# AN EOQ INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING PRODUCTS WITH ADVERTISEMENT AND PRICE SENSITIVE DEMAND UNDER ORDER QUANTITY DEPENDENT TRADE CREDIT

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**ABSTRACT**
This research work discusses an inventory model in which a supplier provides to the retailer different credit periods associated with the order quantity. In the inventory model, it is considered that the product's deterioration rate is non-instantaneous in nature. Additionally, an advertisement and price sensitive demand is modelled; this kind of demand is appropriate for the products for which the demand is influenced by the advertising and price. The main aim of this research works is to determine the optimal ordering policy which maximizes the retailer's profit. In order to illustrate the proposed inventory model, some numerical examples are solved and a sensitivity analysis is presented.

**KEYWORDS:** EOQ, non-instantaneous deterioration, advertisement and price sensitive demand, order quantity dependent trade credit.

**MSC**: 90B05

**ABSTRACT**
Esta investigación discute un modelo de inventarios en el cual el proveedor ofrece al comprador diferentes periodos de crédito asociados con la cantidad de la orden. En el modelo de inventario es considerado que la naturaleza de la tasa de deterioro del producto es no instantánea. Adicionalmente, se modela que la demanda es sensible a la propaganda y al precio; este tipo de demanda es apropiada para productos para los cuales la demanda es influenciada por la propaganda y el precio. El objetivo principal de esta investigación es determinar la política óptima de ordenar que maximiza la utilidad

**PALABRAS CLAVE:** EOQ, deterioro no-instantáneo, demanda sensible a propaganda y precio, crédito de la transacción dependiente del tamaño de la orden

## 1. INTRODUCTION

The economic order quantity (EOQ) inventory model is based on the supposition that a purchaser pays in a cash on manner when he or she receives the lot size. On the other hand, if the purchaser has the opportunity of paying more later without interest charges then the purchaser is encouraged to procure products. For the duration of the permissible delay in payment period, the purchaser obtains interest on the vended products. It is important to mention that the delay in payment permits to clients manage more effectively the short term cash flow. It is relevant to remark that the trade credit usually generates conditions in which sometimes the organizations are incapable to cover the debt responsibilities. Possibly, Goyal (1985) was the first academic to study the effect of trade credit on optimal inventory policies. Based on the work of Goyal (1985) several researchers developed inventory models under trade credit. For example, Jaggi et al. (2008) determined the retailer's optimal policy when the credit is linked to demand with permissible delay in payments. Soni et al. (2010) presented an excellent review paper related to inventory models that include trade credit policy. Lou and Wang (2013) studied the seller's decisions when there exists a delay payment period.
The majority of the mentioned research works related to inventory models in conditions of permissible delay in payments consider the situation in which the delay in payment does not dependent of the quantity

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purchased. With the purpose of motivating the retailer to purchase large quantities, the supplier gives a permissible delay of payment for large quantities but obliges an immediate payment for small quantities. As a result, the supplier states a predefined order quantity below which a delay in payment is not permitted and the payment must be done immediately. The trade credit period is allowed only for order quantities above this predetermined order quantity. Khouja and Mehrez (1996) investigated the influence of supplier credit strategies on the optimal lot size. Two kinds of supplier credit strategies are given in Khouja and Mehrez (1996). The first kind considers that the credit terms do not dependent of the quantity bought. The second kind takes into account that the credit terms are associated to the order size. Shinn and Hwang (2003) derived an inventory model which determines jointly the selling price and order quantity when the order size is dependent on the delay in payments. Chang et al. (2003) formulated an EOQ inventory model for a deteriorating product when supplier associates the trade credit to the order quantity. Chung and Liao (2004) determined the optimal replenishment cycle time for an exponentially deteriorating item for the case when the delay in payments depends on the quantity purchased. In the same line of research of Chung and Liao (2004), there are the following interesting research works Chung et al. (2005), Liao (2007), Ouyang et al. (2008), Ouyang et al. (2009), Chang et al. (2009), and Yang et al. (2010). Other recent research investigations related to deteriorating products are the works of Tripathy et al. (2010), Mishra and Tripathy (2015), Mishra (2017), Mishra et al. (2018, and Vandana and Sana (2019). Tripathy et al. (2010) developed an economic production quantity (EPQ) inventory model for linear deteriorating product when the holding cost is variable. Mishra and Tripathy (2015) built an inventory model for products that deteriorate according to a Weibull function considering that there is salvage value. Mishra (2017) presented an inventory model for Weibull deterioration when the demand is influence by both the price and the stock. Mishra et al. (2018) determined the optimal ordering, pricing, and preservation technology investment policies for a deteriorating product when there is permissible delay in payments. Recently, Vandana and Sana (2019) formulated a two-echelon inventory model for ameliorating/deteriorating products for a supply chain comprised of a single vendor and multi-buyers.

Chang et al. (2014) introduced an inventory model for non-instantaneous deteriorating products in an environment where the supplier offers to the retailer a trade credit associated to the order quantity.

It is well-known that the products loss their properties thru time, this happens due to a phenomenon named as deterioration. Ghare and Schrader (1963) introduced the effect of deterioration into an inventory model. There exists several review papers related to inventory models with deteriorating products, see for instance, Raafat (1991), Shah and Shah (2000), Goyal and Giri (2001) and Bakker et al. (2012). The researchers and academicians have considered two kinds of deterioration: constant rate and Weibull rate. All the above mentioned models suppose that the deterioration of products begins in the instant in which these enter into warehouse. In many situations of real life there exists a time length in which most suppliers preserve the quality or original conditions of the products, this means that during a short period of time no deterioration happens. After this period, some of the products begins to deteriorate. Wu et al. (2006) defined this phenomenon as non-instantaneous deterioration. Chang et al. (2010) derived the best replenishment policies for non-instantaneous deteriorating products when demand is stock dependent. Geetha and Uthayakumar (2010) constructed an EOQ inventory model for non-instantaneous deteriorating products with permissible delay in payments and partial backlogging. Maihami and Kamalabadi (2012) presented an inventory model for non-instantaneous deteriorating products when the demand function is dependent on both price and time. Recently, Saha et al. (2019) introduced some promotional coordination mechanisms for when the demand is dependent on both sales efforts and price. Navarro et al. (2019) derives a collaborative economic production quantity inventory model for a three-echelon supply chain with multiple products taking into account that the demand is dependent on marketing effort. Taleizadeh et al. (2019) optimized the selling price decision in an inventory model for complementary and substitutable products.

This paper develops an EOQ inventory model for non-instantaneous deteriorating products when occurs an advertisement and price sensitive demand and there exists an order quantity dependent trade credit.

## 2. NOTATION AND ASSUMPTIONS

The following notation and assumptions are used in the development of EOQ inventory model.

