THE RAYLEIGH LINDLEY DISTRIBUTION: A NEW GENERALIZATION OF RAYLEIGH DISTRIBUTION WITH PHYSICAL APPLICATIONS

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ABSTRACT
In this paper, we introduce a new generalized form of Rayleigh distribution which offers a more flexible and reliable model for lifetime data. Several statistical properties of the new distribution are explored. Density expansion, moments, order statistics, information measures, Bonferroni and Lorenz curves, hazard and reverse hazard functions are studied. Furthermore, we obtain model parameters estimation using the maximum likelihood and real data applications which illustrate the performance of the distribution and its excellence over other distributions based on some information criterion are also given.

KEYWORDS: Rayleigh distribution; Hazard function; Order statistics; Maximum likelihood estimation; Information measures.
MSC: 62F10; 62F15; 62F25; 62G30.

RESUMEN
En este trabajo se introduce una nueva forma generalizada de la distribución de Rayleigh que ofrece un modelo más flexible y confiable para describir datos del tipo tiempo de vida. Se exploran las propiedades estadísticas de esta nueva distribución como son la expansión de densidad, los momentos, estadísticos de orden, medidas de información las curvas de Bonferroni y de Lorenz, y las funciones de riesgo y de riesgo reverso. Además se obtienen los modelos de estimación de parámetros usando máxima verosimilitud. Se muestran los resultados correspondientes a datos que surgen de aplicaciones reales. Los resultados obtenidos ilustran como la nueva distribución supera las existentes de acuerdo a ciertos criterios de información

PALABRAS CLAVE: Distribución de Rayleigh, función de riesgo, estadístico de orden, estimación máximo verosímil, medidas de información.

1. INTRODUCTION

The amount of data available for analysis has been growing increasingly, requiring new statistical distributions that enables us to better describe each phenomenon or experiment under study. Defining these new distributions is a very significant problem in statistics when a researcher aims to predict more accurate future behaviors of the data based on an observed set of data. Many attempts have been made by several authors to define new distributions or new families of distributions to provide more flexibility in modeling data under investigation. One such example is a family of univariate

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distributions generated by Stacy's generalized gamma variable proposed by Zografos and Balakrishnan (2008) [38].
The Rayleigh distribution was introduced by Rayleigh in 1880 whose probability density function (pdf) of the Rayleigh distribution has the form

$$h(x;\alpha)=\frac{x}{\alpha^2}e^{-\frac{x^2}{2\alpha^2}}, \qquad x>0, \tag{1.1}$$

where $\alpha>0$ is a scale parameter. The cumulative distribution function (cdf) of Rayleigh distribution is

$$H(x)=1-e^{-\frac{x^2}{2\alpha^2}}. \tag{1.2}$$

Rayleigh distribution has several desirable properties and nice physical interpretations. It is commonly used to analyze lifetime data, the reader is referred to Johnson et al. (1994) [18], for more details. The origin and other aspects of this distribution can be found in Siddiqui (1962) [33], and Miller and Sackrowttz (1967) [26]. Unfortunately, Rayleigh distribution has an increasing failure rate and hence many authors are interested in defining new generalized families of Rayleigh distributions. This provides greater flexibility in modeling skewed lifetime data. Adding parameters to a well established distribution is an effective way for obtaining more flexible new distributions. Kumaraswamy (1980) [19] has introduced a generalization of Beta distribution, where he has found that the new one was much better suited than Beta distribution. Later in 1997, a new method has been proposed by Marshall and Olkin [25], their idea of obtaining a new distribution has based on adding a new parameter to the original distribution. The new family of distributions includes the original distribution as a special case, and it gives more flexibility to the original model. Eugene et al. (2002) [12], Zografos and Balakrishnan (2009) [38], Bdair (2012) [5], Ristic and Balakrishnan (2011) [31], Cordeiro and de Castro (2011) [9], Cordeiro et al. (2013) [10], Bdair and Haj Ahmad (2019) [6], Andrade et al. (2015) [4] and Haj Ahmad et al. (2017) [16] are some authors who have studied families of generalized distributions.
Some earlier studies have considered general forms of Rayleigh distribution, Surles and Padgett (2001, 2004) [35, 36] have introduced two parameter generalized Rayleigh distribution, which was a particular member of the exponentiated Weibull distribution. Kundu and Raqab (2005) [20] have studied different estimation methods for a generalized Rayleigh distribution. Abu Awwad et al. (2018) [2] have studied the prediction of progressively censored Rayleigh distribution.
Our work is based on introducing a generalized form of Rayleigh distribution by replacing x with $\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\beta}$ in the cdf formula, where ξ is the vector of parameters of the baseline distribution with cdf G, and survival function $\overline{G}=1-G$. By using a suitable baseline distribution, we improve the original distribution and make it more flexible and reliable to some real life data. Bourguignon et al.(2014) [8] have used a special case of this generalization method, they introduced a new family of univariate distributions with two additional parameters using the Weibull generator applied to the odds ratio $G(x)/(1-G(x))$. For more details about different methods in generating families of continuous distributions, the reader may refer to Lee et al. (2013) [21]. In our study, without loss of generality, we consider $\beta=1$ and hence the cdf for the new Rayleigh generalization is:

$$\begin{aligned} F_{RG}(x;\alpha,\xi) &= \int_0^{\frac{G(x;\xi)}{\overline{G}(x;\xi)}} \frac{t}{\alpha^2}e^{-\frac{t^2}{2\alpha^2}}\,dt \\ &= 1-\exp\left[\frac{-1}{2\alpha^2}\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]. \end{aligned} \tag{1.3}$$

The pdf of this new generalization is

$$f_{RG}(x;\alpha,\xi)=\frac{1}{\alpha^2}\frac{G(x;\xi)}{\overline{G}^3(x;\xi)}g(x;\xi)\exp\left[\frac{-1}{2\alpha^2}\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right], \tag{1.4}$$

where $g(x;\xi)$ is the pdf of the baseline distribution. Equation (1.4) can be more tractable when $G(x;\xi)$ and $g(x;\xi)$ have simple analytic expressions. The random variable X with density function (1.4) is denoted by $X \sim RG(\alpha,\xi)$. To define this mixed distribution, we consider Lindley distribution as a baseline distribution. Although Lindley distribution has an increasing failure rate, our suggested mixture between Rayleigh and Lindley distributions has also an increasing hazard rate function with increasing-decreasing density.

