2 - Y)\left((X_1 + X_2 - 2Y)^2 - (X_1 - Y)^2\right) \\ + (X_3 - Y)\left((X_1 + X_2 + X_3 - 3Y)^2 - (X_1 + X_2 - 2Y)^2\right) \end{array}\right\}T_3^2 \\ + Y(X_1 + X_2 + X_3 - 3Y)^2\left(T - T_3\right)^2 \end{array}\right]$ (48)

Optimality conditions

a) $\frac{\partial}{\partial T_3} TC(C) = 0$ and $\frac{\partial^2}{\partial T_3^2} TC(C) = 0$

b) $\frac{\partial}{\partial T}TC(C)=0$ and $\frac{\partial^2}{\partial T^2}TC(C)=0$

The equation (48) partially differentiate w.r.t. T_3

$$\frac{\partial}{\partial T_3}TC(T)=\begin{bmatrix} 2\begin{Bmatrix}(X_1-Y)^3+(X_2-Y)\left((X_1+X_2-2Y)^2-(X_1-Y)^2\right) \\ +(X_3-Y)\left((X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)^2\right)\end{Bmatrix}T_3 \\ +2Y(X_1+X_2+X_3-3Y)^2(T-T_3)(-1)\end{bmatrix}=0$$

On simplification

$$T_3=\frac{Y(X_1+X_2+X_3-3Y)^2T}{\begin{bmatrix}(X_1-Y)^3+(X_2-Y)\left[(X_1+X_2-2Y)^2-(X_1-Y)^2\right] \\ +(X_3-Y)\left((X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)\right)^2+Y(X_1+X_2+X_3-3Y)^2\end{bmatrix}} \tag{49}$$

The equation (48) partially differential w.r.t. T,

$$\frac{\partial}{\partial T}TC(T)=\begin{bmatrix}\frac{-S_C}{T^2} \\ +\frac{H_C+\mu D_C}{2T^2(X_1+X_2+X_3-3Y)}\begin{bmatrix}\begin{pmatrix}-(X_1-Y)^3-(X_2-Y)\left[(X_1+X_2-2Y)^2-(X_1-Y)^2\right] \\ -(X_3-Y)\left((X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)^2\right)\end{pmatrix}T_3^2 \\ +Y(X_1+X_2-2Y)^2(T^2-T_3^2)\end{bmatrix}\end{bmatrix}=0$$

$$\begin{bmatrix}\begin{pmatrix}-(X_1-Y)^3-(X_2-Y)\left[(X_1+X_2-2Y)^2-(X_1-Y)^2\right] \\ -(X_3-Y)\left((X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)^2\right)\end{pmatrix}T_3^2 \\ +Y(X_1+X_2+X_3-3Y)^2(T^2-T_3^2)\end{bmatrix}=\frac{2C_0(X_1+X_2+X_3-3Y)^2}{H_C+\mu D_C}$$

On some mathematical simplification

$$T^2=\frac{2H_C\begin{bmatrix}(X_1-Y)^3+(X_2-Y)\left\{(X_1+X_2-2Y)^2-(X_1-Y)^2\right\} \\ +(X_3-Y)\left(X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)^2\right)+Y(X_1+X_2-2Y)^2\end{bmatrix}}{Y(H_C+\mu D_C)\begin{bmatrix}(X_1-Y)^3+(X_2-Y)\left((X_1+X_2-2Y)^2-(X_1-Y)^2\right) \\ +(X_3-Y)\left((X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)^2\right)\end{bmatrix}}$$

Therefore, the optimal cycle time

$$T^*=\sqrt{\frac{2H_C\begin{bmatrix}(X_1-Y)^3+(X_2-Y)\left\{(X_1+X_2-2Y)^2-(X_1-Y)^2\right\} \\ +(X_3-Y)\left(X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)^2\right)+Y(X_1+X_2-2Y)^2\end{bmatrix}}{Y(H_C+\mu D_C)\begin{bmatrix}(X_1-Y)^3+(X_2-Y)\left((X_1+X_2-2Y)^2-(X_1-Y)^2\right) \\ +(X_3-Y)\left((X_1+X_2+X_3-3Y)^2-(X_1+X_2-2Y)^2\right)\end{bmatrix}}} \tag{50}$$