### 2.1. Notation

**Parameters:**

| | |
|---|---|
| $C_o$ | Ordering cost ($/order) |
| $C_p$ | Purchase cost ($/unit) |
| $s$ | Selling price where $s > C_p$ ($/unit) |
| $h$ | Inventory holding cost ($/unit/unit of time) |
| $\theta$ | Deterioration rate, $0 \le \theta < 1$. |
| $I_e$ | Interest earned (%/unit of time) |
| $I_c$ | Interest charged for unsold stock by the supplier where $I_c > I_e$ (%/unit of time) |
| $D(A,s)$ | Advertisement and price sensitive demand, $D(A,s) = A^{\gamma}(a-bs)$ |
| $t_d$ | Time at which deterioration starts (unit of time) |
| $M_j$ | Permissible delay period (unit of time) $j = 1, 2, \ldots, n$ |
| $I(t)$ | Inventory level at time *t*, where $0 \le t \le T_j$ $j = 1, 2, \ldots, n$ |
| $Q_j$ | Order quantity (units) $j = 1, 2, \ldots, n$ |
| $Z_{i,j}(T_j)$ | Total profit ($/unit time) $i = 1, 2, \ldots, 6$ $j = 1, 2, \ldots, n$ |

**Decision variable:**

| | |
|---|---|
| $T_j$ | Length of replenishment cycle when the permissible delay period is $M_j$, $j = 1, 2, \ldots, n$ |

### 2.2. Assumptions

a) The inventory model is for a single product.

b) The demand rate is as follows $D(A,s) = A^{\gamma}(a-bs)$ which is a function of advertising and price.

Here, $s$ is the selling price per unit, $a > 0$ is the scale demand, $b > 0$ represents the linear rate of change of the demand with respect to the selling price.

c) The supplier provides the order dependent credit time in the following structure:

$$M = \begin{cases} M_1 & Q_1 \le Q < Q_2 \\ M_2 & Q_2 \le Q < Q_3 \\ \vdots & \vdots \\ M_k & Q_k \le Q < Q_{k+1} \end{cases} \qquad (1)$$

where, $0 < Q_1 < Q_2 < \ldots < Q_k < Q_{k+1}$ and $0 \le M_1 < M_2 < \ldots < M_k$.

d) The non-instantaneous deterioration is taken into consideration.

e) The capital opportunity cost occurs if $T_j > M_j$. The interest earned $(I_e)$ from sales revenue is gained within the interval $[0, M_j]$.

## 3. MATHEMATICAL INVENTORY MODEL

Two cases are studied: (i) $T_j \le t_d$, and (ii) $T_j \ge t_d$.

In the first case ($T_j \le t_d$): the replenishment cycle is less than or equal to the length of time at which the item does not deteriorate. In consequence, there is not deterioration in the replenishment cycle. In this case, the order quantity is given by

$$Q_1 = \int_0^{T_j} D(A,s)\,dt, \tag{2}$$

and the inventory level declines simply due to the demand that occurs thru the interval $\left[0,T_j\right]$. For that reason, the inventory level, $I(t)$, at time $t \in \left[0,T_j\right]$ is determined by

$$I(t) = Q_1 - \int_0^{t} D(A,s)\,dt = Q_1 - A^{\gamma}(a-bs)t, \quad 0 \le t \le T_j. \tag{3}$$

For the period of the interval $\left[0,t_d\right]$ when $T_j \ge t_d$ the inventory level drops merely attributable to demand that happens during this time. As a result, the inventory level, $I_1(t)$ at time $t \in \left[0,t_d\right]$ is computed with

$$I_1(t) = Q_j - \int_0^{t} D(A,s)\,dt, \qquad 0 \le t \le t_d \tag{4}$$

and in the interval $\left[t_d,T_j\right]$, the inventory level, $I_2(t)$ drops caused by both demand and deterioration. Consequently, the change of inventory level is governed by this differential equation:

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -D(A,s), \quad t_d < t < T_j, \tag{5}$$

with the boundary condition $I_2(T_j) = 0$. Hence, the solution of equation (5) is

$$I_2(t) = \frac{D}{\theta}\left(e^{\theta(T_j - t)} - 1\right), \qquad t_d \le t \le T_j. \tag{6}$$

Taking into account the continuity of $I_1(t)$ and $I_2(t)$ at time $t = t_d$. This means that $I_1(t_d) = I_2(t_d)$. It follows from equation (4) and equation (6) that $Q_j - \int_0^{t_d} D(A,s)\,dt = \frac{D(A,s)}{\theta}\left(e^{\theta(T_j - t)} - 1\right)$. Thus, the order quantity is calculated with

$$Q_j = A^{\gamma}(a-bp)t_d + \frac{D(A,s)}{\theta}\left(e^{\theta(T_j - t_d)} - 1\right). \tag{7}$$

Replacing equation (7) into equation (4): the following result is obtained

$$I_1(t) = D(A,s)(t_d - t) + \frac{D(A,s)}{\theta}\left(e^{\theta(T_j - t_d)} - 1\right), \; 0 \le t \le t_d, \tag{8}$$

The cost components and total profit per time unit for different scenarios are given in Table 1.

## 4. SOLUTION PROCEDURE

This section presents the solution procedure.

**Scenario 1:** $T_j \le M_j \le t_d$

$$Z_{1,j}(T_j) = \frac{1}{T_j}\left\{SR - CP - OC - HC - IC + IE\right\}$$

$$Z_{1,j}(T_j)=\frac{1}{T_j}\left[sD(A,s)T_j+sI_e\left\{D(A,s)T_jM_j-\frac{D(A,s)T_j^2}{2}\right\}-\frac{hD(A,s)T_j^2}{2}-C_pD(A,s)T_j-C_o\right]$$

$$Z_{1,j}(T_j)=\{sD(A,s)+D(A,s)M_j-C_pD(A,s)\}-\frac{D(A,s)\{sI_e+h\}}{2}T_j-\frac{C_o}{T_j} \tag{9}$$

Differentiating equation (9) with respect to $T_j$

$$\frac{dZ_{1,j}(T_j)}{dT_j}=-\frac{D(A,s)\{sI_e+h\}}{2}+\frac{C_o}{T_j^2} \tag{10}$$

The derivative of equation (10) with respect to $T_j$ is

$$\frac{d^2Z_{1,j}(T_j)}{dT_j^2}=-\frac{2C_o}{T_j^3}<0 \tag{11}$$

Equation (11) shows that the objective function has a maximum value at $t=T_j$ .
From equation (10):

$$\frac{dZ_{1,j}(T_j)}{dT_j}=0$$

$$-\frac{D(A,s)\{sI_e+h\}}{2}+\frac{C_o}{T_j^2}=0$$

$$T_j=\sqrt{\frac{2C_o}{D(A,s)\{sI_e+h\}}}$$

$$Q_1=D(A,s)T_j=D(A,s)\sqrt{\frac{2C_o}{D(A,s)\{sI_e+h\}}}=\sqrt{\frac{2C_oD(A,s)}{\{sI_e+h\}}}$$

The objective function is

$$Z_{1,j}(T_j)=\left[\begin{array}{l}\{sD(A,s)+D(A,s)M_j-C_pD(A,s)\}\\ -\frac{D(A,s)\{sI_e+h\}}{2}\sqrt{\frac{2C_o}{D(A,s)\{sI_e+h\}}}-\frac{C_o}{\sqrt{\frac{2C_o}{D(A,s)\{sI_e+h\}}}}\end{array}\right]$$

$$Z_{1,j}(T_j)=\{sD(A,s)+D(A,s)M_j-C_pD(A,s)\}-\sqrt{\frac{C_oD(A,s)\{sI_e+h\}}{2}}-\sqrt{\frac{C_oD(A,s)\{sI_e+h\}}{2}}$$