The rest of the paper is organized as follows. In Section 2, Rayleigh-Lindley distribution is defined, expansion formula for its density function is obtained and its monotonicity property is studied and hazard and reverse hazard rate are derived. Moments are obtained in Section 3 and Reliability is presented in Section 4. In Section 5, we derive the quantile function of the new proposed distribution and an algorithm to generate the random data is presented. In Section 6, we discuss the maximum likelihood estimation and the use a Monte Carlo simulation experiment to evaluate the maximum likelihood estimates (MLEs) of the model parameters. Benferroni and Lorenz curves are studied in Section 7. Order statistics and measures of uncertainty are discussed in Section 8. Section 9 presents two real life examples which are used as applications on Rayleigh-Lindley distribution. Finally, some concluding remarks are made in Section 10.

2. RAYLEIGH-LINDLEY DISTRIBUTION

Lindley distribution is a well known distribution that can be used in a wide variety of fields, including biology, engineering and medicine; see Ghitany et al. (2008) [14]. Lindley distribution is a mixture of exponential (θ) and gamma $(2,\theta)$ distributions. Although the survival function of gamma distribution can not be expressed in closed form, and the hazard rate of exponential distribution is constant, Lindley distribution has an advantage since its hazard rate is increasing.

Lindley distribution has been generalized, extended, and its applications in reliability and other fields of science have been introduced by several authors. One may refer to Hussain (2006) [17], Zakerzadeh and Dolati (2009) [37], Nadarajah et al. (2011) [27], Deniz and Ojeda (2011) [11], Ghitany et al. (2013) [14], Oluyede and Yang (2015) [29], Alkarni (2015) [3] and Abouammoh et al. (2015) [1] and the references therein.

Even though many researchers have used Lindley distribution to model lifetime data, such as Hussain (2006) [17], who has showed that the Lindley distribution is important for studying stress-strength reliability modeling, there are many situations in the modeling of real lifetime data where the Lindley distribution may not be suitable from a theoretical or applied point of view. Therefore, it is necessary to obtain a new distribution which is more flexible than the Lindley distribution for modeling lifetime data. Here we introduce a new distribution by considering a mixture of Rayleigh and Lindley distributions. In this section we assume that the baseline distribution is Lindley distribution given to be Equation (1.4) whose pdf and cdf are

$$g(x;\theta)=\frac{\theta^2}{(1+\theta)}(1+x)e^{-\theta x}, \qquad x>0, \tag{2.1}$$

$$G(x;\theta)=1-\frac{(1+\theta+\theta x)}{1+\theta}e^{-\theta x}, \tag{2.2}$$

respectively, where $\theta>0$ is a scale parameter.

Using $g(x;\theta)$ and $G(x;\theta)$ defined in Equations (2.1) and (2.2) and substituting them in Equation (1.4), we obtain the density of Rayleigh-Lindley distribution (RL)

$$\begin{aligned} f_{RL}(x;\alpha,\theta) &= \frac{\theta^2}{\alpha^2}(\theta+1)(x+1)e^{\theta x}\left[\frac{e^{\theta x}(\theta+1)-(1+\theta+\theta x)}{(1+\theta+\theta x)^3}\right] \\ &\quad \times \exp\left[\frac{-1}{2\alpha^2}\left(\frac{(1+\theta)e^{\theta x}}{(1+\theta+\theta x)}-1\right)^2\right], \qquad x>0, \end{aligned} \tag{2.3}$$

where $\alpha>0$, $\theta>0$. The cdf of RL distribution is given by

$$F_{RL}(x;\alpha,\theta) = 1-\exp\left[\frac{-1}{2\alpha^2}\left(\frac{(1+\theta)e^{\theta x}}{(1+\theta+\theta x)}-1\right)^2\right]. \tag{2.4}$$

Our suggested new distribution has some privileges over the original distributions and many other distributions in fitting some data sets. We present in Section 9 two types of real data where the RL distribution fits these data better than other distributions. In fact, we have noticed during our study that RL distribution is better in dealing with data which have small values than large values data. Now, we study some properties of the new RL distribution with its hazard and reverse hazard rate functions.

2.1. Expansion of density

Using some series expansion techniques such as binomial and power series expansion we obtain the following:

$$\frac{1}{\overline{G}^3(x;\xi)} = \frac{1}{(1-G(x;\xi))^3} = \sum_{k=0}^{\infty}\frac{\Gamma(k+3)}{\Gamma(3)k!}G^k(x;\xi), \tag{2.5}$$

$$\exp\left\{\frac{-1}{2\alpha^2}\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right\} = \sum_{j=0}^{\infty}\frac{(-1)^j(\frac{1}{2\alpha^2})^j}{j!}\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{2j}, \tag{2.6}$$

$$\frac{1}{\overline{G}^{2j}(x;\xi)} = \frac{1}{(1-G(x;\xi))^{2j}} = \sum_{i=0}^{\infty}\frac{\Gamma(i+2j)}{\Gamma(2j)i!}G^i(x;\xi), \quad j\neq 0, \tag{2.7}$$

where Γ is the gamma function. Using Equation ((2.5), (2.6) and (2.7)) we can rewrite Equation (1.4) as

$$f_{RG}(x;\alpha,\xi) = g(x;\xi)\sum_{j=1}^{\infty}\sum_{i,k=0}^{\infty}p_{ijk}(G(x;\xi))^{i+2j+k+1}, \tag{2.8}$$

where the coefficient p_{ijk} is defined by:

$$p_{ijk} = \frac{(k+2)(k+1)(-1)^j\Gamma(i+2j)}{i!k!\Gamma(2j)(2\alpha^2)^{j+1}}.$$

The series representation of the pdf of RL distribution is given by

$$f_{RL}(x;\alpha,\theta) = \frac{\theta^2}{(1+\theta)}(1+x)e^{-\theta x}\sum_{j=1}^{\infty}\sum_{i,k=0}^{\infty}p_{ijk}\left(1-\frac{(1+\theta+\theta x)}{1+\theta}e^{-\theta x}\right)^{i+2j+k+1}. \tag{2.9}$$

Series representation of RL is useful in finding moments, reliability, Benferroni and Lorenz curves and Rényi entropy.