For example, X_1 = 4000, X_2 =5000, X_3= 6000, Y = 3500, S_C =110, H_C = 11, μ = 0.01, D_C =110, P_C = 110

Solution: Optimum cycle time = 0.1146, Optimal Quantity = 401.11, Production times T_1 = 0.0076, T_2 = 0.0307 and T_3 = 0. 0692, maximum inventory levels I_1 = 3.84 and I_2 = 46.18, I_3 = 173.18, setup cost = 957.82, production cost = 385,000, holding cost = 872.51, deteriorative cost = 87.24 and total cost = 386646.98

Note: substituting $X_1=X_2=X_3=X$ then the above T value from the equation (50) is reduced to the standard production inventory model which is given below

$$T=\sqrt{\frac{2XS_C}{Y(X-Y)(H_C+\mu D_C)}}$$

Table 5. Rate of Deteriorative items with the Inventory costs in three rates of production inventory model

Rate of Deteriorative	T	Q	I_1	I_2	I_3	T_1	T_2	T_3	Setup cost	Holding cost	Deteriorative cost	Total cost
0.01	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
0.02	0.1097	384.03	3.68	44.21	165.81	0.0073	0.0294	0.0663	1002.50	835.42	167.08	387005.01
0.03	0.1054	368.97	3.54	42.48	159.31	0.0070	0.0283	0.0637	1043.44	802.64	240.79	387086.88
0.04	0.1015	355.54	3.41	40.93	153.51	0.0068	0.0272	0.0614	1082.83	773.45	309.38	387165.66
0.05	0.0981	343.49	3.29	39.54	148.30	0.0066	0.0263	0.0593	1120.83	747.22	373.61	387241.67
0.06	0.0950	332.58	3.19	38.29	143.60	0.0063	0.0255	0.0574	1157.59	723.49	434.09	387315.19
0.07	0.0921	322.65	3.09	37.15	139.31	0.0061	0.0247	0.0557	1193.22	701.89	491.32	387386.44
0.08	0.0895	313.56	3.01	36.10	135.38	0.0060	0.0240	0.0541	1227.87	682.12	545.69	387455.63
0.09	0.0872	305.20	2.92	35.14	131.77	0.0058	0.0234	0.0527	1261.46	663.92	597.53	387522.92
0.10	0.0849	297.47	2.85	34.25	128.44	0.0057	0.0228	0.0513	1294.23	647.11	647.11	387588.46

Production cost = 385,000

From table 5, it is observed that there is a positive relationship between increases in the rate of deteriorative items with cost of setup, cost of deteriorative items and total cost. There is a negative relationship between increases in the rate of deteriorative items with cycle time (T): optimum quantity (Q): maximum inventory (I_1): (I_2) and (I_3): production times (T_1): (T_2) and (T_3) and cost of holding inventory.

Sensitivity Analysis:

Table 6. Effect of Demand and cost parameters on optimal values in three rate of Production inventory model

