# OPTIMUM COST-TIME TRADE-OFF PAIRS IN A FRACTIONAL PLUS FRACTIONAL CAPACITATED TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

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**ABSTRACT**
This paper provides the optimum cost-time trade-off pairs to the manager of a trading firm, D. M. Chemicals, who deals in the trade of soap stone. The problem of the manager is to determine the quantity (in tons) of soap stone that the firm should purchase from different sellers and sell to the different buyers such that the ratio of actual cartage to standard cartage plus ratio of purchasing cost to profit is minimized provided the demand and supply conditions are satisfied keeping the reserve stocks for emergency situations. Moreover, the manager wishes to minimize the maximum time of transporting goods. The problem under consideration is modeled as a fractional plus linear fractional capacitated transportation problem with restricted flow. The data is taken from the account keeping books of the firm. The solution so obtained by using the developed algorithm is compared with the existing data. Moreover, the solution obtained is verified by a computing software Excel Solver.

**KEYWORDS:** capacitated, transportation problem, trade-off, restricted flow, related transportation problem.
**MSC:** 90C08; 90B06

**RESUMEN**
Este documento proporciona la relación costo-beneficio óptima de los pares al gerente de una empresa comercializadora, D.M Chemicals, que se dedica al comercio de piedra de jabón. El problema del gerente es determinar la cantidad (en toneladas) de jabón piedra que la empresa debe comprar a diferentes vendedores y vender a los diferentes compradores de tal manera que la relación entre el acarreo real y el acarreo estándar más la relación costo de compra se minimice siempre que se satisfagan las condiciones de oferta y demanda manteniendo existencias de reserva para situaciones de emergencia. Además, el gerente desea minimizar el tiempo máximo de transporte de mercancías. El problema bajo consideración se modela como un problema fraccionario más un transporte fraccional lineal con restricciones de capacidad y flujo restringido. Los datos se toman de los libros de contabilidad de la empresa. La solución así obtenida mediante el uso del algoritmo desarrollado se compara con los datos existentes. Además, la solución obtenida es verificada por un software informático Excel Solver.

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**PALABRAS CLAVE:** Problema de transtporte con restriccoines de capacidad, intercambio, flujo restringido, problema de transporte relacionado

## 1. INTRODUCTION

Transportation problems with fractional objective function are widely used as performance measures in many real life situations such as in the analysis of financial aspects of transportation enterprises and undertaking and in transportation management situations where an individual or a group of people is confronted with the hurdle of maintaining good ratios between some important and crucial parameters concerned with the transportation of commodities from certain sources to various destinations. Optimization of a ratio of criteria often describes some kind of an efficiency measure for a system. Fractional objective function include ratio of actual transportation cost to standard transportation cost, total return to total investment, ratio of risk assets to capital, total sales tax to total public expenditure on a commodity, amount of raw material wasted to amount of raw material used, resource allocation problem, routing problem for ships and planes, cargo loading problem, inventory problem, stock cutting problem etc. Sometimes there may exist emergency situations such as fire services, ambulance services, police services etc, when the time of transportation is more important than cost of transportation. This gives rise to time minimization transportation problem. Pandian et al. [11], Gupta et al. [6] and many other researchers have contributed a lot in the field of time minimizing transportation problem. Dan et al. [3] studied paradox in sum of a linear and a linear fractional transportation problem. In 2004, Arora et al. [1] also studied time-cost trade-off pairs in a three dimensional fixed charge indefinite quadratic transportation problem. Xie et al. [15] developed a technique for duration and cost minimization for transportation problem.
If the total flow in a transportation problem with bounds on rim conditions is also specified, the resulting problem makes the transportation problem more realistic. Moreover, if the total capacity of each route is also specified then the optimal solution of such problem is of great importance which gives rise to capacitated transportation problems. Many researchers such as Dahiya et al. [2], Gupta et al. [7, 8] have contributed a lot in the field of capacitated transportation problem. The standard transportation problem is concerned with transporting a homogenous commodity from each of the factories to a number of markets at a minimum cost. Quite frequently, it may so happen that reserve stocks are to be kept at factories for emergencies. This gives rise to restricted flow problem where the total flow is restricted to a known specified level. Khurana et al. [10] have studied restricted flow problems. In today's competitive scenario, demand in the market is highly dynamic and volatile in nature. Production volumes are directly dependent on demand of finished goods which, in turn, determines the order allocation of different parts to suppliers. Demand of finished products varies on continuous basis, making selection of right suppliers a challenging task [14]. Many researchers have worked on the application of different optimization tools to study the real life problems of industry. For instance, Dao et al. [4] proposed an integrated production scheduling model for multi-product orders in virtual computer integrated manufacturing systems. Perez et al. [12] used a data mining algorithm to derive a decision tree that determine the best method for comparison based on the characteristics of Truck and Trailer Routing Problem with fuzzy demands and capacities. Gupta et al. [5] solved the problem of an industry by finding paradox in fractional plus fractional capacitated

transportation problem. Kar [9] used a mixed methodological approach to reinvestigate the vendor selection criteria in the iron and steel industry. Tinnila and Kallio [13] illustrates the challenges of purchasing public services from private markets by analyzing the bus tenders in a European city area with a population of approximately one million.