2.2. Monotonicity of RL distribution

In this section, we study the monotonicity of RL distribution, for that purpose let $V(x) = G(x;\theta) = 1 - \frac{(1+\theta+\theta x)}{1+\theta} e^{-\theta x}$, then Equation (2.2) can be written as

$$f_{RL}(x;\alpha,\theta) = \frac{\theta^2}{\alpha^2(1+\theta)} V(x)(1-V(x))^{-3}(x+1)e^{-\theta x} \exp\{\frac{-1}{2\alpha^2}(\frac{V(x)}{1-V(x)})^2\}, \tag{2.10}$$

where $x > 0, \alpha > 0, \theta > 0$. Applying the logarithmic function of both sides leads to

$$\begin{aligned} \log f_{RL}(x;\alpha,\theta) &= 2\log\left(\frac{\theta^2}{\alpha^2(1+\theta)}\right) + \log V(x) - 3\log(1-V(x)) \\ &+ \log(x+1) - \theta x - \frac{1}{2\alpha^2}(\frac{V(x)}{1-V(x)})^2. \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{dx}\log f_{RL}(x;\alpha,\theta) &= \frac{V'(x)}{V(x)} + 3\frac{V'(x)}{1-V(x)} + \frac{1}{x+1} - \theta \\ &- \frac{1}{\alpha^2}(\frac{V(x)}{1-V(x)})\{\frac{(1-V(x))V'(x) + V(x)V'(x)}{(1-V(x))^2}\} \\ &= \frac{1}{x+1} - \theta + V'(x)\left(\frac{(1-V(x))^2 - \frac{1}{\alpha^2}V^2(x)}{V(x)(1-V(x))^3}\right). \end{aligned} \tag{2.11}$$

Let us now discuss the quantities appeared in Eq. (2.11); $V'(x) = (\theta^2/1+\theta)(1+x)e^{-\theta x} > 0, \forall x > 0$, $V(x)$ is monotonically increasing from 0 to 1, $V'(x)$ is decreasing form $\theta^2/1+\theta$ to 0, $0 < V(x) < 1$, $0 < 1 - V(x) < 1$ and $0 < 1/1 + x < 1$.
If $\theta < (1/1+x) < 1$ and $\alpha^2 > (\frac{V(x)}{1-V(x)})^2$, then $\frac{d}{dx}\log f_{RL}(x;\alpha,\theta) > 0$ and hence $f_{RL}(x;\alpha,\theta)$ is increasing.
If $\theta > ((1+\sqrt{5})/2)$, then $V'(x) > 1$ which yields $V'(x)/(V(x)(1-V(x))^3) > 1$. Now, if $\alpha^2 < (\frac{V(x)}{1-V(x)})^2$ and $\theta > (1/1+x)$, then $\frac{d}{dx}\log f_{RL}(x;\alpha,\theta) < 0$ and hence $f_{RL}(x;\alpha,\theta)$ is decreasing.
If $0 < \theta < ((1+\sqrt{5})/2)$, then $V'(x) < 1$ which yields $V'(x)/(V(x)(1-V(x))^3) < 1$. Now, if $(\frac{V(x)}{1+(1-V(x))^2}) < \alpha^2 < (\frac{V(x)}{1-V(x)})^2$ and $\theta > (1/1+x)$, then $\frac{d}{dx}\log f_{RL}(x;\alpha,\theta) < 0$ and hence $f_{RL}(x;\alpha,\theta)$ is decreasing.
Figure 1 illustrates the shape of the density function for some selected parameters' values.

2.3. Hazard rate and reverse hazard rate functions

In this section, we provide the hazard rate (failure rate) and the reverse hazard rate functions of RL distribution. We also present some graphs of these functions for some values of parameters α and θ that illustrates their properties. The hazard and reverse hazard functions of RL distribution are:

$$r_{RL}(x;\alpha,\theta) = \frac{\theta^2}{\alpha^2}(\theta+1)(x+1)e^{\theta x}\{\frac{e^{\theta x}(\theta+1) - (1+\theta+\theta x)}{(1+\theta+\theta x)^3}\},$$

and

$$\rho_{RL} = \frac{\frac{\theta^2}{\alpha^2}(\theta+1)(x+1)e^{\theta x}\{\frac{e^{\theta x}(\theta+1)-(1+\theta+\theta x)}{(1+\theta+\theta x)^3}\}}{(\exp\{\frac{1}{2\alpha^2}(\frac{(1+\theta)e^{\theta x}}{(1+\theta+\theta x)} - 1)^2\} - 1)},$$

for $x > 0$, $\alpha > 0$, $\theta > 0$, respectively.

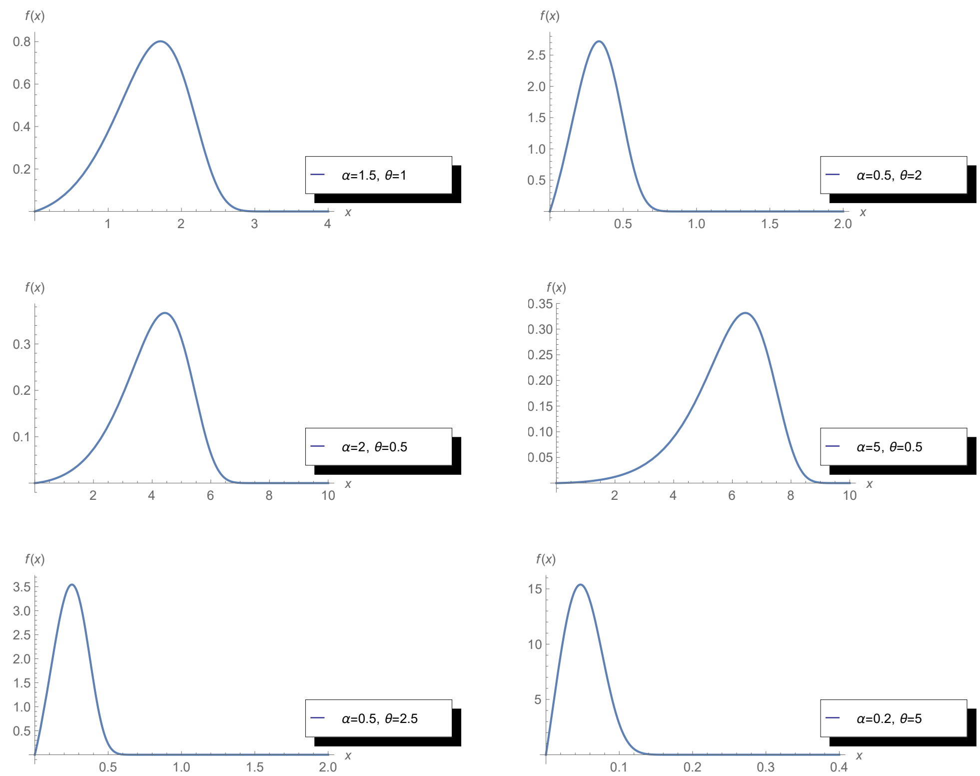

Figure 1: Density Plots of RL distribution.