Parameters		Optimum Values											
		T	Q	I_1	I_2	I_3	T_1	T_2	T_3	Setup cost	Holding cost	Deteriorative cost	Total cost
X_1	3800	0.1141	399.60	1.43	43.12	171.68	0.0047	0.0287	0.0688	963.45	875.86	87.58	386926.90
	3900	0.1143	400.35	2.51	44.67	172.43	0.0062	0.0297	0.0689	961.63	874.21	87.42	386923.27
	4000	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
	4100	0.1148	401.05	5.44	47.63	173.92	0.0091	0.0317	0.0695	958.06	870.97	87.09	386916.13
	4200	0.1150	402.55	7.28	49.04	174.62	0.0010	0.0326	0.0698	956.38	869.44	86.94	386912.77
X_2	4800	0.1147	401.71	4.04	37.82	173.78	0.0081	0.0291	0.0695	958.40	871.27	87.12	386916.81
	4900	0.1147	401.47	3.94	41.96	173.55	0.0079	0.0299	0.0694	958.95	871.78	87.17	386917.91
	5000	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
	5100	0.1144	400.62	3.75	50.45	172.69	0.0075	0.0315	0.0691	961.00	873.64	87.63	386922.01
	5200	0.1142	400.01	3.66	54.77	172.08	0.0073	0.0322	0.0688	962.47	874.97	87.49	386924.95
X_3	5800	0.1174	411.12	4.25	51.09	168.48	0.0085	0.0340	0.0732	936.45	851.32	85.13	386872.91
	5900	0.1160	405.98	4.04	48.55	170.91	0.0081	0.0323	0.0712	948.31	862.11	86.21	386896.62
	6000	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
	6100	0.1132	396.49	3.66	43.97	175.31	0.0073	0.0293	0.0674	971.00	882.73	88.27	386942.01
	6200	0.1120	392.12	3.49	41.91	177.29	0.0070	0.0279	0.0656	981.84	892.58	89.25	386963.68

μ	0.01	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
	0.02	0.1097	384.03	3.68	44.21	165.81	0.0073	0.0294	0.0663	1002.50	835.42	167.08	387005.01
	0.03	0.1054	368.97	3.54	42.48	159.31	0.0070	0.0283	0.0637	1043.44	802.64	240.79	387086.88
	0.04	0.1015	355.54	3.41	40.93	153.51	0.0068	0.0272	0.0614	1082.83	773.45	309.38	387165.66
	0.05	0.0981	343.49	3.29	39.54	148.30	0.0066	0.0263	0.0593	1120.83	747.22	373.61	387241.67
S_C	90	0.1036	362.82	3.48	41.77	156.65	0.0069	0.0278	0.0626	868.19	789.27	78.92	386736.39
	100	0.1092	382.44	3.66	44.03	165.12	0.0073	0.0293	0.0660	915.16	831.96	83.19	386830.32
	110	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
	120	0.1196	418.94	4.02	48.23	180.88	0.0080	0.0321	0.0723	1002.50	911.37	91.13	387005.01
	130	0.1245	436.05	4.18	50.20	188.27	0.0083	0.0334	0.0753	1043.44	948.58	94.85	387086.88
H_C	9	0.1254	439.03	4.21	50.54	189.56	0.0084	0.0336	0.0758	876.92	781.41	95.50	386753.48
	10	0.1196	418.79	4.02	48.22	180.82	0.0080	0.0321	0.0723	919.31	828.20	91.10	386838.62
	11	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
	12	0.1101	385.49	3.69	44.38	166.44	0.0073	0.0295	0.0665	998.70	914.84	83.86	386997.40
	13	0.1061	371.57	3.56	42.78	160.43	0.0071	0.0285	0.0641	1036.12	955.28	80.83	387072.24
D_C	90	0.1155	404.47	3.88	46.57	174.63	0.0077	0.0310	0.0698	951.86	879.87	71.98	386903.72
	100	0.1150	402.78	3.86	46.37	173.91	0.0077	0.0309	0.0695	955.85	876.19	79.65	386911.71
	110	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	386919.68
	120	0.1141	399.46	3.83	45.99	172.47	0.0076	0.0306	0.0689	963.78	868.98	94.79	386927.57
	130	0.1136	397.83	3.81	45.80	171.77	0.0076	0.0305	0.0687	967.72	865.45	102.28	386935.45
P_C	90	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	315000.00 316919.65
	100	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	350000.00 351919.65
	110	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	385000.00 386919.65
	120	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	420000.00 421919.65
	130	0.1146	401.11	3.84	46.18	173.18	0.0076	0.0307	0.0692	957.82	872.51	87.24	455000.00 456919.65

