DYNAMIC DEFORMATION OF A THICK-WALLED IDEALLY ELASTIC HOLLOW SPHERE WITH AN ARBITRARY LOAD.

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ABSTRACT

The stress-strain state of an ideally elastic hollow ball under the action of arbitrary time-dependent loads is investigated using the defining relations of the mechanics of continuous media. Approximate analytical solutions are obtained for the displacement field in the elastic region. The dependences of elastic displacements on time and type of load are constructed.

KEYWORDS: analytical solution, equations of motion, Hooke's law.

MSC: 74B99

RESUMEN

Utilizando las relaciones definitorias de la mecánica continua, se investiga el estado de tensión-deformación de una bola hueca idealmente elástica bajo la acción de cargas dependientes del tiempo arbitrarias. Se obtienen soluciones analíticas aproximadas para el campo de desplazamiento en la región elástica. Se construyen las dependencias de los desplazamientos elásticos en el tiempo y tipo de carga.

PALABRAS CLAVE: solución analítica , ecuaciones de movmiento, Ley de Hooke.

1. INTRODUCTION

In [1], in the static formulation using the theory of small elastic-plastic deformations, the problem of loss of stability of a thick-walled spherical shell under uniform pressure was considered. In [2], using the theory of flow with the assumption of incompressibility of the material, the problem of complex media was investigated. In [3], the problem of a three-axis tension of an elastic-plastic space weakened by a spherical cavity was solved in the dynamic formulation under the action of periodic loads. In this paper, we consider the dynamic deformation of a thick-walled hollow ideal-elastic ball of radius R_1, with a radius of the internal cavity R_0 (Fig. 1). The distributed load P_1 acts on the external surface of the ball, and the load P_0 - on the contour of the internal cavity. The forces P_0, P_1 are arbitrary, uniformly acting on the time interval $t \in [t_0, t_1]$. The purpose of this work is to build with sufficient accuracy an approximate analytical solution of the problem under consideration, as well as an analysis of the solution obtained.

The problem is solved in a spherical coordinate system in dimensionless quantities. Variables having length dimensions are related to the radius of the internal cavity of the ball R_0, variables having stress dimensions are referred to the shear modulus μ.

We divide the segment $[t_0, t_1]$ into n parts by points t_k, k = 1..n-1 and suppose that the loads P_0, P_1 on each of the segments $[t_k, t_{k+1}]$ are representable as

$$P_0 = A_0^{(k)} + A_1^{(k)}t + A_2^{(k)}t^2 + A_3^{(k)}t^3, \quad (1)$$
$$P_1 = B_0^{(k)} + B_1^{(k)}t^1 + B_2^{(k)}t^2 + B_3^{(k)}t^3.$$

Here $A_i^{(k)}$, $B_i^{(k)}$ are known constants.

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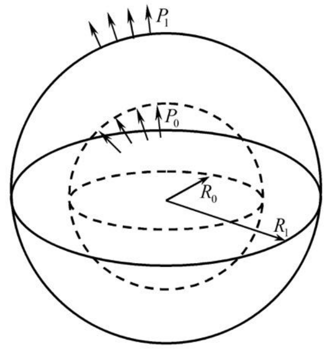

Fig. 1. Thick-walled ball under the action of external and internal dynamic loads

Given the axial symmetry, the defining relations of the problem under consideration are of the form [2]:
equations of motion

$$\frac{\partial \sigma_r}{\partial r}+\frac{2}{r}(\sigma_r-\sigma_\theta)-\rho_0\frac{\partial^2 u}{\partial t^2}=0\ ; \qquad (2)$$

$\rho_0=\rho/\mu$ - dimensionless material density, $u_r=u$;

Cauchy relationship

$$\varepsilon_r=\frac{\partial u}{\partial r}\ ;\ \varepsilon_\theta=\varepsilon_\varphi=\frac{u}{r}\ ; \qquad (3)$$

Hooke's law for stresses in the elastic region

$$\sigma_r=2\varepsilon_r+\lambda_0(\varepsilon_r+2\varepsilon_\theta),$$
$$\sigma_\theta=\sigma_\varphi=2\varepsilon_\theta+\lambda_0(\varepsilon_r+2\varepsilon_\theta), \qquad (4)$$

$\lambda_0=\lambda/\mu$; λ, μ - Lame parameters.