Motivated by the available literature on minimizing cost and time in a capacitated transportation problem with flow constraints, an attempt has been made to develop an algorithm to find optimum cost-time trade-off pairs when the objective function is the sum of two fractional functions. We apply the developed algorithm on data taken from the account keeping books of a trading firm D. M. Chemicals, Delhi. This firm deals in the trading of soap stone across various states in India. We contacted the manager of the firm and asked him about the business transactions, sellers, buyers, cartage, cost price per unit, selling price per unit etc. The manager told us that the firm wishes to find the cost-time trade off pairs especially when reserve stocks are to be kept for emergencies. Conversation that we had with the manager and the data that we obtain from the books of the firm motivated us to study capacitated transportation problem with restricted flow and apply the developed algorithm on the data of the firm to solve the problem of the manager of the firm.

This paper is organized as : In section 2, problem of the manager of a trading firm D. M. Chemicals, Delhi is described. The data taken from the account keeping books of the firm is shown. In section 3, fractional plus linear fractional capacitated transportation problem with restricted flow is formulated. To solve this restricted flow problem, a related transportation problem is also formed and the two problems are shown to be equivalent. In section 4, optimality criterion for the solution of fractional plus fractional capacitated transportation problem is developed. In section 5, an algorithm is presented to find optimum cost-time trade-off pairs in a fractional plus fractional capacitated transportation problem with restricted flow. In section 6, problem of the manager of D.M Chemicals is formulated and solved by the developed algorithm and in section 7, the solution so obtained is compared with the existing data.

### 1.1. Objective

The primary objective of the present paper is to suggest different shipping schedules to the manager of a trading firm that would provide a trade-off between time and cost, thereby, maximizing the total profit of the firm. The decision maker may choose any of the trade-off pairs depending upon the conditions prevailing in the market. This objective is achieved by formulating the available data as a bi-criterion capacitated transportation problem with fractional plus fractional objective function. Physically, this objective function is the ratio of actual cartage to standard cartage plus the ratio of purchasing cost to profit. In addition to this, time of transporting the goods is also considered in the objective function. This objective is subjected to certain constraints such as bounded demand and supply conditions, bounded decision variables and restricted flow constraint.

## 2. PROBLEM OF THE MANAGER OF THE TRADING FIRM D.M. CHEMICALS, DELHI

D.M. Chemicals is a trading firm which deals in trading of soap stone across various states of India. Books of the firm provides the following information.
The firm purchases soap stone (in tons) from three sellers-

- Shree Shyam Grinding Udyog, RIICO Industrial Area, Ajitgarh, Rajasthan
- Neejal Industries, 16 Duniya village, Halol 389350, District - Panchmahal, Gujarat.
- Kev Minerals, 37, Alindra Malav Road, Ta Kalol District- Panchmahal, Vadodra, Gujarat.

The firm sells its product (in tons) to three buyers-

- Jindal Mechno Bricks Pvt Ltd, VPO - Badli District, Jhajjar, Haryana.
- Poplon Chemie, Jalandhar
- Maheshwari Industries, 73, third cross behind LVK, Kalyan Mandap Kamakshi Pallya, Bangalore.

Goods (soap stone) are supplied by two types of trucks- A large truck that has a maximum capacity of supplying 50 tons of goods in one run and a small truck that has a maximum capacity of supplying 20 tons in one run. But the truck driver will not carry the goods in his truck if the quantity of goods to be supplied is less than 5 tons. D.M chemicals purchases a minimum of 20 tons of soap stone per month from each of the sellers. Moreover, each buyer has a minimum monthly demand of 20 tons of soap stone. Maximum availability of soap stone at Neejal Industries, Shree Shyam grinding Udyog and Kev Minerals is $50, 70$ and $50$ tons respectively. Jindal mechno bricks demanded a maximum of 90 tons of soap stone monthly where as Poplon Chemie and Maheshwari Industries demanded a maximum of 50 and 100 tons of soap stone monthly. Cost price per ton, Selling price per ton, Standard cartage per ton and actual Cartage per ton are shown in Table (1). Time (in hours) of transporting soap stone from different sellers to different buyers is given in Table (2). Sometimes, reserve stocks are to be kept at factories for emergencies. At that time, the firm has to restrict the total flow to 70 tons per month. The manager of the company wishes to determine how many tons of soap stone per month, the firm should purchase from each seller and sell it to the different buyers so that the ratio of cost price to profit plus the ratio of actual cartage to standard cartage is minimum and the reserve stocks may also be kept whenever situation arises. The firm would be benefited if maximum time of transporting goods is also minimized. Data from the books of D.M. Chemicals shows that the firm did the following business transactions-

- Purchased 40 tons of soap stone from Neejal industries and sold it to Jindal mechno bricks.
- Purchased 5 tons of soap stone from Neejal industries and sold it to Poplon Chemie.
- Purchased 5 tons of soap stone from Neejal industries and sold it to Maheshwari Industries.

- Purchased 30 tons of soap stone from Shree Shyam Grinding Udyog and sold it to Jindal mechno bricks.
- Purchased 20 tons of soap stone from Shree Shyam Grinding Udyog and sold it to Poplon Chemie.
- Purchased 20 tons of soap stone from Shree Shyam Grinding Udyog and sold it to Maheshwari Industries.
- Purchased 5 tons of soap stone from Kev Minerals and sold it to Jindal mechno bricks.
- Purchased 5 tons of soap stone from Kev Minerals and sold it to Poplon Chemie.
- Purchased 10 tons of soap stone from Kev Minerals and sold it to Maheshwari Industries.