Production cost = 385,000

Managerial insights: A sensitivity analysis is performed to study the effects of change in the system parameters, ordering cost (S_C), holding cost (H_C) on optimal values that is optimal cycle time (T): optimal quantity (Q): production time (T_1), (T_2) and (T_3): maximum inventory (I_1),(I_2) and (I_3): setup cost, holding cost, deteriorative cost and total cost. The sensitivity analysis is performed by changing (increasing or decreasing) the parameter taking at a time, keeping the remaining parameters at their original values. The following influences can be obtained from sensitivity analysis based on table 6.

1) there is a positive relationship between increase in one rate of production P_1 with optimal cycle time (T): optimum quantity (Q): production times (T_1), (T_2) and(T_3): maximum inventories (I_1),(I_2) and (I_3) . There is a negative relationship between increases in X_1, with cost of holding inventory, cost of deteriorative items, total cost and production rate X_2 .

2) there is positive relationship between with the increase in two rate of production X_2 with production time (T_2): maximum inventory (I_2): setup cost, holding cost, deteriorative cost . There is negative relationship between increases in two rates of production P_2 with optimal cycle time (T): optimum quantity (Q): Production times $(T_1)$and (T_3): maximum inventories (I_1) and (I_3) production time (X_3).

3) there is positive relationship between increase in three rate of production X_3 with maximum inventories(I_3): cost of holding inventory, cost of setup, cost of deteriorative items and total cost. There is a negative relationship between with the increase in three rate of production P_3 with optimal cycle time (T): optimum quantity (Q): maximum inventory (I_1) and (I_2): production time(T_1), (T_2) and(T_3).

4) there is a positive relationship between with the increase in rate of deteriorating item (μ) with setup cost, cost of deteriorating and total cost. There is a negative relationship between increases in rate of deteriorating items (μ) with optimal cycle time T, optimum quantity (Q): maximum inventory level (I_1),(I_2) and (I_3): production time (T_1), (T_2) and(T_3) .

5)there is a positive relationship between with the increase in setup cost per unit (C_0) with optimum quantity (Q*): cycle time (T): production time (T_1, T_2, T_3), maximum inventory(I_1, I_2, I_3), setup cost, holding cost, deteriorative cost and total cost.

6) there is a positive relationship with the increase in cost of holding in inventory per unit per unit time (H_C) with the setup cost, cost of holding inventory and total cost. There is a negative relationship between the increases of cost of holding inventory per unit per unit time$(H_C)$with optimal cycle time (T) and optimal lot size (Q): production time (T_1) , (T_2) and(T_3): maximum inventory(I_1) ,(I_2) and (I_3): deteriorative cost.

7) Similarly, other parameters deteriorative cost per unit (D_C), production cost per unit (P_C), can also be observed from the table 6.

4. COMPARATIVE STUDY

The details of cost of holding inventory in different rates of production are given in the following table:

Holding cost	Production rate X_3	Production rate X_2	Production rate X_1	Consumption period
One rate of production	428.97	518.62	461.47	61.42
Two rates of production	11.47	63.69	-	218.19
Three rates of production	1.38	-	-	346.22

From the above table, it is observed that in model three, three rates of production inventory model, the holding cost during one rate, two rates and three rates are 428.97, 11.47 and 1.38 respectively. It is reduced in each rate of production. In model two, one rate of production and two rates of productions, the holding cost are 518.62 and 63.69. It is also gradually reduced. In consumption period, the holding cost are gradually increased. It is benefited to the concern and so as to reduce the cost of production and the concern can earn maximum profit and initial investment low in one rate of production. Initially, heavy investment can be avoided.