**Table 1.** Retailer’s cost components and total profit

| | Ordering cost | Sales revenue | Holding cost | Purchasing cost | Interest earned | Interest charged | Total profit |
|---|---|---|---|---|---|---|---|

| | | $OC$ | $SR$ | $HC$ | $CP$ | $IE$ | $IC$ | $Z_{i,j}(T_j)=\begin{pmatrix} SR-CP-OC \\ -HC-IC+IE \end{pmatrix}$ |
|---|---|---|---|---|---|---|---|---|
| Scenarios | $T_j \le M_j \le t_d$ | $\frac{C_o}{T_j}$ | $\frac{s}{T_j} Q_j$ | $\frac{h}{T_j}\int_0^{T_j} I(t)\,dt$ | $\frac{C_p}{T_j} Q_j$ | $\frac{sI_e}{T_j}\begin{pmatrix}\int_0^{T_j} D(A,s)t\,dt \\ +D(A,s)T_j(M_j-T_j)\end{pmatrix}$ | 0 | $Z_{1,j}(T_j)$ |
| | $M_j \le T_j \le t_d$ | | $\frac{s}{T_j} Q_j$ | $\frac{h}{T_j}\int_0^{T_j} I(t)\,dt$ | $\frac{C_p}{T_j} Q_j$ | $\frac{sI_e}{T_j}\int_0^{M_j} D(A,s)t\,dt$ | $\frac{C_pI_c}{T_j}\int_{M_j}^{T_j} I(t)\,dt$ | $Z_{2,j}(T_j)$ |
| | $M_j \le t_d \le T_j$ | | $\frac{s}{T_j} Q_j$ | $\frac{h}{T_j}\begin{bmatrix}\int_0^{t_d} I_1(t)\,dt \\ +\int_{t_d}^{T_j} I_2(t)\,dt\end{bmatrix}$ | $\frac{C_p}{T_j} Q_j$ | $\frac{sI_e}{T_j}\int_0^{M_j} D(A,s)t\,dt$ | $\frac{C_pI_c}{T_j}\begin{bmatrix}\int_{M_j}^{t_d} I_1(t)\,dt \\ +\int_{t_d}^{T_j} I_2(t)\,dt\end{bmatrix}$ | $Z_{3,j}(T_j)$ |
| | $T_j \le t_d \le M_j$ | | $\frac{s}{T_j} Q_j$ | $\frac{h}{T_j}\int_0^{T_j} I(t)\,dt$ | $\frac{C_p}{T_j} Q_j$ | $\frac{sI_e}{T_j}\begin{pmatrix}\int_0^{T_j} D(A,s)t\,dt \\ +D(A,s)T_j(M_j-T_j)\end{pmatrix}$ | 0 | $Z_{4,j}(T_j)$ |
| | $t_d \le T_j \le M_j$ | | $\frac{s}{T_j} Q_j$ | $\frac{h}{T_j}\begin{bmatrix}\int_0^{t_d} I_1(t)\,dt \\ +\int_{t_d}^{T_j} I_2(t)\,dt\end{bmatrix}$ | $\frac{C_p}{T_j} Q_j$ | $\frac{sI_e}{T_j}\begin{pmatrix}\int_0^{T_j} D(A,s)t\,dt \\ +D(A,s)T_j(M_j-T_j)\end{pmatrix}$ | 0 | $Z_{5,j}(T_j)$ |
| | $t_d \le M_j \le T_j$ | | $\frac{s}{T_j} Q_j$ | $\frac{h}{T_j}\begin{bmatrix}\int_0^{t_d} I_1(t)\,dt \\ +\int_{t_d}^{T_j} I_2(t)\,dt\end{bmatrix}$ | $\frac{C_p}{T_j} Q_j$ | $\frac{sI_e}{T_j}\int_0^{M_j} D(A,s)t\,dt$ | $\frac{C_pI_c}{T_j}\int_{M_j}^{T_j} I_2(t)\,dt$ | $Z_{6,j}(T_j)$ |

Detailed calculations are shown in Appendix A.

$$Z_{1,j}(T_j)=\{sD(A,s)+D(A,s)M_j-C_pD(A,s)\}-2\sqrt{\frac{C_oD(A,s)\{sI_e+h\}}{2}}$$

$$Z_{1,j}(T_j)=\{sD(A,s)+D(A,s)M_j-C_pD(A,s)\}-\sqrt{2C_oD(A,s)\{sI_e+h\}} \qquad (12)$$

**Scenario 2:** $M_j \le T_j \le t_d$

$$Z_{2,j}(T_j)=\frac{1}{T_j}\{SR-CP-OC-HC-IC+IE\}$$

$$Z_{2,j}(T_j)=\frac{1}{T_j}\left[(s-C_p)D(A,s)T_j+C_pI_cD(A,s)M_jT_j+\{sI_eD-C_pI_c\}\frac{D(A,s)M_j^2}{2}-\frac{\{C_pI_cD(A,s)+hD(A,s)\}T_j^2}{2}-C_o\right]$$

$$Z_{2,j}\left(T_j\right)=\left\{\left(s-C_p\right)D\left(A,s\right)+C_pI_cD\left(A,s\right)M_j\right\}-\frac{1}{2}\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}T_j$$
$$-\frac{1}{T_j}\left\{C_o-\left\{sI_eD-C_pI_c\right\}\frac{D\left(A,s\right)M_j^2}{2}\right\} \tag{13}$$

The derivative of equation (13) with respect to $T_j$ is

$$\frac{dZ_{2,j}\left(T_j\right)}{dT_j}=-\frac{1}{2}\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}+\frac{1}{T_j^2}\left\{C_o-\left\{sI_eD+C_pI_c\right\}\frac{D\left(A,s\right)M_j^2}{2}\right\} \tag{14}$$

The derivative of equation (14) with respect to $T_j$ is

$$\frac{d^2Z_{2,j}\left(T_j\right)}{dT_j^2}=-\frac{2}{T_j^3}\left\{C_o-\left(sI_eD-C_pI_c\right)\frac{D\left(A,s\right)M_j^2}{2}\right\}<0 \tag{15}$$

Equation (15) indicates that the objective function has a maximum value at $t=T_j$.

From equation (14):

$$\frac{dZ_{2,j}\left(T_j\right)}{dT_j}=0$$

$$-\frac{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}}{2}+\frac{\left[C_o-\left\{sI_eD-C_pI_c\right\}\frac{D\left(A,s\right)M_j^2}{2}\right]}{T_j^2}=0$$

$$T_j=\sqrt{\frac{2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2}{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}}} \tag{16}$$

$$Q_2=D\left(A,s\right)T_j=D\left(A,s\right)\sqrt{\frac{2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2}{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}}}=\sqrt{D\left(A,s\right)\left[\frac{2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2}{\left\{C_pI_c+h\right\}}\right]}$$

$$Z_{2,j}\left(T_j\right)=\left[\left\{\left(s-C_p\right)D\left(A,s\right)+C_pI_cD\left(A,s\right)M_j\right\}-\frac{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}\sqrt{\frac{2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2}{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}}}}{2}-\frac{\left[C_o-\left\{sI_eD-C_pI_c\right\}\frac{D\left(A,s\right)M_j^2}{2}\right]}{\sqrt{\frac{2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2}{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}}}}\right]$$

$$Z_{2,j}\left(T_j\right)=\left[\begin{array}{l}\left\{\left(s-C_p\right)D\left(A,s\right)+C_pI_cD\left(A,s\right)M_j\right\}-\\ \sqrt{\frac{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}\left[2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2\right]}{2}}\\ -\sqrt{\frac{\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}\left[2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2\right]}{2}}\end{array}\right]$$

$$Z_{2,j}\left(T_j\right)=\left[\begin{array}{l}\left\{\left(s-C_p\right)D\left(A,s\right)+C_pI_cD\left(A,s\right)M_j\right\}-\\ \sqrt{2\left\{C_pI_cD\left(A,s\right)+hD\left(A,s\right)\right\}\left[2C_o-\left\{sI_eD-C_pI_c\right\}D\left(A,s\right)M_j^2\right]}\end{array}\right] \tag{17}$$