Total Purchasing cost $= Rs.173450$
Total Profit earned $= Rs.625895$
Actual Cartage paid in the above transactions $= Rs.260250$
Standard Cartage according to the above transactions $= Rs.228500$
$\frac{Costprice}{Profit} + \frac{ActualCartage}{StandardCartage} = \frac{173450}{625895} + \frac{260250}{228500} = Rs.1.416073$
Maximum time of transporting goods in the above transactions is 31 hours.
Cost-time trade-off pair is $(1.416073, 31)$.

## 3. MATHEMATICAL MODEL OF A BI-CRITERION FRACTIONAL PLUS FRACTIONAL CAPACITATED TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

Let $I = \{1, 2, ....., m\}$ be the index set of $m$ origins.
$J = \{1, 2, ......., n\}$ is the index set of $n$ destinations.
$x_{ij}$ =the number of units shipped from the $i^{\text{th}}$ origin to the $j^{\text{th}}$ destination.
$c_{ij}$ = per unit purchasing cost when goods are supplied from the $i^{\text{th}}$ origin to the $j^{\text{th}}$ destination.
$d_{ij}$ = profit per unit earned when goods are supplied from the $i^{\text{th}}$ origin to the $j^{\text{th}}$ destination.
$e_{ij}$ = actual cartage of transporting one unit of a commodity from $i^{\text{th}}$ origin to the $j^{\text{th}}$ destination.
$f_{ij}$ = standard cartage of transporting one unit of a commodity from $i^{\text{th}}$ origin to the $j^{\text{th}}$ destination.
$l_{ij}$ and $u_{ij}$ are the lower and upper bounds on amount of goods transported from the $i^{\text{th}}$ origin to $j^{\text{th}}$ destination.
$a_i$ and $A_i$ are the lower and upper bounds respectively on the availability of goods at origin $i$
$b_j$ and $B_j$ are the lower and upper bounds respectively on the demand of goods by destination $j$
$P$ = Total flow
$t_{ij}$ = time of transporting goods from the $i^{\text{th}}$ origin to the $j^{\text{th}}$ destination.
Consider a fractional plus fractional capacitated transportation problem with restricted flow constraint given by :

$$(P_1) : \min\{\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}} + \frac{\sum_{i\in I}\sum_{j\in J} e_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} f_{ij}x_{ij}}, \max_{i\in I, j\in J}(t_{ij}|x_{ij} > 0)\}$$

Table 1: Cost in rupees(per ton)

| Sellers↓ | buyers → | Jindal mechno | Poplon Chemie | Maheshwari Ind. |
|---|---|---|---|---|
| Neejal Industries | *actual cartage* → | 2800 | 2500 | 2500 |
| | *standard cartage* → | 2000 | 3000 | 3000 |
| | *C.P* → | 1148 | 1148 | 1148 |
| | *S.P* → | 9690 | 10000 | 8000 |
| | *Profit* → | 8542 | 8852 | 6852 |
| Shree Shyam | *actual cartage* → | 600 | 800 | 1500 |
| | *standard cartage* → | 500 | 1000 | 1300 |
| | *C.P* → | 1075 | 1075 | 1075 |
| | *S.P* → | 1836 | 2000 | 3000 |
| | *Profit* → | 761 | 925 | 1925 |
| Kev Minerals | *actual cartage* → | 2850 | 3000 | 3000 |
| | *standard cartage* → | 2000 | 2500 | 3500 |
| | *C.P* → | 2040 | 2040 | 2040 |
| | *S.P* → | 8333 | 9000 | 8000 |
| | *Profit* → | 6293 | 6960 | 5960 |

Table 2: Time in hours

| Sellers↓buyers → | Jindal mechno | Poplon Chemie | Maheshwari Ind. |
|---|---|---|---|
| Neejal Industries | 15 | 20 | 19 |
| Shree Shyam | 8 | 11 | 31 |
| Kev Minerals | 11 | 14 | 30 |

subject to

$$a_i \leq \sum_{j \in J} x_{ij} \leq A_i, \forall i \in I \tag{3.1}$$

$$b_j \leq \sum_{i \in I} x_{ij} \leq B_j, \forall j \in J \tag{3.2}$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \quad \text{and integers}, \quad \forall i \in I, \forall j \in J \tag{3.3}$$

$$\sum_{i \in I} \sum_{j \in J} x_{ij} = P(< \min(\sum_{i \in I} A_i, \sum_{j \in J} B_j)) \tag{3.4}$$

It is assumed that $c_{ij}$, $d_{ij}$, $e_{ij}$, $f_{ij}$, $t_{ij}$, $a_i$, $A_i$, $b_j$, $B_j$, $l_{ij}$ and $u_{ij}$ all are non-negative. Further, it is assumed that $\sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} > 0$ and $\sum_{i \in I} \sum_{j \in J} f_{ij} x_{ij} > 0$ for every feasible solution. The flow constraint given by (3.4) in the problem $(P_1)$ implies that a total of $(\sum_{i \in I} A_i - P)$ of the reserves has to be kept at the various sellers and a total of $(\sum_{j \in J} B_j - P)$ of slacks is to be retained at the various buyers. Therefore, an extra buyer to receive the sellers reserves and an extra seller to fill up the buyers slacks are introduced. In order to solve the problem $(P_1)$, we separate it into two problems $(P_2)$ and $(P_3)$ where