In order to determine the monotonicity of the hazard function $r_{RL}(x;\alpha,\theta)$, it's enough to determine the monotonicity of $\log r_{RL}(x;\alpha,\theta)$. The first derivative of $\log r_{RL}(x;\alpha,\theta)$ with respect to x is given by

$$\frac{\partial}{\partial x}\log r_{RL}(x;\alpha,\theta) = \frac{1}{x+1} + \theta + \frac{\theta(\theta+1)e^{\theta x} - \theta}{(\theta+1)e^{\theta x} - (1+\theta+\theta x)} - \frac{3\theta}{1+\theta+\theta x}. \tag{2.12}$$

After some mathematical simplifications, Eq. (2.12) can be written as

$$\frac{\partial}{\partial x}\log r_{RL}(x;\alpha,\theta) = \frac{1}{x+1} + \theta - 2 + (1+\theta x)\left(\frac{1}{(\theta+1)e^{\theta x} - (1+\theta+\theta x)} + \frac{3}{1+\theta+\theta x}\right). \tag{2.13}$$

Using the Taylor series expansion of $e^{\theta x}$, we can insure that the term $(\theta+1)e^{\theta x} - (1+\theta+\theta x)$ is strictly positive. And after some mathematical manipulation, Eq. (2.13) is positive and consequently the hazard rate function is always increasing.
Figure 2 gives the graphs of the first derivative of the hazard function and also the hazard function, it can be easily noticed that the first derivative is always positive and as a result the hazard function is increasing function.
The time plot of the hazard function of RL distribution for different values of α and θ are illustrated in Figure 3, where it is clear that the hazard rate is increasing.

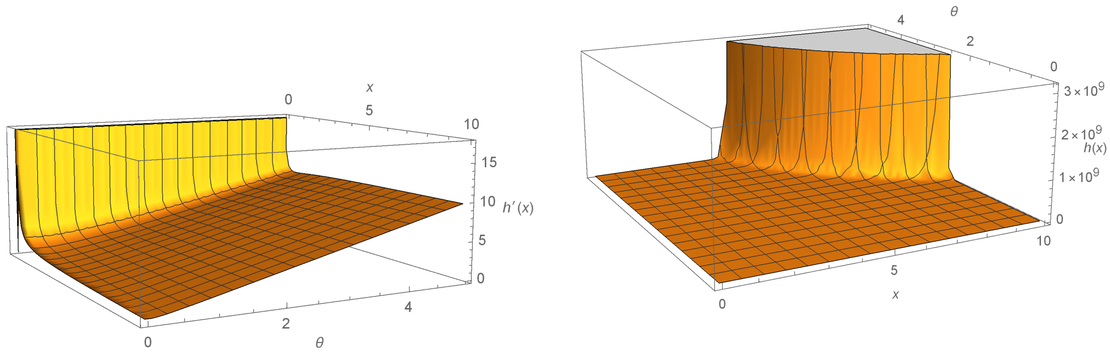

Figure 2: Plots of first derivative of hazard function (left) and the hazard function (right) for different values of θ.

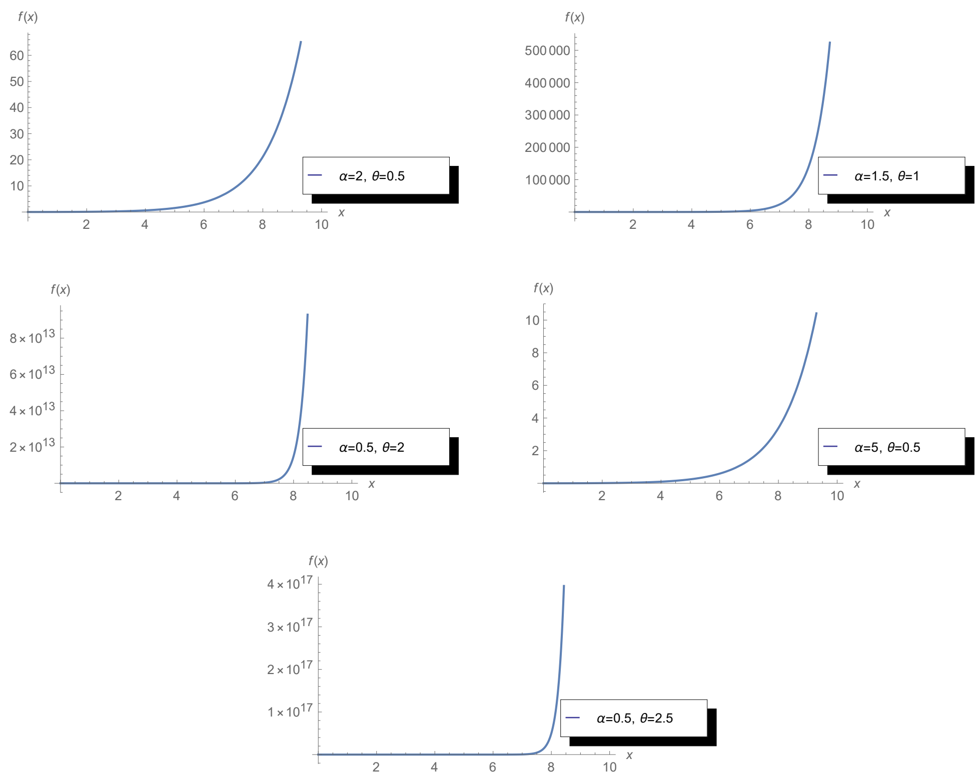

Figure 3: Plots of the hazard function for different values of α and θ.

3. MOMENTS

Using the pdf form of the RG distribution given in Equation (2.8), the s^{th} moment is written as

$$\begin{aligned} \mu'_s &= \int_0^{\infty} x^s f_{RG}(x;\alpha,\xi)dx \\ &= \sum_{j=1}^{\infty}\sum_{i,k=0}^{\infty} p_{ijk}\int_0^{\infty} x^s (G(x;\xi))^{i+2j+k+1} g(x;\xi)dx. \end{aligned}$$

Consider the Lindley distribution as a baseline function, the s^{th} moment can readily be

$$\mu_s' = \frac{\theta^2}{(1+\theta)} \sum_{j=1}^{\infty} \sum_{i,k=0}^{\infty} p_{ijk} \int_0^{\infty} x^s (1+x) e^{-\theta x} (1 - \frac{(1+\theta+\theta x)}{1+\theta} e^{-\theta x})^t dx,$$

where $t = i+2j+k+1$. Using the series representations of $(1-\frac{(1+\theta+\theta x)}{1+\theta}e^{-\theta x})^t$ and $(1-\frac{(1+\theta+\theta x)}{1+\theta}e^{-\theta x})^{k+1}$, the s^{th} moment for the RL distribution is given by

$$\mu_s' = \sum_{j=1}^{\infty} \sum_{i,k=0}^{\infty} p_{ijk} \sum_{l=0}^{t} \sum_{r=0}^{l} \sum_{w=0}^{r+1} \frac{\theta^{r-s-w+1}}{(1+\theta)^{l+1}} \binom{t}{l} \binom{l}{r} \binom{r+1}{w} (-1)^l \frac{\Gamma(s+w+1)}{(l+1)^{s+w+1}}.$$

Table 1 lists the first five moments, variance, skewness and kurtosis for selected values of the parameters of RL distribution.