**Scenario 3:** $M_j\le t_d\le T_j$

$$Z_{3,j}\left(T_j\right)=\frac{1}{T_j}\left\{SR-CP-OC-HC-IC+IE\right\}$$

$$Z_{3,j}\left(T_j\right)=\frac{1}{T_j}\left[\begin{array}{l}\left(s-C_p\right)Q_j-C_o-h\left\{Q_jt_d-\frac{D\left(A,s\right)t_d^2}{2}+\frac{D\left(A,s\right)\left(T_j-t_d\right)^2}{2}\right\}\\-C_pI_c\left\{Q_j\left(t_d-M_j\right)-\frac{D\left(A,s\right)\left(t_d^2-M_j^2\right)}{2}+\frac{D\left(A,s\right)\left(T_j-t_d\right)^2}{2}\right\}+\frac{sI_eD\left(A,s\right)M_j^2}{2}\end{array}\right]$$

$$Z_{3,j}\left(T_j\right)=\frac{1}{T_j}\left[\begin{array}{l}\left\{\left(s-C_p\right)-ht_d-C_pI_c\left(t_d-M_j\right)\right\}Q_j-\left\{\begin{array}{l}C_o-\frac{hD\left(A,s\right)t_d^2}{2}-\frac{C_pI_cD\left(A,s\right)\left(t_d^2-M_j^2\right)}{2}\\-\frac{sI_eD\left(A,s\right)M_j^2}{2}\end{array}\right\}\\-\frac{\left(h+C_pI_c\right)D\left(A,s\right)\left(T_j-t_d\right)^2}{2}\end{array}\right] \tag{18}$$

Setting the value of $Q_j$ into equation (18) and after simplify of the equation (18): we have

$$Z_{3,j}\left(T_j\right)=\frac{1}{T_j}\left\{A_4T_j-A_5-A_6T_j^2\right\}$$

$$Z_{3,j}\left(T_j\right)=A_4-A_6T_j-\frac{A_5}{T_j} \tag{19}$$

The derivative of equation (19) with respect to $T_j$ is

$$\frac{dZ_{3,j}\left(T_j\right)}{dT_j}=0-A_6+\frac{A_5}{T_j^2} \tag{20}$$

The derivative of equation (20) is

$$\frac{d^2Z_{3,j}\left(T_j\right)}{dT_j^2}=-\frac{2A_5}{T_j^3}<0 \tag{21}$$

Equation (21) specifies that the objective function has a maximum value at $t=T_j$.

$$\frac{dZ_{3,j}\left(T_j\right)}{dT_j}=-A_6+\frac{A_5}{T_j^2}=0$$

$$T_j=\sqrt{\frac{A_5}{A_6}} \tag{22}$$

From equation (19): the objective function is written as

$$Z_{3,j}\left(T_j\right)=A_4-A_6\sqrt{\frac{A_5}{A_6}}-\frac{A_5}{\sqrt{\frac{A_5}{A_6}}}$$

$$Z_{3,j}\left(T_j\right)=A_4-\sqrt{A_5A_6}-\sqrt{A_5A_6}$$

$$Z_{3,j}\left(T_j\right)=A_4-2\sqrt{A_5A_6} \tag{23}$$

The value of $Q_j$ is given by

$$Q_j = D(A,s)t_d + D(A,s)\left\{\sqrt{\frac{A_5}{A_6}} - t_d\right\} + \frac{D(A,s)\theta}{2}\left\{\sqrt{\frac{A_5}{A_6}} - t_d\right\}^2 \tag{24}$$

Where

$$A_1 = \left\{(s - C_p) - ht_d - C_p I_c (t_d - M_j)\right\}$$

$$A_2 = \left\{ C_o - \frac{hD(A,s)t_d^2}{2} - \frac{C_p I_c D(A,s)(t_d^2 - M_j^2)}{2} - \frac{sI_e D(A,s) M_j^2}{2} \right\}$$

$$A_3 = \frac{(h + C_p I_c) D(A,s)}{2}$$

$$A_4 = A_1 D(A,s) - A_1 D(A,s)\theta t_d + 2A_3 t_d$$

$$A_5 = A_2 - \frac{A_1 D(A,s) q t_d^2}{2} + A_3 t_d^2$$

$$A_6 = A_3 - \frac{A_1 D(A,s)\theta}{2}$$

**Scenario 4:** $T_j \le t_d \le M_j$

$$Z_{4,j}(T_j) = \frac{1}{T_j}\left\{SR - CP - OC - HC - IC + IE\right\}$$

$$Z_{4,j}(T_j) = \frac{1}{T_j}\left[ sD(A,s)T_j + sI_e\left\{ D(A,s)T_j M_j - \frac{D(A,s)T_j^2}{2} \right\} - \frac{hD(A,s)T_j^2}{2} - C_p D(A,s) T_j - C_o \right]$$

$$Z_{4,j}(T_j) = \left\{ sD(A,s) + D(A,s)M_j - C_p D(A,s) \right\} - \frac{D(A,s)\{sI_e + h\}}{2} T_j - \frac{C_o}{T_j} \tag{25}$$

The derivative of equation (25) with respect to $T_j$ is

$$\frac{dZ_{4,j}(T_j)}{dT_j} = 0 - \frac{D(A,s)\{sI_e + h\}}{2} + \frac{C_o}{T_j^2} \tag{26}$$

The derivative of equation (26) with respect to $T_j$ is

$$\frac{d^2 Z_{4,j}(T_j)}{dT_j^2} = -\frac{2C_o}{T_j^3} < 0 \tag{27}$$

Equation (27) confirms that the objective function has a maximum value at $t = T_j$ .