$(P_2)$ : Minimize the cost function $[\frac{\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}}{\sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}} + \frac{\sum_{i \in I} \sum_{j \in J} e_{ij} x_{ij}}{\sum_{i \in I} \sum_{j \in J} f_{ij} x_{ij}}]$ subject to (3.1), (3.2), (3.3) and (3.4)

$(P_3)$ : Minimize the time function $(\max_{i \in I, j \in J} (t_{ij} | x_{ij} > 0))$subject to (3.1), (3.2), (3.3) and (3.4)

In order to solve the problem $(P_2)$, we convert it into related problem $(P_2^{'})$ with an additional seller and an additional buyer. Let $I^{'} = \{1, 2, ......., m, m+1\}$ and $J^{'} = \{1, 2, ......., n, n+1\}$ be the index set of sellers and buyers respectively.

$$(P_2^{'}) : \min\{\frac{\sum_{i \in I^{'}} \sum_{j \in J^{'}} c_{ij}^{'} y_{ij}}{\sum_{i \in I^{'}} \sum_{j \in J^{'}} d_{ij}^{'} y_{ij}} + \frac{\sum_{i \in I^{'}} \sum_{j \in J^{'}} e_{ij}^{'} y_{ij}}{\sum_{i \in I^{'}} \sum_{j \in J^{'}} f_{ij}^{'} y_{ij}}\}$$

subject to

$$\sum_{j \in J^{'}} y_{ij} = A_i^{'}, \forall i \in I^{'} \tag{3.5}$$

$$\sum_{i \in I^{'}} y_{ij} = B_j^{'}, \forall j \in J^{'} \tag{3.6}$$

$$l_{ij} \leq y_{ij} \leq u_{ij} \text{ and integers}, \forall i \in I, \forall j \in J \tag{3.7}$$

$$0 \leq y_{m+1,j} \leq B_j - b_j, \forall j \in J \tag{3.8}$$

$$0 \leq y_{i,n+1} \leq A_i - a_i, \forall i \in I \tag{3.9}$$

$$y_{m+1,n+1} = 0 \tag{3.10}$$

$$A_i^{'} = A_i, \forall i \in I, \ A_{m+1}^{'} = \sum_{j \in J} B_j - P \tag{3.11}$$

$$B_j^{'} = B_j \ \forall j \in J, \ B_{n+1}^{'} = \sum_{i \in I} A_i - P \tag{3.12}$$

$$c_{ij}^{'} = c_{ij}, c_{m+1,j}^{'} = c_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J, \ c_{m+1,n+1}^{'} = M$$

$$d'_{ij} = d_{ij}, d'_{m+1,j} = d'_{i,n+1} = 0; \forall i \in I, \forall j \in J,\ d'_{m+1,n+1} = M$$

$$e'_{ij} = e_{ij}, e'_{m+1,j} = e'_{i,n+1} = 0; \forall i \in I, \forall j \in J,\ e'_{m+1,n+1} = M$$

$$f'_{ij} = f_{ij}, f'_{m+1,j} = f'_{i,n+1} = 0; \forall i \in I, \forall j \in J,\ f'_{m+1,n+1} = M$$

where M is a large positive integer.

Similarly, to solve problem $(P_3)$, we convert it into related problem $(P'_3)$ given below.

$$(P'_3) : \min T = \max\{t'_{ij}|y_{ij}\} > 0, i \in I', j \in J'\}$$

subject to equation (3.5) to (3.12) such that $t'_{ij} = t_{ij}, t'_{m+1,j} = t'_{i,n+1} = 0; \forall i \in I, \forall j \in J$ and $t'_{m+1,n+1} = 0$. Now, we will show that problems $(P_2)$ and $(P'_2)$ are equivalent. This can be shown by the following set of definitions and theorems.

**Definition 3.1** ***Corner feasible solution:*** *A basic feasible solution $\{y_{ij}\}, i \in I', j \in J'$ to problem $(P'_2)$ is called a corner feasible solution $(cfs)$ if $y_{m+1,n+1} = 0$.*

**Theorem 3.2** *A non-corner feasible solution of problem $(P'_2)$ cannot provide a basic feasible solution to problem $(P_2)$. [8]*

**Lemma 3.3** *There is a one-to-one correspondence between the feasible solution to problem $(P_2)$ and the corner feasible solution to problem $(P'_2)$. [8]*

**Remark 3.4** *If problem $(P'_2)$ has a $cfs$ ,then since $c'_{m+1,n+1} = M = d'_{m+1,n+1} = e'_{m+1,n+1} = f'_{m+1,n+1}$, it follows that non corner feasible solution cannot be an optimal solution of problem $(P_2)$.*

**Lemma 3.5** *The value of the objective function of problem $(P_2)$ at a feasible solution $\{x_{ij}\}_{I\times J}$ is equal to the value of the objective function of problem $(P'_2)$ at its corresponding $cfs$ $\{y_{ij}\}_{I'\times J'}$ and conversely. [5]*

**Lemma 3.6** *There is a one-to-one correspondence between the optimal solution to problem $(P_2)$ and optimal solution among the corner feasible solution to problem $(P'_2)$. [8]*

**Theorem 3.7** *Optimizing problem $(P'_2)$ is equivalent to optimizing problem $(P_2)$ provided problem $(P_2)$ has a feasible solution. [8]*