Table 1: Moments, variance, skewness and kurtosis for $(\alpha, \theta) = (0.5, 1.5), (1.5, 1.5), (1.5, 2.5), (2.5, 1.5)$

μ_s'	$\alpha = 0.5, \theta = 1.5$	$\alpha = 1.5, \theta = 1.5$	$\alpha = 1.5, \theta = 2.5$	$\alpha = 2.5, \theta = 1.5$
μ_1'	0.4761	0.9658	0.5253	1.2616
μ_2'	0.2632	1.0309	0.3070	1.7240
μ_3'	0.1606	1.1762	0.1926	2.4867
μ_4'	0.1053	1.4102	0.1273	3.7362
μ_5'	0.0731	1.7588	0.0876	5.7999
Variance	0.0366	0.0981	0.0310	0.1322
Skewness	22.9047	38.2677	35.1994	51.7429
Kurtosis	78.4811	146.475	132.026	213.828

4. RELIABILITY

Since reliability plays an important role in industry and life testing experiments, in this section we derive the reliability ($\mathbf{R}$) when X and Y have independent RL distributions. Assume $X \sim RL(\alpha_1, \theta_1)$ and $Y \sim RL(\alpha_2, \theta_2)$, in order to define reliability we proceed as follow:

$$\begin{aligned}
\mathbf{R} &= P(X > Y) = \int_0^{\infty} f_X(x; \alpha_1, \theta_1) F_Y(x; \alpha_2, \theta_2) dx \\
&= \int_0^{\infty} \frac{\theta_1^2}{(1+\theta_1)} (1+x) e^{-\theta_1 x} \sum_{i,j,k=0}^{\infty} p_{1ijk} (1 - \frac{(1+\theta_1+\theta_1 x)}{1+\theta_1} e^{-\theta_1 x})^{i+2j+k+1} \\
&\quad \times (1 - \exp\{\frac{-1}{2\alpha_2^2} (\frac{(1+\theta_2) e^{\theta_2 x}}{(1+\theta_2+\theta_2 x)} - 1)^2\}) dx \\
&= \frac{\theta_1^2}{(1+\theta_1)} \sum_{i,j,k=0}^{\infty} p_{1ijk} \int_0^{\infty} (A(x) - B(x)) dx,
\end{aligned}$$

where

$$A(x) = (1+x) e^{-\theta_1 x} (1 - \frac{(1+\theta_1+\theta_1 x)}{1+\theta_1} e^{-\theta_1 x})^{i+2j+k+1},$$

$$B(x) = (1+x) e^{-\theta_1 x} (1 - \frac{(1+\theta_1+\theta_1 x)}{1+\theta_1} e^{-\theta_1 x})^{i+2j+k+1} e^{\frac{-1}{2\alpha_2^2} (\frac{(1+\theta_2) e^{\theta_2 x}}{(1+\theta_2+\theta_2 x)} - 1)^2},$$

and

$$p_{1ijk}=\frac{(k+2)(k+1)(-1)^j\Gamma(i+2j)}{i!j!\Gamma(2j)(2\alpha_1^2)^{j+1}}.$$

To simplify the above integral, we use some series representation and algebraic manipulations, so we have

$$\begin{aligned}
\int_0^\infty A(x)dx &= \sum_{l=0}^{t}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\frac{\theta_1^{r-w-1}}{(1+\theta_1)^l}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}(-1)^l\frac{\Gamma(w+1)}{(l+1)^{w+1}}\\
\int_0^\infty B(x)dx &= \sum_{l=0}^{i+2j+k+1}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\sum_{s,t=0}^{\infty}\sum_{u=0}^{2t+s}\sum_{p=0}^{u}\sum_{z=0}^{p}(-1)^{l+t+u}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}\binom{2t+s-1}{s}\\
&\times\frac{(\frac{1}{2\alpha_2^2})^t}{t!}\binom{2t+s}{u}\binom{u}{p}\binom{p}{z}\frac{\theta_1^r\theta_2^p}{(1+\theta_1)^l(1+\theta_2)^u}\frac{\Gamma(w+z+1)}{(\theta_1(l+1)+\theta_2 u)^{w+z+1}},
\end{aligned}$$

where $t=i+2j+k+1$. Hence the reliability $\mathbf{R}$ can be written as:

$$\begin{aligned}
\mathbf{R} &= \sum_{i,j,k=0}^{\infty}p_{1ijk}\{\sum_{l=0}^{t}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\frac{\theta_1^{r-w+1}}{(1+\theta_1)^{l+1}}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}(-1)^l\frac{\Gamma(w+1)}{(l+1)^{w+1}}\\
&-\sum_{l=0}^{i+2j+k+1}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\sum_{s,t=0}^{\infty}\sum_{u=0}^{2t+s}\sum_{p=0}^{u}\sum_{z=0}^{p}(-1)^{l+t+u}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}\binom{2t+s-1}{s}\\
&\times\frac{(\frac{1}{2\alpha_2^2})^t}{t!}\binom{2t+s}{u}\binom{u}{p}\binom{p}{z}\frac{\theta_1^{r+2}\theta_2^p}{(1+\theta_1)^{l+1}(1+\theta_2)^u}\frac{\Gamma(w+z+1)}{(\theta_1(l+1)+\theta_2 u)^{w+z+1}}\}.
\end{aligned}$$

5. QUANTILE FUNCTION AND GENERATION ALGORITHM

Based on Equation (1.3), the inverse of the new Rayleigh generalization is given by

$$G(x;\xi)=1-\frac{1}{1+\sqrt{-2\alpha^2\log(1-u)}},0<u<1, \tag{5.1}$$

where $G(x;\xi)$ is the cdf of Lindely distribution given in (2.2). Jodra (2010) has showed that the quantile function of the Lindley distribution is given by

$$G^{-1}(u)=-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\left(-\frac{\theta+1}{e^{\theta+1}}(1-u)\right), \tag{5.2}$$

where $W_{-1}(.)$ denotes the negative branch of the Lambert W function (i.e., the solution of the equation $W(z)e^{W(z)}=z$). Since $-\frac{1}{e}<-\frac{\theta+1}{e^{\theta+1}}(1-u)<0$, then $W_{-1}(.)$ is unique and this implies that $G^{-1}(u)$ is also unique. Using Eq.'s (5.1) and (5.2), it readily follows that the quantile function of the Rayleigh-Lindley distribution is given by

$$F_{RL}^{-1}(u)=-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\left(-\frac{\theta+1}{e^{\theta+1}}(1-s)\right), \tag{5.3}$$

where

$$s=1-\frac{1}{1+\sqrt{-2\alpha^2\log(1-u)}},0<u<1.$$

Now, we generating random data from the inverse CDF in (2.4) of the Rayleigh-Lindley distribution according to the following algorithm

Algorithm 1:

Step 1: Generate $U_i \sim Uniform(0,1)$, $i = 1, ..., n$;

Step 2: Compute $S_i = 1 - \frac{1}{1+\sqrt{-2\alpha^2 \log(1-U_i)}}$;

Step 3: Set

$$X_i = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(-\frac{\theta+1}{e^{\theta+1}}(1-S_i)\right).$$

We use Algorithm 1 to generate data from the Rayleigh-Lindley distribution.

6. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we obtain the maximum likelihood estimations (MLEs) of the parameters of RL distribution. Numerical methods are used to solve the nonlinear systems of equations obtained. Consider the pdf of RL distribution written by Equation (1.3), then its log likelihood function denoted by $\ell(x;\alpha,\theta)$ is given by

$$\begin{aligned} \ell(x;\alpha,\theta) &= 2n\log\frac{\theta}{\alpha} + n\log(\theta+1) + \sum_{i=1}^{n}\log(x_i+1) + \theta\sum_{i=1}^{n}x_i - 3\sum_{i=1}^{n}\log(1+\theta+\theta x_i) \\ &+ \sum_{i=1}^{n}\log(e^{\theta x_i}(\theta+1) - (1+\theta+\theta x_i)) - \frac{1}{2\alpha^2}\sum_{i=1}^{n}(\frac{(1+\theta)e^{\theta x_i}}{1+\theta+\theta x_i} - 1)^2. \end{aligned}$$

The first derivatives with respect to the parameters α and θ are:

$$\frac{\partial\ell(x;\alpha,\theta)}{\partial\alpha} = -\frac{2n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{n}(\frac{(1+\theta)e^{\theta x_i}}{1+\theta+\theta x_i} - 1)^2, \tag{6.1}$$

and

$$\begin{aligned} \frac{\partial\ell(x;\alpha,\theta)}{\partial\theta} &= \frac{2n}{\theta} + \frac{n}{1+\theta} + \sum_{i=1}^{n}x_i - 3\sum_{i=1}^{n}\frac{1+x_i}{1+\theta+\theta x_i} + \sum_{i=1}^{n}\frac{e^{\theta x_i}(1+(1+\theta)x_i) - (1+x_i)}{e^{\theta x_i}(\theta+1) - (1+\theta+\theta x_i)} \\ &- \frac{1}{\alpha^2}\sum_{i=1}^{n}(\frac{(1+\theta)e^{\theta x_i}}{1+\theta+\theta x_i} - 1)(\frac{\theta e^{\theta x_i}x_i(1+(1+x_i)(1+\theta))}{(1+\theta+\theta x_i)^2}). \end{aligned} \tag{6.2}$$

Equating the above two equations to zero and solving them with respect to α and θ we have:

$$\widehat{\alpha}^2 = \frac{\sum_{i=1}^{n}(\frac{(1+\theta)e^{\theta x_i}}{1+\theta+\theta x_i} - 1)^2}{2n}, \tag{6.3}$$

substituting Equation (6.3) in Equation (6.2) and using numerical methods we can obtain the MLE of θ, denoted by $\widehat{\theta}$.

The normal approximation of the MLE of vector parameter $\delta = (\alpha, \theta)$ can be used to construct approximate confidence intervals and for testing hypotheses on the parameters α and θ. From the asymptotic property of the MLE we have $\sqrt{n}(\widehat{\delta} - \delta) \underset{\rightarrow}{d} N_2(0, K^{-1}(\delta))$, where $K(\delta)$ is the unit expected information matrix, and $K(\delta) = \lim_{n\to\infty}\frac{1}{n}I_n(\delta)$, here $I_n(\delta)$ is the observed information matrix evaluated at $\widehat{\delta}$. The observed information matrix is given by

$$I_n(\delta) = -\begin{bmatrix} E(\ell_{\alpha\alpha}) & E(\ell_{\alpha\theta}) \\ E(\ell_{\theta\alpha}) & E(\ell_{\theta\theta}) \end{bmatrix},$$

The expected values of the second derivatives can be found by using some integration techniques.

7. BENFERRONI AND LORENZ CURVES

An important application of the first incomplete moment is to determine Bonferroni and Lorenz curves, see Bonferroni (1930) [7], Lorenz (1905) [23] and Gastwirth (1971) [13]. They are commonly used in applied works areas such as economics, reliability, demography, insurance, medicine and others. Benferroni and Lorenz curves are defined by

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx \quad \text{ and } L(p)=\frac{1}{\mu}\int_0^q xf(x)dx,$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

Theorem 1. *Benferroni and Lorenz curves for the RL distribution is given by*

$$B(p) = \frac{1}{p}-\frac{1}{p\mu}\sum_{j=1}^{\infty}\sum_{i,k=0}^{\infty}p_{ijk}\sum_{l=0}^{t}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\frac{\theta^{r-w}}{(1+\theta)^{l+1}}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}(-1)^l\frac{\Gamma(w+2,q)}{(l+1)^{w+2}},$$

and

$$L(p) = 1-\frac{1}{\mu}\sum_{j=1}^{\infty}\sum_{i,k=0}^{\infty}p_{ijk}\sum_{l=0}^{t}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\frac{\theta^{r-w}}{(1+\theta)^{l+1}}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}(-1)^l\frac{\Gamma(w+2,q)}{(l+1)^{w+2}},$$

respectively.

Proof. We use Equation (11), with $s=1$, so that

$$\mu=\sum_{j=1}^{\infty}\sum_{i,k=0}^{\infty}p_{ijk}\sum_{l=0}^{t}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\frac{\theta^{r-w}}{(1+\theta)^{l+1}}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}(-1)^l\frac{\Gamma(w+2)}{(l+1)^{w+2}},$$

and

$$\int_q^{\infty}xf_{RL}(x;\alpha,\beta)dx=\sum_{j=1}^{\infty}\sum_{i,k=0}^{\infty}p_{ijk}\sum_{l=0}^{t}\sum_{r=0}^{l}\sum_{w=0}^{r+1}\frac{\theta^{r-w}}{(1+\theta)^{l+1}}\binom{t}{l}\binom{l}{r}\binom{r+1}{w}(-1)^l\frac{\Gamma(w+2,q)}{(l+1)^{w+2}}$$

where $\Gamma(w+2,q)$ is the upper incomplete gamma function. So the Benferroni and Lorenz curves are obtained directly. □

8. ORDER STATISTICS, MEASURES OF UNCERTAINTY AND INFORMATION

In this section we present an explicit expression for the density function of order statistic $X_{k:n}$ in a random sample of size n with RL distribution. Rényi Measure of uncertainty and information for the RL distribution is also given.