From equation (26):

$$\frac{dZ_{4,j}\left(T_j\right)}{dT_j}=0$$

$$-\frac{D\left(A,s\right)\left\{sI_e+h\right\}}{2}+\frac{C_o}{T_j^2}=0$$

$$T_j=\sqrt{\frac{2C_o}{D\left(A,s\right)\left\{sI_e+h\right\}}}$$

$$Q_4=D\left(A,s\right)T_j=D\left(A,s\right)\sqrt{\frac{2C_o}{D\left(A,s\right)\left\{sI_e+h\right\}}}=\sqrt{\frac{2C_oD\left(A,s\right)}{\left\{sI_e+h\right\}}}$$

The objective function is

$$Z_{4,j}\left(T_j\right)=\left[\left\{sD\left(A,s\right)+D\left(A,s\right)M_j-C_pD\left(A,s\right)\right\}-\frac{D\left(A,s\right)\left\{sI_e+h\right\}}{2}\sqrt{\frac{2C_o}{D\left(A,s\right)\left\{sI_e+h\right\}}}-\frac{C_o}{\sqrt{\frac{2C_o}{D\left(A,s\right)\left\{sI_e+h\right\}}}}\right]$$

$$Z_{4,j}\left(T_j\right)=\left\{sD\left(A,s\right)+D\left(A,s\right)M_j-C_pD\left(A,s\right)\right\}-\sqrt{\frac{C_oD\left(A,s\right)\left\{sI_e+h\right\}}{2}}-\sqrt{\frac{C_oD\left(A,s\right)\left\{sI_e+h\right\}}{2}}$$

$$Z_{4,j}\left(T_j\right)=\left\{sD\left(A,s\right)+D\left(A,s\right)M_j-C_pD\left(A,s\right)\right\}-2\sqrt{\frac{C_oD\left(A,s\right)\left\{sI_e+h\right\}}{2}}$$

$$Z_{4,j}\left(T_j\right)=\left\{sD\left(A,s\right)+D\left(A,s\right)M_j-C_pD\left(A,s\right)\right\}-\sqrt{2C_oD\left(A,s\right)\left\{sI_e+h\right\}} \tag{28}$$

**Scenario 5:** $t_d\le T_j\le M_j$

$$Z_{5,j}\left(T_j\right)=\frac{1}{T_j}\left\{SR-CP-OC-HC-IC+IE\right\}$$

$$Z_{5,j}\left(T_j\right)=\frac{1}{T_j}\left[\left(s-C_p\right)Q_j+sI_e\left\{D\left(A,s\right)T_jM_j-\frac{D\left(A,s\right)T_j^2}{2}\right\}-C_o-h\left\{\begin{array}{l}\frac{D\left(A,s\right)\theta t_d^3}{2}-D\left(A,s\right)\theta t_d^2T_j\\+\left(\frac{D\left(A,s\right)\theta t_d}{2}+\frac{D\left(A,s\right)}{2}\right)T_j^2\end{array}\right\}\right] \tag{29}$$

Putting the value of $Q_j$ into equation (29) and simplifying,

$$Z_{5,j}\left(T_j\right)=\frac{1}{T_j}\left[\begin{array}{l}-\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}+\left\{\begin{array}{l}\left(s-C_p\right)\left(D\left(A,s\right)-D\left(A,s\right)\theta t_d\right)\\+sI_eD\left(A,s\right)M_j+hD\left(A,s\right)\theta t_d^2\end{array}\right\}T_j\\-\left\{\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\right\}T_j^2\end{array}\right]$$

$$Z_{5,j}\left(T_j\right)=\left[\begin{array}{l}\frac{-1}{T_j}\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}+\left\{\begin{array}{l}\left(s-C_p\right)\left(D\left(A,s\right)-D\left(A,s\right)\theta t_d\right)\\+sI_eD\left(A,s\right)M_j+hD\left(A,s\right)\theta t_d^2\end{array}\right\}\\-\left\{\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\right\}T_j\end{array}\right] \quad (30)$$

The derivative of equation (30) with respect to $T_j$ is

$$\frac{dZ_{5,j}\left(T_j\right)}{dT_j}=\frac{1}{T_j^2}\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}-\left\{\begin{array}{l}\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)\\+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\end{array}\right\} \quad (31)$$

The derivative of equation (31) with respect to $T_j$ is

$$\frac{d^2Z_{5,j}\left(T_j\right)}{dT_j^2}=-\frac{2}{T_j^3}\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}<0 \quad (32)$$

From equation (30) the value of $T_j$ is obtained as follows.

$$\frac{dZ_{5,j}\left(T_j\right)}{dT_j}=0$$

$$\left[\frac{1}{T_j^2}\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}-\left\{\begin{array}{l}\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)\\+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\end{array}\right\}\right]=0$$

$$T_{j=}\sqrt{\frac{\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}}{\left\{\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\right\}}}$$

By substituting the value of $T_j$ into equation (30): the closed form of objective function is determined

$$Z_{5,j}\left(T_j\right)=\left[\begin{array}{l}\frac{-\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}}{\sqrt{\frac{\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}}{\left\{\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\right\}}}}\\+\left\{\begin{array}{l}\left(s-C_p\right)\left(D\left(A,s\right)-D\left(A,s\right)\theta t_d\right)\\+sI_eD\left(A,s\right)M_j+hD\left(A,s\right)\theta t_d^2\end{array}\right\}\\-\left\{\begin{array}{l}\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)\\+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\end{array}\right\}\sqrt{\frac{\left\{C_o-\left(s-C_p-ht_d\right)\frac{D\left(A,s\right)\theta t_d^2}{2}\right\}}{\left\{\frac{h}{2}\left(D\left(A,s\right)\theta t_d+D\left(A,s\right)\right)+\frac{sI_eD\left(A,s\right)}{2}-\frac{\left(s-C_p\right)D\left(A,s\right)\theta}{2}\right\}}}\end{array}\right]$$

$$Z_{5,j}(T_j)=\left\{\begin{array}{l}(s-C_p)(D(A,s)-D(A,s)\theta t_d)\\+sI_eD(A,s)M_j+hD(A,s)\theta t_d^2\end{array}\right\}-2\sqrt{\left\{C_o-(s-C_p-ht_d)\frac{D(A,s)\theta t_d^2}{2}\right\}\left\{\begin{array}{l}\frac{h}{2}(D(A,s)\theta t_d+D(A,s))\\+\frac{sI_eD(A,s)}{2}-\frac{(s-C_p)D(A,s)\theta}{2}\end{array}\right\}} \quad (33)$$

and

$$Q_j=D(A,s)t_d+D(A,s)(T_j-t_d)+\frac{D(A,s)\theta(T_j-t_d)^2}{2}$$

**Scenario 6:** $t_d \le M_j \le T_j$

$$Z_{6,j}(T_j)=\frac{(SR-CP-OC-HC-IC+IE)}{T_j}$$

$$Z_{6,j}(T_j)=\frac{\left[\begin{array}{l}(s-C_p)Q_j+(sI_e-C_pI_c)\frac{D(A,s)M_j^2}{2}-C_o-h\left\{\begin{array}{l}\frac{D(A,s)\theta t_d^3}{2}-D(A,s)\theta t_d^2T_j\\+\left(\frac{D(A,s)\theta t_d}{2}+\frac{D(A,s)}{2}\right)T_j^2\end{array}\right\}\\-\frac{C_pI_cD(A,s)}{2}T_j^2+C_pI_cD(A,s)M_jT_j\end{array}\right]}{T_j} \quad (34)$$