Using the Cauchy relation (3) and Hooke's law (4), we obtain the dependence of stress components on displacements in the elastic region

$$\sigma_r=(2+\lambda_0)\frac{\partial u}{\partial r}+2\lambda_0\frac{u}{r}, \qquad (5)$$
$$\sigma_\theta=\lambda_0\frac{\partial u}{\partial r}+2(1+\lambda_0)\frac{u}{r}.$$

Substituting the dependence (5) into the equations of motion (2), we obtain the second-order partial differential equation for determining the displacements

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}-\frac{2}{r^2}u-\frac{\rho_0}{\lambda_0+2}\frac{\partial^2 u}{\partial t^2}=0. \qquad (6)$$

The boundary conditions of the problem under consideration, taking into account axial symmetry, are written in a dimensionless form as follows.

$$\sigma_r\big|_{r=1}=P_0^*,\ \sigma_r\big|_{r=R_1^*}=P_1^*, \qquad (7)$$
$$P_0^*=P_0/\mu,\ P_1^*=P_1/\mu,\ R_1^*=R_1/R_0.$$

According to (1), we represent the desired function in the form

$$u(r,t)=u_0^{(k)}(r)+u_1^{(k)}(r)t+u_2^{(k)}(r)t^2+u_3^{(k)}(r)t^3. \qquad (8)$$

Substituting (7) into (6) and equating the terms with the same powers of t, we derive a system of equations in ordinary derivatives to determine the components of the displacement vector

$$\frac{\partial^2 u_3^{(k)}}{\partial r^2}+\frac{2}{r}\frac{\partial u_3^{(k)}}{\partial r}-\frac{2}{r^2}u_3^{(k)}=0,$$
$$\frac{\partial^2 u_2^{(k)}}{\partial r^2}+\frac{2}{r}\frac{\partial u_2^{(k)}}{\partial r}-\frac{2}{r^2}u_2^{(k)}=0,$$
$$\frac{\partial^2 u_1^{(k)}}{\partial r^2}+\frac{2}{r}\frac{\partial u_1^{(k)}}{\partial r}-\frac{2}{r^2}u_1^{(k)}-\frac{6\rho_0}{(\lambda_0+2)}u_3^{(k)}=0, \qquad (9)$$
$$\frac{\partial^2 u_0^{(k)}}{\partial r^2}+\frac{2}{r}\frac{\partial u_0^{(k)}}{\partial r}-\frac{2}{r^2}u_0^{(k)}-\frac{2\rho_0}{(\lambda_0-2)}u_2^{(k)}=0.$$

Solving this system, we obtain the values of the components of the displacement vector

$$u_3^{(k)} = d_1^{(k)} r + \frac{d_2^{(k)}}{r^2},\; u_2^{(k)} = d_3^{(k)} r + \frac{d_4^{(k)}}{r^2},$$

$$u_1^{(k)} = d_5^{(k)} r + \frac{d_6^{(k)}}{r^2} + \frac{3\rho_0}{5(2+\lambda_0)}\left(d_1^{(k)} r^3 - 5d_2^{(k)}\right), \tag{10}$$

$$u_0^{(k)} = d_7^{(k)} r + \frac{d_8^{(k)}}{r^2} + \frac{\rho_0}{5(2+\lambda_0)}\left(d_3^{(k)} r^3 - 5d_4^{(k)}\right).$$

Here $d_i^{(k)}$ are the integration constants i = 1..4.