## 4. OPTIMALITY CRITERIA FOR A FRACTIONAL PLUS FRACTIONAL CAPACITATED TRANSPORTATION PROBLEM

**Theorem 4.1** *[5] Let $X^0 = \{x^0_{ij}\}_{I'\times J'}$ be the feasible solution of problem $(P'_2)$. Let $C^0 = \sum_{i\in I'}\sum_{j\in J'} c'_{ij}x^0_{ij}$; $D^0 = \sum_{i\in I'}\sum_{j\in J'} d'_{ij}x^0_{ij}$; $E^0 = \sum_{i\in I'}\sum_{j\in J'} e'_{ij}x^0_{ij}$; $F^0 = \sum_{i\in I'}\sum_{j\in J'} f'_{ij}x^0_{ij}$. Let $B$ be the set of cells $(i, j)$ which* *are basic and $N_1$ and $N_2$ denotes the set of non- basic cells $(i,j)$ which are at their lower bounds and upper bounds respectively. Let $u_i^1, u_i^2, u_i^3, u_i^4, v_j^1, v_j^2, v_j^3, v_j^4; i \in I^{'}, j \in J^{'}$ be the dual variables such that $u_i^1 + v_j^1 = c_{ij}^{'}, \forall(i,j) \in B; u_i^2 + v_j^2 = d_{ij}^{'}, \forall(i,j) \in B; u_i^3 + v_j^3 = e_{ij}^{'}, \forall(i,j) \in B; u_i^4 + v_j^4 = f_{ij}^{'}, \forall(i,j) \in B; u_i^1 + v_j^1 = z_{ij}^1, \forall(i,j) \notin B; u_i^2 + v_j^2 = z_{ij}^2, \forall(i,j) \notin B; u_i^3 + v_j^3 = z_{ij}^3, \forall(i,j) \notin B; u_i^4 + v_j^4 = z_{ij}^4, \forall(i,j) \notin B$. Then a feasible solution $X^0 = \{x_{ij}^0\}_{I' \times J'}$ of problem $(P_2^{'})$ with objective function value $\frac{C^0}{D^0} + \frac{E^0}{F^0}$ will be an optimal solution if and only if the following conditions holds:*

$$\delta_{ij}^1 = \frac{\theta_{ij}[D^0(c_{ij}^{'} - z_{ij}^1) - C^0(d_{ij}^{'} - z_{ij}^2)]}{D^0[D^0 + \theta_{ij}(d_{ij}^{'} - z_{ij}^2)]} + \frac{\theta_{ij}[F^0(e_{ij}^{'} - z_{ij}^3) - E^0(f_{ij}^{'} - z_{ij}^4)]}{F^0[F^0 + \theta_{ij}(f_{ij}^{'} - z_{ij}^4)]} \geq 0; \forall(i,j) \in N_1 \quad (4.1)$$

$$\delta_{ij}^2 = \frac{-\theta_{ij}[D^0(c_{ij}^{'} - z_{ij}^1) - C^0(d_{ij}^{'} - z_{ij}^2)]}{D^0[D^0 - \theta_{ij}(d_{ij}^{'} - z_{ij}^2)]} - \frac{\theta_{ij}[F^0(e_{ij}^{'} - z_{ij}^3) - E^0(f_{ij}^{'} - z_{ij}^4)]}{F^0[F^0 - \theta_{ij}(f_{ij}^{'} - z_{ij}^4)]} \geq 0; \forall(i,j) \in N_2 \quad (4.2)$$