Putting the value of $Q_j$ into equation (34) and simplifying,

$$Z_{6,j}(T_j)=\frac{\left[\begin{array}{l}-\left\{C_o-(sI_e-C_pI_c)\frac{D(A,s)M_j^2}{2}-(s-C_p-ht_d)\frac{D(A,s)\theta t_d^2}{2}\right\}+\left\{\begin{array}{l}(s-C_p)(D(A,s)-D(A,s)\theta t_d)\\+hD(A,s)\theta t_d^2+C_pI_cD(A,s)M_j\end{array}\right\}T_j\\-\left\{\frac{h}{2}(D(A,s)\theta t_d+D(A,s))+\frac{C_pI_cD(A,s)}{2}-\frac{(s-C_p)D(A,s)\theta}{2}\right\}T_j^2\end{array}\right]}{T_j}$$

$$Z_{6,j}(T_j)=\left[\begin{array}{l}\frac{-\left\{C_o-(sI_e-C_pI_c)\frac{D(A,s)M_j^2}{2}-(s-C_p-ht_d)\frac{D(A,s)\theta t_d^2}{2}\right\}}{T_j}+\left\{\begin{array}{l}(s-C_p)(D(A,s)-D(A,s)\theta t_d)\\+hD(A,s)\theta t_d^2+C_pI_cD(A,s)M_j\end{array}\right\}\\-\left\{\frac{h}{2}(D(A,s)\theta t_d+D(A,s))+\frac{C_pI_cD(A,s)}{2}-\frac{(s-C_p)D(A,s)\theta}{2}\right\}T_j\end{array}\right] \quad (35)$$

The first derivative of equation (35) with respect to $T_j$ is

$$\frac{dZ_{6,j}(T_j)}{dT_j}=\left[\begin{array}{l}\frac{\left\{C_o-(sI_e-C_pI_c)\frac{D(A,s)M_j^2}{2}-(s-C_p-ht_d)\frac{D(A,s)\theta t_d^2}{2}\right\}}{T_j^2}-\\\left\{\frac{h}{2}(D(A,s)\theta t_d+D(A,s))+\frac{C_pI_cD(A,s)}{2}-\frac{(s-C_p)D(A,s)\theta}{2}\right\}\end{array}\right] \quad (36)$$

The derivative of equation (36) with respect to $T_j$ is

$$\frac{d^2 Z_{6,j}\left(T_j\right)}{dT_j^2} = -\frac{2\left\{C_o - \left(sI_e - C_p I_c\right)\frac{D(A,s)M_j^2}{2} - \left(s - C_p - ht_d\right)\frac{D(A,s)\theta t_d^2}{2}\right\}}{T_j^3} < 0 \qquad (37)$$

From equation (36): we can find the value of $T_j$ .

$$\frac{dZ_{6,j}\left(T_j\right)}{dT_j} = 0$$

$$\left[\frac{\left\{C_o - \left(sI_e - C_p I_c\right)\frac{D(A,s)M_j^2}{2} - \left(s - C_p - ht_d\right)\frac{D(A,s)\theta t_d^2}{2}\right\}}{T_j^2} - \left\{\frac{h}{2}\left(D(A,s)\theta t_d + D(A,s)\right) + \frac{C_p I_c D(A,s)}{2} - \frac{\left(s - C_p\right)D(A,s)\theta}{2}\right\}\right] = 0$$

$$T_j = \sqrt{\frac{\left\{C_o - \left(sI_e - C_p I_c\right)\frac{D(A,s)M_j^2}{2} - \left(s - C_p - ht_d\right)\frac{D(A,s)\theta t_d^2}{2}\right\}}{\left\{\frac{h}{2}\left(D(A,s)\theta t_d + D(A,s)\right) + \frac{C_p I_c D(A,s)}{2} - \frac{\left(s - C_p\right)D(A,s)\theta}{2}\right\}}}$$

By replacing the value of $T_j$ into equation (35): the closed form of objective function is

$$Z_{6,j}\left(T_j\right) = \left[\begin{array}{l} \dfrac{-\left\{C_o - \left(sI_e - C_p I_c\right)\frac{D(A,s)M_j^2}{2} - \left(s - C_p - ht_d\right)\frac{D(A,s)\theta t_d^2}{2}\right\}}{\sqrt{\dfrac{\left\{C_o - \left(sI_e - C_p I_c\right)\frac{D(A,s)M_j^2}{2} - \left(s - C_p - ht_d\right)\frac{D(A,s)\theta t_d^2}{2}\right\}}{\left\{\frac{h}{2}\left(D(A,s)\theta t_d + D(A,s)\right) + \frac{C_p I_c D(A,s)}{2} - \frac{\left(s - C_p\right)D(A,s)\theta}{2}\right\}}}} + \left\{\begin{array}{l}\left(s - C_p\right)\left(D(A,s) - D(A,s)\theta t_d\right) \\ + hD(A,s)\theta t_d^2 + C_p I_c D(A,s) M_j\end{array}\right\} \\ -\left\{\frac{h}{2}\left(D(A,s)\theta t_d + D(A,s)\right) + \frac{C_p I_c D(A,s)}{2} - \frac{\left(s - C_p\right)D(A,s)\theta}{2}\right\}\sqrt{\dfrac{\left\{C_o - \left(sI_e - C_p I_c\right)\frac{D(A,s)M_j^2}{2} - \left(s - C_p - ht_d\right)\frac{D(A,s)\theta t_d^2}{2}\right\}}{\left\{\frac{h}{2}\left(D(A,s)\theta t_d + D(A,s)\right) + \frac{C_p I_c D(A,s)}{2} - \frac{\left(s - C_p\right)D(A,s)\theta}{2}\right\}}} \end{array}\right]$$

$$Z_{6,j}\left(T_j\right) = \left\{\begin{array}{l}\left(s - C_p\right)\left(D(A,s) - D(A,s)\theta t_d\right) \\ + hD(A,s)\theta t_d^2 + C_p I_c D(A,s) M_j\end{array}\right\} - 2\sqrt{\frac{\left\{C_o - \left(sI_e - C_p I_c\right)\frac{D(A,s)M_j^2}{2} - \left(s - C_p - ht_d\right)\frac{D(A,s)\theta t_d^2}{2}\right\}}{\left\{\frac{h}{2}\left(D(A,s)\theta t_d + D(A,s)\right) + \frac{C_p I_c D(A,s)}{2} - \frac{\left(s - C_p\right)D(A,s)\theta}{2}\right\}}} \qquad (38)$$

$$Q_j = D(A,s)t_d + D(A,s)\left(T_j - t_d\right) + \frac{D(A,s)\theta\left(T_j - t_d\right)^2}{2}$$

## 2. NUMERICAL EXAMPLE

With the aim of illustrating the proposed inventory model, this section presents and solves some numerical examples.

**Example 1. For scenario 1:** $T_j \le M_j \le t_d$

The data are:

$C_0 = \$200$ / order, $C_p = \$20$ / unit, $s = \$30$ / unit, $h = \$2$ / unit / unit of time, $I_e = 9\%$ / year, $I_p = 12\%$ / year, $A = 10$ / year, $\gamma = 0.1$, $a = 500$, $b = 0.5$, $t_d = 0.8$ year, $M_j = 0.5$ year,

The solution is $Q_j = 227.9565$ units, $T_j = 0.3733$ year, $Z_{1,j}\left(T_j\right) = \$5858.674$ / year,

**Example 2. For scenario 2:** $M_j \le T_j \le t_d$

The data are:

$C_0 = \$200$ / order, $C_p = \$20$ / unit, $s = \$30$ / unit, $h = \$2$ / unit / unit of time, $I_e = 9\%$ / year, $I_p = 12\%$ / year, $A = 10$ / year, $\gamma = 0.1$, $a = 500$, $b = 0.5$, $t_d = 0.6$ year, $M_j = 0.15$ year,

The solution is $Q_j = 234.3826$ units, $T_j = 0.3838$ year, $Z_{2,j}\left(T_j\right) = \$4867.141$ / year,

**Example 3. For scenario 3:** $M_j \le t_d \le T_j$

The data are:

$C_0 = \$200$ / order, $C_p = \$20$ / unit, $s = \$30$ / unit, $h = \$2$ / unit / unit of time, $I_e = 9\%$ / year, $I_p = 12\%$ / year, $\theta = 0.05$, $A = 10$ / year, $\gamma = 0.2$, $a = 1500$, $b = 0.5$, $t_d = 0.25$ year, $M_j = 0.2$ year,

The values for $A_1 = 9.38, A_2 = 283.5516, A_3 = 5177.846, A_4 = 24389.42, A_5 = 572.6725, A_6 = 4625.935$.