5. CONCLUSIONS

In production inventory system, there are situations, in which it is not possible to have single rate of production throughout the production period. Items are produced at different rates during sub periods so as to meet various constraints that arise due to change in demand pattern, market fluctuations, etc.,In this paper, we have dealt with a continuous production inventory models for deteriorating items in which multi rates (one, two and three) of production are available and it is possible that production started at one rate and after some time it may be switched over to another rate. Such a situation is desirable in the sense that by starting at a low rate of production, a large quantum stock of manufacturing items at the initial stage is avoided, leading to reduction in the holding cost. Three models are considered; one rate of production inventory model is studied first, in second model two rates of production inventory model and three rates of production inventory models is investigated in finally. A suitable mathematical model is developed and the optimal production lot size which minimizes the total cost is derived. The global optimal solution is derived and an illustrative example is provided and numerically verified. The validation of result in this model was coded in Microsoft Visual Basic 6.0
The proposed model can assist the manufacturer and retailer in accurately determining the optimal quantity, cycle time and annual total cost. Moreover, the proposed inventory model can be used in inventory control of certain items such as food items, fashionable commodities, stationary stores and others. For further research, this model can be extended in several ways. For instance, time value of money, price discounts, quantity discounts and rework of defective items. However, success depends on the correctness of the estimation of the input parameters. However, in reality management is most likely to be uncertain of the true values of these parameters. Moreover, their values may be changed over time due to their complex structures. Therefore, it is more reasonable to assume that these parameters are known only within some given ranges.

Working notes

In One rate Production Inventory model

1. Holding cost for production period= $\frac{H_C}{2}\int_0^{T_1}\frac{X_1-Y}{\mu}\left(1-e^{-\mu t}\right)dt = \frac{(X_1-Y)H_C T_1^2}{2T}$ = 428.97

2. Holding cost for consumption period= $\frac{H_C}{2}\int_{T_1}^{T}\frac{Y}{\mu}\left(e^{\mu(T-t)}-1\right)dt = \frac{YH_C}{2T}(T-T_1)^2$ = 61.42

In Tworates Production Inventory Model

1. Holding cost for production period X_1= $\frac{H_C}{2}\int_0^{T_1}\frac{X_1-Y}{\mu}\left(1-e^{-\mu t}\right)dt = \frac{(X_1-Y)H_C T_1^2}{2T}$ = **11.47**

2. Holding cost for production period X_2= $\frac{H_C}{2}\int_{T_1}^{T_2}\frac{X_2-Y}{\mu}\left(1-e^{-\mu t}\right)dt = \frac{(X_2-Y)H_C (T_2^2-T_1^2)}{2T}$ = **518.62**

3. Holding cost for consumption period= $\frac{H_C}{2}\int_{T_2}^{T}\frac{Y}{\mu}\left(e^{\mu(T-t)}-1\right)dt = \frac{YD_C}{2T}(T-T_2)^2$ = **218.19**

In Three rates Production Inventory Model

1. Holding cost for production period X_1= $\frac{H_C}{2}\int_0^{T_1}\frac{X_1-Y}{\mu}\left(1-e^{-\mu t}\right)dt = \frac{(X_1-Y)H_C T_1^2}{2T}$ = **1.38**

2. Holding cost for production period X_2= $\frac{H_C}{2}\int_{T_1}^{T_2}\frac{X_2-Y}{\mu}\left(1-e^{-\mu t}\right)dt = \frac{(X_2-Y)H_C (T_2^2-T_1^2)}{2T}$ = **63.69**

3. Holding cost for production period X_3= $\frac{H_C}{2}\int_{T_2}^{T_3}\frac{X_3-Y}{\mu}\left(1-e^{-\mu t}\right)dt = \frac{(X_3-Y)H_C (T_3^2-T_2^2)}{2T}$ = **461.47**

4. Holding cost for consumption period= $\frac{H_C}{2}\int_{T_3}^{T}\frac{Y}{\mu}\left(e^{\mu(T-t)}-1\right)dt = \frac{YH_C}{2T}(T-T_3)^2$ = **346.22**

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