## 5. ALGORITHM

**Step 1:** Given problem $(P_1)$, separate the problem $(P_1)$ in to two problems $(P_2)$ and $(P_3)$. Form the related transportation problems $(P_2^{'})$ and $(P_3^{'})$.
**Step 2:** Find an initial basic feasible solution to the problem $(P_2^{'})$. Let $B$ be its corresponding basis.
**Step 3:** Calculate dual variables $u_i^1, u_i^2, u_i^3, u_i^4, v_j^1, v_j^2, v_j^3, v_j^4; i \in I^{'}, j \in J^{'}$ by using the equations given below and taking one of the $u_i^{'}s$ *or* $v_j^{'}s$ as zero.
$u_i^1 + v_j^1 = c_{ij}^{'}; u_i^2 + v_j^2 = d_{ij}^{'}; u_i^3 + v_j^3 = e_{ij}^{'}; u_i^4 + v_j^4 = f_{ij}^{'}, \forall(i,j) \in B$
$u_i^1 + v_j^1 = z_{ij}^1; u_i^2 + v_j^2 = z_{ij}^2; u_i^3 + v_j^3 = z_{ij}^3; u_i^4 + v_j^4 = z_{ij}^4, \forall(i,j) \in N_1$ and $N_2$.
$N_1$ and $N_2$ denotes the set of non- basic cells $(i,j)$ which are at their lower bounds and upper bounds respectively.
**Step 4:** Calculate $\theta_{ij}, c_{ij}^{'} - z_{ij}^1; d_{ij}^{'} - z_{ij}^2; e_{ij}^{'} - z_{ij}^3; f_{ij}^{'} - z_{ij}^4; \forall i \in I^{'}, j \in J^{'}$ for all non- basic cells and also calculate $C^0 = \sum_{i \in I'} \sum_{j \in J'} c_{ij}^{'} y_{ij}; D^0 = \sum_{i \in I'} \sum_{j \in J'} d_{ij}^{'} y_{ij}; E^0 = \sum_{i \in I'} \sum_{j \in J'} e_{ij}^{'} y_{ij}; F^0 = \sum_{i \in I'} \sum_{j \in J'} f_{ij}^{'} y_{ij}$.
**Step 5:** Calculate $\delta_{ij}^1$ and $\delta_{ij}^2$ given by (4.1) and (4.2) respectively. If $\delta_{ij}^1 \geq 0; \forall(i,j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall(i,j) \in N_2$, then the current solution so obtained is the optimal solution to problem $(P_2^{'})$ and subsequently to problem $(P_2)$. Then go to step 6. Otherwise some $(i,j) \in N_1$ for which $\delta_{ij}^1 \leq 0$ or some $(i,j) \in N_2$ for which $\delta_{ij}^2 \leq 0$ will enter the basis. Go to step 3.
**Step 6:** Find the optimal cost $Z_1 = \frac{C^0}{D^0} + \frac{E^0}{F^0}$ yielded by the basic feasible solution $\{y_{ij}\}$. Read the time with respect to the minimum cost $Z_1$ where time $T_1$ is given by problem $(P_3^{'})$. Find all alternate solutions to the problem $(P_2^{'})$ with the same value of the objective function. Let these solutions be $X_1, X_2, .......X_n$. Corresponding to these solutions, find the time $T_1^* = \min_{X_1, X_2, .......X_n} \max_{i \in I^{'}, j \in J^{'}} \{t_{ij}^{'} | y_{ij}\} > 0\}$. Then $(Z_1^*, T_1^*)$ is called the first cost-time trade-off pair. Go to step 7.
**Step 7:** Now set $y_{ij} = l_{ij}$ if $t_{ij}^{'} \geq T_1^*$ in problem $(P_2^{'})$ and find the optimal solution to this new problem. Also find all feasible alternate solutions. Let the new value of $Z$ be $Z_2^*$ and read the time with respect to this new solution leaving the cell for $t_{ij}^{'} \geq T_1^*$. Let the corresponding time is $T_2^*$, then $(Z_2^*, T_2^*)$ is the second cost-time trade-off pair. Repeat this process. Suppose that after $q^{th}$ iteration, the problem becomes infeasible. Thus, we get the complete set of cost-time trade-off pairs as $(Z_1^*, T_1^*)$, $(Z_2^*, T_2^*)$,........$(Z_q^*, T_q^*)$ where $Z_1^* \geq Z_2^* \geq Z_3^* \geq ....... \geq Z_q^*$ and $T_1^* < T_2^* < T_3^* < ........ < T_q^*$. If $Z_r^* = Z_s^*$ and $T_r^* > T_s^*$ for any two pairs then drop the pair $(Z_r^*, T_r^*)$ as the decision maker will surely

choose the pair with less cost and less time.

**Remark 5.1** *The algorithm will terminate after a finite number of steps because we are moving from one extreme point to another extreme point and the problem becomes infeasible after a finite number of steps.*

## 6. MATHEMATICAL MODEL OF THE PROBLEM OF THE MANAGER OF D.M. CHEMICALS, DELHI

**Assumptions**

- Shortages are not allowed.
- None of the sellers will refuse to supply goods.
- Quantity of soap stone is a positive integer.
- Loss in weight due to powdery nature of soap stone is negligible.

Let the three sellers- Neejal Industries, Shree Shyam Grinding Udyog, Kev Minerals be denoted by three origins given by $O_1, O_2$ and $O_3$ respectively. Let the three buyers- Jindal mechno bricks, Poplon Chemie, Maheshwari Industries be denoted by three destinations given by $D_1, D_2$ and $D_3$ respectively.
Let $I = \{1, 2, 3\}$ be the index set of 3 sellers.
$J = \{1, 2, 3\}$ is the index set of 3 buyers.
$x_{ij}$ = quantity of soap stone (in tons) purchased from the $i^{\text{th}}$ seller and sold to $j^{\text{th}}$ buyer.
$a_i$ = Minimum quantity of soap stone (in tons) available (per month) at seller $i$
$A_i$ = Maximum availability of soap stone at seller $i$
$b_j$ = Minimum quantity of soap stone (in tons) demanded (per month) by buyer $j$
$B_j$ = Maximum quantity of soap stone demanded by buyer $j$
$c_{ij}$ = cost price paid per ton when soap stone is purchased from the $i^{\text{th}}$ seller and sold to the $j^{\text{th}}$ buyer.
$d_{ij}$ = profit per ton from the $j^{\text{th}}$ buyer when the soap stone purchased from the $i^{\text{th}}$ seller are supplied.
$e_{ij}$ = actual cartage paid per ton when soap stone is purchased from the $i^{\text{th}}$ seller and sold to the $j^{\text{th}}$ buyer.
$f_{ij}$ = standard cartage per ton when soap stone is purchased from the $i^{\text{th}}$ seller and sold to the $j^{\text{th}}$ buyer.
$l_{ij}$ = minimum quantity of soap stone (in tons) that can be supplied by the large and small truck from $i^{\text{th}}$ seller to $j^{\text{th}}$ buyer.
$u_{ij}$ = maximum quantity of soap stone (in tons) that can be supplied by the large and small truck from $i^{\text{th}}$ seller to $j^{\text{th}}$ buyer.
$P$ = Total flow of the soap stone.
$t_{ij}$ = time of transporting soap stone from $i^{\text{th}}$ seller to the $j^{\text{th}}$ buyer.