8.1. Distribution of Order Statistics

Let $X_1,...,X_n$ be a random sample of size n from continuous pdf $f(x)$. Let $X_{1:n}<X_{2:n}<...<X_{n:n}$ denote the corresponding order statistics. If $X_1,...,X_n$ is a random sample of RL distribution, it follows from Equations (1.3) and (1.4) that the pdf of the k^{th} order statistics $Y_k=X_{k:n}$ is given by

$$\begin{aligned}f_k(y_k) &= \frac{n!}{(k-1)!(n-k)!}\frac{\theta^2}{\alpha^2}(\theta+1)(1-\exp\{\frac{-1}{2\alpha^2}(\frac{(1+\theta)e^{\theta y_k}}{(1+\theta+\theta y_k)}-1)^2\})^{k-1}\\ &\times\exp\{\frac{-(n-k+1)}{2\alpha^2}(\frac{(1+\theta)e^{\theta y_k}}{(1+\theta+\theta y_k)}-1)^2\}(y_k+1)e^{\theta y_k}\{\frac{e^{\theta y_k}(\theta+1)-(1+\theta+\theta y_k)}{(1+\theta+\theta y_k)^3}\},\end{aligned}$$

and the corresponding cdf of Y_k is

$$F_k(y_k) = \sum_{j=k}^{n} \binom{n}{j} (1 - \exp\{\frac{-1}{2\alpha^2}(\frac{(1+\theta)e^{\theta y_k}}{(1+\theta+\theta y_k)} - 1)^2\})^j \exp\{\frac{-(n-j)}{2\alpha^2}(\frac{(1+\theta)e^{\theta y_k}}{(1+\theta+\theta y_k)} - 1)^2\}$$

8.2. Rényi entropy

Rényi entropy is an extension of shannon entropy, see Rényi (1961) [30]. Rényi entropy is defined as

$$H_\gamma(f_{RL}(x;\alpha,\theta)) = \frac{1}{1-\gamma} \log \int_0^\infty f_{RL}^\gamma(x;\alpha,\theta)dx,$$

where $\gamma > 0$ and $\gamma \neq 1$. Rényi entropy becomes shannon entropy when $\gamma \to 1$. To find Rényi entropy for RL distribution we proceed as follow

$$\begin{aligned} \int_0^\infty f_{RL}^\gamma(x;\alpha,\theta)dx &= (\frac{\theta}{\alpha})^{2\gamma}(\theta+1)^{-\gamma} \int_0^\infty (x+1)^\gamma e^{-\theta\gamma x} V^\gamma(x)(1-V(x))^{-3\gamma} \\ &\quad \times \exp\{\frac{-\gamma}{2\alpha^2}(\frac{V(x)}{1-V(x)})^2\}dx, \end{aligned} \tag{8.1}$$

where $V(x) = G(x;\theta) = 1 - \frac{(1+\theta+\theta x)}{1+\theta}e^{-\theta x}$. Note that

$$\begin{aligned} V^\gamma(x) &= \sum_{k=0}^{\gamma} (-1)^k \binom{\gamma}{k} \frac{(1+\theta+\theta x)^k}{(1+\theta)^k} e^{-\theta k x} \\ &= \sum_{k=0}^{\gamma} (-1)^k \binom{\gamma}{k} \frac{\sum_{j=0}^{k} \binom{k}{j} \theta^j (1+x)^j}{(1+\theta)^k} e^{-\theta k x}, \end{aligned} \tag{8.2}$$

$$\begin{aligned} (1-V(x))^{-3\gamma} &= \sum_{m=0}^{\infty} \binom{3\gamma+m-1}{m} V^m(x) \\ &= \sum_{m=0}^{\infty} \binom{3\gamma+m-1}{m} \sum_{n=0}^{m} \binom{m}{n} (-1)^n \frac{(1+\theta+\theta x)^n}{(1+\theta)^n} e^{-\theta n x} \\ &= \sum_{m=0}^{\infty} \binom{3\gamma+m-1}{m} \sum_{n=0}^{m} \binom{m}{n} (-1)^n \frac{\sum_{z=0}^{n} \binom{n}{t} \theta^z (1+x)^z}{(1+\theta)^n} e^{-\theta n x}, \end{aligned} \tag{8.3}$$

and

$$
\begin{aligned}
\exp\{\frac{-\gamma}{2\alpha^2}(\frac{V(x)}{1-V(x)})^2\} &= \sum_{r=0}^{\infty}\frac{(-1)^r(\frac{\gamma}{2\alpha^2})^r}{r!}\left(\frac{V(x)}{1-V(x)}\right)^{2r} \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r(\frac{\gamma}{2\alpha^2})^r}{r!}\sum_{s=0}^{2r}\binom{2r}{s}(-1)^s\frac{(1+\theta+\theta x)^s}{(1+\theta)^s}e^{-\theta sx} \\
&\quad\times\sum_{v=0}^{\infty}\binom{2r+v-1}{v}\sum_{w=0}^{v}\binom{v}{w}(-1)^w\frac{(1+\theta+\theta x)^w}{(1+\theta)^w}e^{-\theta wx} \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{2r}\sum_{v=0}^{\infty}\sum_{w=0}^{v}\frac{(-1)^{r+s+w}(\frac{\gamma}{2\alpha^2})^r}{r!}\binom{2r}{s}\binom{2r+v-1}{v}\binom{v}{w} \\
&\quad\times\frac{(1+\theta+\theta x)^{s+w}}{(1+\theta)^{s+w}}e^{-\theta(s+w)x} \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{2r}\sum_{v=0}^{\infty}\sum_{w=0}^{v}\frac{(-1)^{r+s+w}(\frac{\gamma}{2\alpha^2})^r}{r!}\binom{2r}{s}\binom{2r+v-1}{v}\binom{v}{w} \\
&\quad\times\frac{\sum_{u=0}^{s+w}\binom{s+w}{u}\theta^u(1+x)^u}{(1+\theta)^{s+w}}e^{-\theta(s+w)x} \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{2r}\sum_{v=0}^{\infty}\sum_{w=0}^{v}\sum_{u=0}^{s+w}\frac{(-1)^{r+s+w}(\frac{\gamma}{2\alpha^2})^r}{r!}\binom{2r}{s}\binom{2r+v-1}{v} \\
&\quad\times\binom{v}{w}\binom{s+w}{u}\frac{\theta^u(1+x)^u}{(1+\theta)^{s+w}}e^{-\theta(s+w)x} \qquad (8.4)
\end{aligned}
$$