Using the decomposition (1) and relations (5), (9), we obtain the expression for the components of the stress tensor

$$\sigma_{r_3}{}^{(k)} = (2+3\lambda_0)d_1^{(k)} - \frac{4d_2^{(k)}}{r^3},\; \sigma_{r_2}{}^{(k)} = (2+3\lambda_0)d_3^{(k)} - \frac{4d_4^{(k)}}{r^3},$$

$$\sigma_{r_1}{}^{(k)} = (2+3\lambda_0)d_5^{(k)} - \frac{4d_6^{(k)}}{r^3} + \frac{3(6+5\lambda_0)}{5(2+\lambda_0)}\rho_0 r^2 d_1^{(k)} - \frac{6\lambda_0\rho_0}{(2+\lambda_0)r} d_2^{(k)},$$

$$\sigma_{r_0}{}^{(k)} = (2+3\lambda_0)d_7^{(k)} - \frac{4d_8^{(k)}}{r^3} + \frac{6+5\lambda_0}{5(2+\lambda_0)}\rho_0 r^2 d_3^{(k)} - \frac{2\lambda_0\rho_0}{(2+\lambda_0)r} d_4^{(k)},$$

$$\sigma_{\theta_3}{}^{(k)} = (2+3\lambda_0)d_1^{(k)} + \frac{2d_2^{(k)}}{r^3},\; \sigma_{\theta_2}{}^{(k)} = (2+3\lambda_0)d_3^{(k)} + \frac{2d_4^{(k)}}{r^3}, \tag{11}$$

$$\sigma_{\theta_1}{}^{(k)} = (2+3\lambda_0)d_5^{(k)} + \frac{2d_6^{(k)}}{r^3} + \frac{3(2+5\lambda_0)}{5(2+\lambda_0)}\rho_0 r^2 d_1^{(k)} - \frac{6(1+\lambda_0)\rho_0}{(2+\lambda_0)r} d_2^{(k)},$$

$$\sigma_{\theta_0}{}^{(k)} = (2+3\lambda_0)d_7^{(k)} + \frac{2d_8^{(k)}}{r^3} + \frac{2+5\lambda_0}{5(2+\lambda_0)}\rho_0 r^2 d_3^{(k)} - \frac{2(1+\lambda_0)\rho_0}{(2+\lambda_0)r} d_4^{(k)}.$$

To determine the integration constants, we use the boundary conditions (7).

Considering the representation of functions (1) and relations (11), the system of equations for determining the integration constants $d_i^{(k)}$, i = 1..8, k = 1..n-1 takes the form

$$(2+3\lambda_0)d_1^{(k)} - 4d_2^{(k)} = A_3^{(k)},\; (2+3\lambda_0)d_3^{(k)} - 4d_4^{(k)} = A_2^{(k)},$$

$$(2+3\lambda_0)d_5^{(k)} - 4d_6^{(k)} + \frac{3(6+5\lambda_0)}{5(2+\lambda_0)}\rho_0 d_1^{(k)} - \frac{6\lambda_0\rho_0}{2+\lambda_0} d_2^{(k)} = A_1^{(k)},$$

$$(2+3\lambda_0)d_7^{(k)} - 4d_8^{(k)} + \frac{6+5\lambda_0}{5(2+\lambda_0)}\rho_0 d_3^{(k)} - \frac{2\lambda_0\rho_0}{2+\lambda_0} d_4^{(k)} = A_1^{(k)},$$

$$(2+3\lambda_0)d_1^{(k)} - \frac{4d_2^{(k)}}{R_1^{*3}} = B_3^{(k)},\; (2+3\lambda_0)d_3^{(k)} - \frac{4d_4^{(k)}}{R_1^{*3}} = B_2^{(k)}, \tag{12}$$

$$(2+3\lambda_0)d_5^{(k)} - \frac{4d_6^{(k)}}{R_1^{*3}} + \frac{3(6+5\lambda_0)}{5(2+\lambda_0)}\rho_0 R_1^{*2} d_1^{(k)} - \frac{6\lambda_0\rho_0}{(2+\lambda_0)R_1^*} d_2^{(k)} = B_1^{(k)},$$

$$(2+3\lambda_0)d_7^{(k)} - \frac{4d_8^{(k)}}{R_1^{*3}} + \frac{6+5\lambda_0}{5(2+\lambda_0)}\rho_0 R_1^{*2} d_3^{(k)} - \frac{2\lambda_0\rho_0}{(2+\lambda_0)R_1^*} d_4^{(k)} = B_0^{(k)}.$$