The problem under consideration can be formulated mathematically as a fractional plus fractional capacitated transportation problem with restricted flow constraint as-

$$(P_1):\min\{\frac{\sum_{i=1}^{3}\sum_{j=1}^{3}c_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{3}d_{ij}x_{ij}}+\frac{\sum_{i=1}^{3}\sum_{j=1}^{3}e_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{3}f_{ij}x_{ij}},\max_{i,j=1,2,3}(t_{ij}|x_{ij}>0)\}$$

subject to

$$20\le\sum_{j=1}^{3}x_{1j}\le 50;20\le\sum_{j=1}^{3}x_{2j}\le 70;20\le\sum_{j=1}^{3}x_{3j}\le 50$$

$$20\le\sum_{i=1}^{3}x_{i1}\le 90;20\le\sum_{i=1}^{3}x_{i2}\le 50;20\le\sum_{i=1}^{3}x_{i3}\le 100$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3}x_{ij}=P(<min(\sum_{i=1}^{3}A_i=170,\sum_{j=1}^{3}B_j=240))$$

$$5\le x_{11}\le 50;5\le x_{12}\le 20;5\le x_{13}\le 50$$

$$5\le x_{21}\le 50;5\le x_{22}\le 20;5\le x_{23}\le 50$$

$$5\le x_{31}\le 50;5\le x_{32}\le 20;5\le x_{33}\le 50$$

Figures for cost per ton, Profit per ton, actual cartage per ton, standard cartage per ton denoted by $c_{ij}, d_{ij}, e_{ij}, f_{ij}$ and time denoted by $t_{ij}$ are given in table (3). In order to solve the problem $(P_1)$, we separate it into two problems $(P_2)$ and $(P_3)$. Then introduce a dummy source and a dummy destination in problem $(P_2)$ and $(P_3)$ to covert them into problem $(P_2^{'})$ and $(P_3^{'})$ with $c_{i4}=0=d_{i4}=e_{i4}=f_{i4}\ for\ i=1,2,3.$ and $c_{4j}=0=d_{4j}=e_{4j}=f_{4j}\ for\ j=1,2,3.$ and $c_{44}=d_{44}=e_{44}=f_{44}=M$ where $M$ is a large positive integer. Also, $0\le y_{14}\le 30;0\le y_{24}\le 50;0\le y_{34}\le 30;0\le y_{41}\le 70;0\le y_{42}\le 30;0\le y_{43}\le 80;y_{44}=0;A_1^{'}=50;A_2^{'}=70;A_3^{'}=50;A_4^{'}=\sum_{j=1}^{3}B_j-P=240-70=170;B_1^{'}=90;B_2^{'}=50;B_3^{'}=100;B_4^{'}=\sum_{i=1}^{3}A_i-P=170-70=100$. Also, $t_{14}=t_{24}=t_{34}=t_{41}=t_{42}=t_{43}=t_{44}=0$. Find an initial basic feasible solution to the $(P_2^{'})$ so formed. This solution is shown in Table (4) and is tested for optimality. In Table (4), $C^0=96740;D^0=401735;E^0=153250;F^0=159000$

Since in table (5), $\delta_{ij}^1\ge 0;\forall(i,j)\in N_1$ and $\delta_{ij}^2\ge 0;\forall(i,j)\in N_2$ , therefore the solution in table (4)is an optimal solution of problem $(P_2^{'})$ and hence yields an optimal solution of $(P_2)$ with minimum cost $=Z_1=\frac{96740}{401735}+\frac{153250}{159000}=1.204642$. Read the time in the time matrix given in Table (2) corresponding to this optimal solution. We get $T_1=31$. After considering all alternate optimum solutions, we get the first cost-time trade-off pair as $(1.204642,31)$. Now we set $y_{23}=l_{23}=5$ where time $t_{23}=31$ and again

Table 3: Cost(per ton), bounds and time (in hours)

| Sellers↓ | buyers → | Jindal mechno | Poplon Chemie | Maheshwari Ind. |
|---|---|---|---|---|
| Neejal Ind. | $(c_{ij}, d_{ij}) \rightarrow$ | (1148,8542) | (1148,8852) | (1148,6852) |
| | $(e_{ij}, f_{ij}) \rightarrow$ | (2800,2000) | (2500,3000) | (2500,3000) |
| | $t_{ij} \rightarrow$ | 15 | 20 | 19 |
| Shree Shyam | $(c_{ij}, d_{ij}) \rightarrow$ | (1075,761) | (1075,925) | (1075,1925) |
| | $(e_{ij}, f_{ij}) \rightarrow$ | (600,500) | (800,1000) | (1500,1300) |
| | $t_{ij} \rightarrow$ | 8 | 11 | 31 |
| Kev Minerals | $(c_{ij}, d_{ij}) \rightarrow$ | (2040,6293) | (2040,6960) | (2040,5960) |
| | $(e_{ij}, f_{ij}) \rightarrow$ | (2850,2000) | (3000,2500) | (3000,3500) |
| | $t_{ij} \rightarrow$ | 11 | 14 | 30 |