The solution is $Q_j = 828.7048$ units, $T_j = 0.3518$ year, $Z_{3,j}\left(T_j\right) = \$21134.18$ / year,

**Example 4. For scenario 4:** $T_j \le t_d \le M_j$

The data are:

$C_0 = \$200$ / order, $C_p = \$20$ / unit, $s = \$30$ / unit, $h = \$1$ / unit / unit of time, $I_e = 9\%$ / year, $I_p = 12\%$ / year, $A = 10$ / year, $\gamma = 0.1$, $a = 1200$, $b = 0.5$, $t_d = 0.3$ year, $M_j = 0.6$ year,

The solution is $Q_j = 401.595$ units, $T_j = 0.2692$ year, $Z_{4,j}\left(T_j\right) = \$15849.12$ / year,

**Example 5. For scenario 5:** $t_d \le T_j \le M_j$

The data are:

$C_0 = \$200$ / order, $C_p = \$20$ / unit, $s = \$30$ / unit, $h = \$2$ / unit / unit of time, $I_e = 9\%$ / year, $I_p = 12\%$ / year, $\theta = 0.05$, $A = 10$ / year, $\gamma = 0.2$, $a = 1700$, $b = 0.5$, $t_d = 0.1$ year, $M_j = 0.4$ year,

The solution is $Q_j = 496.3247$ units, $T_j = 0.1853$ year, $Z_{5,j}\left(T_j\right) = \$27423.75$ / year,

**Example 6. For scenario 6:** $t_d \le M_j \le T_j$

The data are:

$C_0 = \$100$ / order, $C_p = \$20$ / unit, $s = \$30$ / unit, $h = \$1$ / unit / unit of time, $I_e = 9\%$ / year, $I_p = 12\%$ / year, $\theta = 0.05$, $A = 10$ / year, $\gamma = 0.2$, $a = 2000$, $b = 0.4$, $t_d = 0.1$ year, $M_j = 0.25$ year, The solution is $Q_j = 1301.325$ units, $T_j = 0.4106$ year, $Z_{6,j}\left(T_j\right) = \$29011.28$ / year,

## 3. SENSITIVITY ANALYSIS

The numerical example 6 is utilized to investigate the effect of under or over estimation of the input parameters on the optimal values for cycle length ($T_j$): order quantity ($Q_j$) and total profit of the inventory system. The results are shown in Table 2.

**Table 2. Sensitivity analysis for numerical example 6**

| Parameters | % change of parameter | % change in | | |
|---|---|---|---|---|
| | | $T_j^*$ | $Q_j^*$ | Total profit* |
| $C_o$ | -20 | -1.32 | -1.32 | 0.17 |
| | -10 | -0.66 | -0.66 | 0.08 |
| | 10 | 0.63 | 0.65 | -0.08 |
| | 20 | 1.27 | 1.30 | -0.17 |
| $C_p$ | -20 | -1.00 | -0.99 | 45.38 |
| | -10 | -0.44 | -0.43 | 22.69 |
| | 10 | 0.32 | 0.34 | -22.69 |
| | 20 | 0.61 | 0.61 | -45.38 |
| $S$ | -20 | -1.32 | -1.19 | -65.95 |
| | -10 | -0.68 | -0.62 | -32.95 |
| | 10 | 0.73 | 0.69 | 32.92 |
| | 20 | 1.58 | 1.46 | 65.79 |
| $h$ | -20 | 3.63 | 3.69 | 0.46 |
| | -10 | 1.75 | 1.79 | 0.23 |
| | 10 | -1.70 | -1.70 | -0.22 |
| | 20 | -3.29 | -3.32 | -0.44 |
| $\theta$ | -20 | -1.58 | -1.69 | -0.12 |
| | -10 | -0.80 | -0.86 | -0.06 |
| | 10 | 0.80 | 0.88 | 0.06 |
| | 20 | 1.63 | 1.78 | 0.13 |
| $A$ | -20 | 0.29 | -4.08 | -4.40 |
| | -10 | 0.14 | -1.95 | -2.10 |
| | 10 | -0.12 | 1.80 | 1.94 |
| | 20 | -0.24 | 3.47 | 3.74 |
| $\gamma$ | -20 | 0.61 | -8.23 | -8.87 |
| | -10 | 0.29 | -4.21 | -4.54 |
| | 10 | -0.29 | 4.40 | 4.75 |
| | 20 | -0.58 | 9.02 | 9.73 |
| $a$ | -20 | 1.61 | -18.82 | -20.29 |
| | -10 | 0.71 | -9.40 | -10.14 |
| | 10 | -0.61 | 9.40 | 10.15 |
| | 20 | -1.10 | 18.80 | 20.29 |
| $b$ | -20 | -0.02 | 0.11 | 0.12 |
| | -10 | -0.02 | 0.06 | 0.06 |
| | 10 | 0.00 | -0.06 | -0.06 |
| | 20 | 0.00 | -0.11 | -0.12 |
| $t_d$ | -20 | 0.13 | 0.28 | 0.09 |
| | -10 | 0.10 | 0.14 | 0.04 |
| | 10 | -0.12 | -0.15 | -0.04 |
| | 20 | -0.24 | -0.31 | -0.08 |
| $M_j$ | -20 | -6.26 | -6.30 | -0.17 |
| | -10 | -2.90 | -2.92 | -0.11 |
| | 10 | 2.46 | 2.50 | 0.17 |
| | 20 | 4.55 | 4.61 | 0.39 |

## 5. CONCLUSION

This paper develops an inventory model for non-instantaneous deteriorating products in conditions where the supplier offers to the retailer a trade credit linked to order quantity. In this inventory model, the demand is dependent of both the advertisement and price. There are some interesting research topics to conduct in the near future, for example to consider the nonlinear holding cost, power demand pattern, partial backlogging, two or three level trade credit, among others.