Solving the system (12) we get the final form of the integration constants for each of the time intervals $[t_k, t_{k+1}]$, k = 1..n-1.

$$d_1^{(k)} = \frac{B_3^{(k)} R_1^{*3} - A_3^{(k)}}{(2+3\lambda_0)\left(R_1^{*3}-1\right)},\; d_2^{(k)} = \frac{\left(B_3^{(k)} - A_3^{(k)}\right)R_1^{*3}}{4\left(R_1^{*3}-1\right)},\; d_3^{(k)} = \frac{B_2^{(k)} R_1^{*3} - A_2^{(k)}}{(2+3\lambda_0)\left(R_1^{*3}-1\right)},\; d_4^{(k)} = \frac{\left(B_2^{(k)} - A_2^{(k)}\right)R_1^{*3}}{4\left(R_1^{*3}-1\right)},$$

$$d_5^{(k)} = \frac{B_1^{(k)} R_1^{*3} - A_1^{(k)} + 3q_1\left(R_1^{*5}-1\right)d_{11}^{(k)} + 3q_2\left(R_1^{*2}-1\right)d_{12}^{(k)}}{(2+3\lambda_0)\left(R_1^{*3}-1\right)}, \tag{13}$$

$$d_6^{(k)} = \frac{\left(A_1^{(k)} - B_1^{(k)}\right)R_1^* + 3q_1 R_1^*\left(R_1^{*2}-1\right)d_{11}^{(k)} + 3q_2\left(R_1^*-1\right)d_{12}^{(k)}}{4\left(1-R_1^{*3}\right)} R_1^{*2}$$

$$d_7^{(k)} = \frac{B_0^{(k)} R_1^{*3} - A_0^{(k)} + q_1\left(R_1^{*5} - 1\right) d_{13}^{(k)} + q_2\left(R_1^{*2} - 1\right) d_{14}^{(k)}}{\left(2 + 3\lambda_0\right)\left(R_1^{*3} - 1\right)}, \quad d_8^{(k)} = \frac{\left(A_0^{(k)} - B_0^{(k)}\right) R_1^* + q_1 R_1^*\left(R_1^{*2} - 1\right) d_{13}^{(k)} + q_2\left(R_1^* - 1\right) d_{14}^{(k)}}{4\left(1 - R_1^{*3}\right)} R_1^{*2}$$

$$q_1 = \frac{\left(6 + \lambda_0\right)}{5\left(2 + \lambda_0\right)}\rho_0, \quad q_2 = \frac{2\lambda_0\rho_0}{2 + \lambda_0}.$$

Thus, relations (8), (10) - (11), (13) completely determine the components of the stress tensor and displacement vector of the problem under consideration, with arbitrary dynamic loading.

The solution obtained can be used as a zero approximation when solving problems of elastic deformation with a geometry close to an ideal ball. It is also possible to use the relations obtained as a solution in the elastic region of some elastic-viscoplastic problems, or problems with complex rheology.

The results of the numerical experiment are presented in Fig.2 - Fig. 5. The change in the dimensionless displacement u/R_0 in time is shown for $r = \left(1 + R_1 / R_0\right)/2$.

The materials considered in numerical modeling and their physicomechanical parameters are presented in table 1:

Table 1. The values of the physicomechanical parameters for the considered materials [5]

	E/GPa	ν	ρ/(kg/m^3)	λ/GPa	μ/GPa
Stainless steels	189	0.3	7600	109,038	72,692
Aluminum alloys	68	0.32	2700	45,791	25,758
Cooper alloys	112	0.33	8930	81,734	42,105
Titanium alloys	90	0.36	4600	85,084	33,088

Here: E is the modulus of elasticity, ν is the Poisson's ratio, ρ is the density of the material.