Table 4: Corner feasible solution of problem $(P_2^{'})$

| $x_{ij}$ | D1 | D2 | D3 | D4 | $u_i^1$ | $u_i^2$ | $u_i^3$ | $u_i4$ |
|---|---|---|---|---|---|---|---|---|
| $O1$ | $\underline{5}$ | $\overline{20}$ | $\underline{5}$ | **20** | 2040 | 5960 | 3000 | 3500 |
| $O2$ | **10** | $\underline{5}$ | $\underline{5}$ | $\overline{50}$ | 1075 | 761 | 600 | 500 |
| $O3$ | $\underline{5}$ | $\underline{5}$ | **10** | **30** | 2040 | 5960 | 3000 | 3500 |
| $O4$ | **70** | **20** | **80** | $\underline{0}$ | 0 | 0 | 0 | 0 |
| $v_j^1$ | 0 | 0 | 0 | -2040 | | | | |
| $v_j^2$ | 0 | 0 | 0 | -5960 | | | | |
| $v_j^3$ | 0 | 0 | 0 | -3000 | | | | |
| $v_j^4$ | 0 | 0 | 0 | -3500 | | | | |

Notes.Entries of the form $\underline{a}$ and $\overline{b}$ represent non- basic cells which are at their lower and upper bounds respectively. Entries in bold are basic cells.

solve the new problem as before. We find that the second trade-off pair is $(1.204642, 30)$. Since the cost is same in both first and second trade-off pairs but the time in first trade-off pair is more than the time in second trade-off pair, therefore, the manager will surely choose the second pair with less time. So we will drop the first pair and consider $(1.204642, 30)$ as first trade-off pair. Proceeding like this, we get the second cost-time trade-off pair as $(1.23897, 20)$ with $y_{11} = 5, y_{12} = 15, y_{13} = 10, y_{14} = 20, y_{21} = 10, y_{22} = 5, y_{23} = 5, y_{24} = 50, y_{31} = 5, y_{32} = 10, y_{33} = 5, y_{34} = 30, y_{41} = 70, y_{42} = 20, y_{43} = 80$. Third cost-time trade-off pair is $(1.25732, 19)$ with $y_{11} = 5, y_{12} = 5, y_{13} = 19, y_{14} = 21, y_{21} = 10, y_{22} = 6, y_{23} = 5, y_{24} = 49, y_{31} = 5, y_{32} = 10, y_{33} = 5, y_{34} = 30, y_{41} = 70, y_{42} = 29, y_{43} = 71$. After that the problem becomes infeasible and algorithm terminates here. We also verified this optimal solution by using a computing software Excel Solver.

Table 5: Computation of $\delta^1_{ij}$ and $\delta^2_{ij}$

| $NB$ | $O1D1$ | $O1D2$ | $O1D3$ | $O2D2$ | $O2D3$ | $O2D4$ | $O3D1$ | $O3D2$ |
|---|---|---|---|---|---|---|---|---|
| $\theta_{ij}$ | 0 | 10 | 0 | 0 | 0 | 0 | 0 | 0 |
| $c^{'}_{ij} - z^1_{ij}$ | -892 | -892 | -892 | 0 | 0 | 965 | 0 | 0 |
| $d^{'}_{ij} - z^2_{ij}$ | 2582 | 2892 | 892 | 164 | 1164 | 5199 | 333 | 1000 |
| $e^{'}_{ij} - z^3_{ij}$ | -200 | -500 | -500 | 200 | 900 | 2400 | -150 | 0 |
| $f^{'}_{ij} - z^4_{ij}$ | -1500 | -500 | -500 | 500 | 800 | 3000 | -1500 | -1000 |
| $\delta^1_{ij}$ and $\delta^2_{ij}$ | 0 | 0.037986 | 0 | 0 | 0 | 0 | 0 | |

## 7. COMPARISON WITH THE EXISTING DATA

In section 2, we have shown that the data from the account keeping books of the firm shows that cost-time trade-off pair is $(1.416073, 31)$. The trade-off pairs obtained by the developed algorithm are $(1.204642, 30), (1.23897, 20), (1.25732, 19)$ . Therefore, the firm would save $\frac{1.416073-1.204642}{1.416073} \times 100 =$ 14.93% and time $=31 - 30 = 1$ hour if the business transactions are done according to first trade off pair. Similarly, the firm would save 12.5% and time of 11 hours if the manager choose second trade off pair and savings would be 11.2% and 12 hours if business transactions are done according to third cost-time trade-off pair. It is up to the manager of the firm to choose any of the trade-off pair according to prevailing business circumstances.

## 8. CONCLUSION

The data taken from the accounts keeping books of a firm is studied and the problem of the manager of the firm is formulated as a fractional plus fractional capacitated transportation problem with restricted flow. In order to find the optimum cost-time trade-off pairs in a fractional plus fractional capacitated transportation problem with restricted flow, we first separate the given problem into two problems. Then a related transportation problem is formulated to solve the cost minimization problem which possesses a corner feasible solution. Optimal solution to cost minimization problem can be determined from optimal corner feasible solution to related transportation problem. Then the corresponding time is read from the time minimization problem to form the trade off pair. After that the cost problem is modified and solved again to find the next trade off pair. This process continues till the problem gets infeasible. In this way, we get various cost-time trade-off pairs in a fractional plus fractional capacitated transportation problem with restricted flow. Existing data is compared with the solution obtained using the proposed method and it is found that the firm could save in terms of cost and time both if they would follow the transportation schedule using any of the trade-off pairs obtained by using the developed model.

As future work, it is intended to apply proposed algorithm to a sum of n fractional functions when the decision variables are bounded. Moreover, the developed algorithm can also be applied in a solid fixed charge capacitated transportation problem, indefinite quadratic transportation problem, fuzzy transportation problem, with or without flow constraint. Developed algorithms can be applied on the problems of the real world.

## 9. DECLARATION

I am not getting any financial support from any funding agency and no one is having any competing interest in this paper.

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