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## Appendix A

**Scenario 1:** $T_j \le M_j \le t_d$

Holding cost (*HC*) $= \dfrac{h}{T_j}\displaystyle\int_0^{T_j} I(t)\,dt = \dfrac{h}{T_j}\int_0^{T_j}\left[D(A,s)T_j - D(A,s)t\right]dt$

$$= \frac{h}{T_j}\left[D(A,s)T_j^2 - \frac{D(A,s)T_j^2}{2}\right] = \frac{hD(A,s)T_j^2}{2T_j}$$

Interest earned (*IE*) $= \dfrac{sI_e}{T_j}\left(\displaystyle\int_0^{T_j} D(A,s)t\,dt + D(A,s)T_j\left(M_j - T_j\right)\right)$

$$= \frac{sI_e}{T_j}\left[\frac{D(A,s)T_j^2}{2} + D(A,s)T_jM_j - D(A,s)T_j^2\right]$$

$$= \frac{sI_e}{T_j}\left[D(A,s)T_jM_j - \frac{D(A,s)T_j^2}{2}\right]$$

**Scenario 2:** $M_j \le T_j \le t_d$

Holding cost ($HC$) $= \frac{h}{T_j}\int_0^{T_j} I(t)\,dt = \frac{h}{T_j}\int_0^{T_j}\left[D(A,s)T_j - D(A,s)t\right]dt$

$$= \frac{h}{T_j}\left[D(A,s)T_j^2 - \frac{D(A,s)T_j^2}{2}\right] = \frac{hD(A,s)T_j^2}{2T_j}$$

Interest earned ($IE$) $= \frac{sI_e}{T_j}\int_0^{M_j} D(A,s)t\,dt = \frac{sI_eD(A,s)M_j^2}{2T_j}$

Interest paid ($IP$) $= \frac{C_pI_c}{T_j}\int_{M_j}^{T_j} I(t)\,dt = \frac{C_pI_c}{T_j}\int_{M_j}^{T_j}\left\{D(A,s)T_j - D(A,s)t\right\}dt$

$$= \frac{C_pI_c}{T_j}\left[D(A,s)T_j\left(T_j - M_j\right) - \frac{D(A,s)T_j^2}{2} + \frac{D(A,s)M_j^2}{2}\right]$$

$$= \frac{C_pI_c}{T_j}\left[\frac{D(A,s)T_j^2}{2} - D(A,s)T_jM_j + \frac{D(A,s)M_j^2}{2}\right]$$

**Scenario 3:** $M_j \le t_d \le T_j$

Holding cost ($HC$) $= \frac{h}{T_j}\left\{\int_0^{t_d} I_1(t)\,dt + \int_{t_d}^{T_j} I_2(t)\,dt\right\}$

$$= \frac{h}{T_j}\left[\int_0^{t_d}\left(Q_j - D(A,s)t\right)dt + \int_{t_d}^{T_j}\frac{D(A,s)}{\theta}\left\{e^{\theta(T_j-t)} - 1\right\}dt\right]$$

$$= \frac{h}{T_j}\left[Q_jt_d - \frac{D(A,s)t_d^2}{2} + \frac{D(A,s)}{\theta^2}\left\{e^{\theta(T_j-t_d)} - \theta\left(T_j - t_d\right) - 1\right\}\right]$$

(A.1)

Expanding equation by Taylor series expansion of equation (A.1):

Holding cost ($HC$) $= \frac{h}{T_j}\left[Q_jt_d - \frac{D(A,s)t_d^2}{2} + \frac{D(A,s)\left(T_j - t_d\right)^2}{2}\right]$

Interest earned ($IE$) $= \frac{sI_e}{T_j}\int_0^{M_j} D(A,s)t\,dt = \frac{sI_eD(A,s)M_j^2}{2T_j}$

Interest paid ($IP$) $= \frac{C_p I_c}{T_j}\left\{\int_{M_j}^{t_d} I_1(t)\,dt + \int_{t_d}^{T_j} I_2(t)\,dt\right\}$

$$= \frac{C_p I_c}{T_j}\left[\int_{M_j}^{t_d} Q_j - D(A,s)t\,dt + \int_{t_d}^{T_j} \frac{D(A,s)}{\theta}\left\{e^{\theta(T_j-t)} - 1\right\}dt\right]$$

$$= \frac{C_p I_c}{T_j}\left[Q_j(t_d - M_j) - \frac{D(A,s)(t_d^2 - M_j^2)}{2} + \frac{D(A,s)}{\theta^2}\left\{e^{\theta(T_j-t_d)} - \theta(T_j - t_d) - 1\right\}\right]$$

(A.2)

Expanding equation by Taylor series expansion of equation (A.2):

Interest paid ($IP$) $= \frac{C_p I_c}{T_j}\left[Q_j(t_d - M_j) - \frac{D(A,s)(t_d^2 - M_j^2)}{2} + \frac{D(A,s)(T_j - t_d)^2}{2}\right]$

**Scenario 4:** $T_j \le t_d \le M_j$

Holding cost ($HC$) $= \frac{h}{T_j}\int_0^{T_j} I(t)\,dt = \frac{h}{T_j}\int_0^{T_j}\{D(A,s)T_j - D(A,s)t\}\,dt = \frac{h}{T_j}\left[D(A,s)T_j^2 - \frac{D(A,s)T_j^2}{2}\right] = \frac{hD(A,s)T_j^2}{2T_j}$

Interest earned ($IE$) $= \frac{sI_e}{T_j}\left(\int_0^{T_j} D(A,s)t\,dt + D(A,s)T_j(M_j - T_j)\right)$

$$= \frac{sI_e}{T_j}\left[\frac{D(A,s)T_j^2}{2} + D(A,s)T_jM_j - D(A,s)T_j^2\right]$$

$$= \frac{sI_e}{T_j}\left[D(A,s)T_jM_j - \frac{D(A,s)T_j^2}{2}\right]$$

**Scenario 5:** $t_d \le T_j \le M_j$

Holding cost ($HC$) $= \frac{h}{T_j}\left[Q_jt_d - \frac{D(A,s)t_d^2}{2} + \frac{D(A,s)(T_j - t_d)^2}{2}\right]$

Interest earned ($IE$) $= \frac{sI_e}{T_j}\left(\int_0^{T_j} D(A,s)t\,dt + D(A,s)T_j(M_j - T_j)\right)$

$$= \frac{sI_e}{T_j}\left[\frac{D(A,s)T_j^2}{2} + D(A,s)T_jM_j - D(A,s)T_j^2\right]$$

$$= \frac{sI_e}{T_j}\left[D(A,s)T_jM_j - \frac{D(A,s)T_j^2}{2}\right]$$

**Scenario 6:** $t_d \le M_j \le T_j$

Holding cost ($HC$) $= \frac{h}{T_j}\left[Q_jt_d - \frac{D(A,s)t_d^2}{2} + \frac{D(A,s)(T_j - t_d)^2}{2}\right]$

Interest earn ($IE$) $= \frac{sI_e}{T_j}\int_0^{M_j} D(A,s)tdt = \frac{sI_e D(A,s) M_j^2}{2T_j}$

Interest paid ($IP$) $= \frac{C_p I_c}{T_j}\int_{M_j}^{T_j} I_2(t)dt = \frac{C_p I_c}{T_j}\int_{M_j}^{T_j}\frac{D(A,s)}{\theta}\left\{e^{\theta(T_j-t)}-1\right\}dt = \frac{C_p I_c}{T_j}\frac{D(A,s)}{\theta^2}\left\{e^{\theta(T_j-M_j)}-\theta(T_j-M_j)-1\right\}$ (A.3)

Expanding equation by Taylor series expansion of equation (A.3):

Interest paid ($IP$) $= \frac{C_p I_c}{T_j}\frac{D(A,s)(T_j-M_j)^2}{2}$