To determine the accuracy of the obtained approximate analytical solution, this problem was solved numerically by the method of a centered implicit scheme using the computer algebra system MAPLE 12.

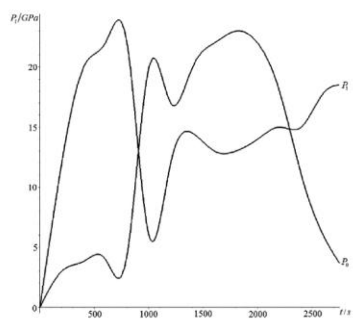

Fig. 2. The change in time of the external P_1 and internal P_0 loads.

Analysis of the obtained results shows that the relative difference between the solutions $\Delta = \frac{\left|u_{\text{approximate}} - u_{\text{numerical}}\right|}{u_{\text{numerical}}}$ is in the interval $\Delta \in [0\ ,\ 0{,}0048]$.

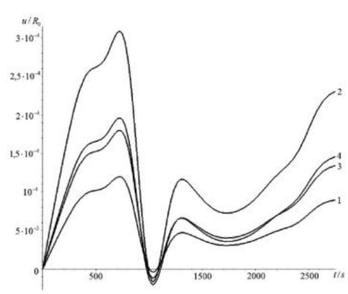

Fig. 3. The change in time of the dimensionless displacement u / R_0, for loads P_0, P_1 (Fig. 2.) for materials: 1 – stainless steels; 2 – aluminum alloys; 3 – cooper alloys; 4 – titanium alloys.

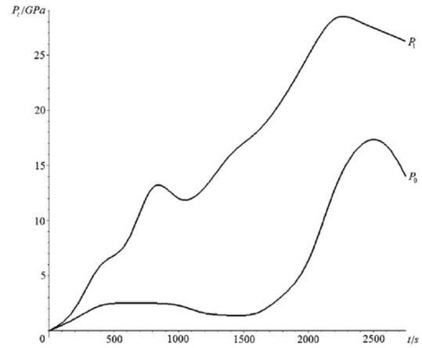

Fig. 4. The change in time of the external P_1 and internal P_0 loads.

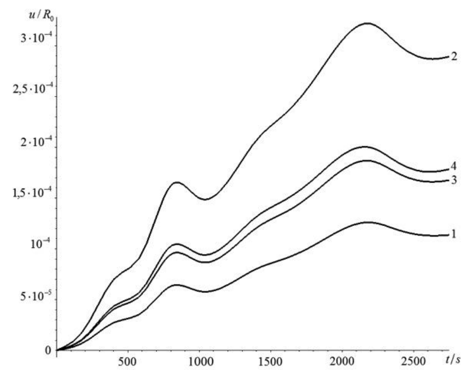

Fig. 5. The change in time of the dimensionless displacement u / R_0, for loads P_0, P_1 (Fig. 4.) for materials: 1 – stainless steels; 2 – aluminum alloys; 3 – cooper alloys; 4 – titanium alloys.

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REFERENCES

[1] ERSHOV, L.V. (1960): On the axisymmetric loss of stability of a thick-walled spherical shell under uniform pressure. **Applied mechanics and technical physics**, 4, 81-82.

[2] SPORYKHIN, A.N. (1997): **The perturbation method in problems of stability of complex environments**. Voronezh. State Univ., Voronezh.

[3] SEMYKINA, T.D. (1963): On the triaxial tension of an elastoplastic space weakened by a spherical cavity. **Izv. Academy of Sciences of the USSR. Mechanics and Engineering**, 1, 17-21.

[4] SPORYKHIN A.N. and SHASHKIN, A.I. (2004): **Equilibrium stability of spatial bodies and problems of rock mechanics**. M. Fizmatlit,

[5] Author? (2003): **Materials. Data. Book**. Cambridge University Engineering Department, Cambridge.