Substituting Equations (8.2), (8.3) and (8.4) into Equation (8.1), we obtain

$$
\begin{aligned}
\int_0^{\infty}f_{RL}^{\gamma}(x;\alpha,\theta)dx &= (\frac{\theta}{\alpha})^{2\gamma}(\theta+1)^{-\gamma}\sum_{k,m,r,v=0}^{\infty}\sum_{j=0}^{k}\sum_{n=0}^{m}\sum_{z=0}^{n}\sum_{s=0}^{2r}\sum_{w=0}^{v}\sum_{u=0}^{s+w}\frac{(-1)^{k+n+r+s+w}\theta^{j+z+u}}{(\theta+1)^{k+n+s+w}}\frac{(\frac{\gamma}{2\alpha^2})^r}{r!} \\
&\quad\times\binom{\gamma}{k}\binom{k}{j}\binom{3\gamma+m-1}{m}\binom{m}{n}\binom{n}{z}\binom{2r}{s}\binom{2r+v-1}{v}\binom{v}{w}\binom{s+w}{u} \\
&\quad\times\int_0^{\infty}(1+x)^{z+j+\gamma+u}e^{-\theta(k+\gamma+n+s+w)x}dx \\
&= (\frac{\theta}{\alpha})^{2\gamma}(\theta+1)^{-\gamma}\sum_{k,m,r,v=0}^{\infty}\sum_{j=0}^{k}\sum_{n=0}^{m}\sum_{z=0}^{n}\sum_{s=0}^{2r}\sum_{w=0}^{v}\sum_{u=0}^{s+w}\sum_{a=0}^{z+j+\gamma+u}\frac{(-1)^{k+n+r+s+w}\theta^{j+z+u}}{(\theta+1)^{k+n+s+w}} \\
&\quad\times\frac{(\frac{\gamma}{2\alpha^2})^r}{r!}\binom{\gamma}{k}\binom{k}{j}\binom{3\gamma+m-1}{m}\binom{m}{n}\binom{n}{z}\binom{2r}{s}\binom{2r+v-1}{v}\binom{v}{w} \\
&\quad\times\binom{s+w}{u}\binom{z+j+\gamma+u}{a}\frac{\Gamma(a+1)}{(\theta(k+\gamma+n+s+w))^{a+1}}.
\end{aligned}
$$

Consequently, Rényi entropy for RL distribution is

$$
\begin{aligned}
H_\gamma(f_{RL}(x;\alpha,\theta)) = \quad & \frac{1}{1-\gamma}\log\left((\frac{\theta}{\alpha})^{2\gamma}(\theta+1)^{-\gamma}\right) \\
& +\frac{1}{1-\gamma}\log\{\sum_{k,m,r,v=0}^{\infty}\sum_{j=0}^{k}\sum_{n=0}^{m}\sum_{z=0}^{n}\sum_{s=0}^{2r}\sum_{w=0}^{v}\sum_{u=0}^{s+w}\sum_{a=0}^{z+j+\gamma+u}\frac{(-1)^{k+n+r+s+w}\theta^{j+z+u}}{(\theta+1)^{k+n+s+w}} \\
& \times\frac{(\frac{\gamma}{2\alpha^2})^r}{r!}\binom{\gamma}{k}\binom{k}{j}\binom{3\gamma+m-1}{m}\binom{m}{n}\binom{n}{z}\binom{2r}{s}\binom{2r+v-1}{v}\binom{v}{w} \\
& \times\binom{s+w}{u}\binom{z+j+\gamma+u}{a}\frac{\Gamma(a+1)}{(\theta(k+\gamma+n+s+w))^{a+1}}\}.
\end{aligned}
$$

9. APPLICATIONS

In this section, applications of the RL distribution including the estimation of the parameters via the method of maximum likelihood estimation for the comparison of the RL distribution with other models for given data sets are presented. We provide examples to illustrate the flexibility of the RL distribution in contrast to other models for data modelling purposes.

The MLEs of the parameters α and θ are computed by maximizing the objective function via the R package (bbmle). The estimated values of the parameters, standard error, -2log-likelihood statistic, Akaike Information Criterion; $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion; $BIC = p\ln(n) - 2\ln(L)$ and the corresponding second order Information Criterion; $AICC = AIC + 2(p(p+1)/(n-p-1))$, where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations and p is the number of estimated parameters are presented in Tables 2 and 3, for the RL distribution and other models including the Weibull, Lindley, Rayeigh, Burr X (generalized Rayleigh) and power Lindley distributions. Kolmogorov–Smirnov ($K-S$) test and the p-value of all mentioned distributions are also computed to complete the comparison process.

Example 1: Let us consider the real life data representing the uncensored data set from Nichols and Padgett (2006) [28] on the breaking stress of carbon fibres (in Gba).

The data are recorded as follows:

```
3.70  2.74  2.73  2.50  3.60  3.11  3.27  2.87  1.47  3.11  3.56
4.42  2.41  3.19  3.22  1.69  3.28  3.09  1.87  3.15  4.90  1.57
2.67  2.93  3.22  3.39  2.81  4.20  3.33  2.55  3.31  3.31  2.85
1.25  4.38  1.84  0.39  3.68  2.48  0.85  1.61  2.79  4.70  2.03
1.89  2.88  2.82  2.05  3.65  3.75  2.43  2.95  2.97  3.39  2.96
2.35  2.55  2.59  2.03  1.61  2.12  3.15  1.08  2.56  1.80  2.53
```

The MLEs of the parameters with standard errors and the values of the statistics ($-2ln(L)$, AIC, $AICC$, BIC, $K-S$ and the p-value) are given in Table 2. The starting points of the iterative processes for the data set for the RL distribution are $(0.64, 0.414)$. Many authors in the literature used this data set to show the privilege of some new distributions over other distributions, for example Oluyede and Yang (2015) [29] used this data set to show that beta-generalized Lindley (BGL) distribution is significantly better than other models and sub-models used in their paper. From the values of the statistics for the carbon fibre data, we note that the RL model is better than the Weibull, Lindley, Rayeigh, Burr X, power Lindley and BGL models in terms of fitting this set of data.

Example 2: The second real life data representing the strength of 1.5cm glass fibers, measured at National physical laboratory, England (see Smith and Naylor (1978) [34]). The data are recorded as follows:

```
0.55  0.93  1.25  1.36  1.49  1.52  1.58  1.61  1.64
```

Table 2: Goodness of fit tests for the real data set in Example 1

	$(\hat{\alpha}, \hat{\theta})$	Std. Error	-2 ln L	AIC	AICc	BIC	K-S	p-value
RL	(0.622, 0.411)	(0.031, 0.045)	172.052	176.053	176.243	180.433	0.086	0.924
Weibull	(3.441, 3.062)	(0.330, 0.115)	172.134	176.135	176.325	180.515	0.082	0.761
Lindley	(- , 0.590)	(- , 0.053)	244.768	246.768	246.958	248.957	0.297	0.0
Rayleigh	(- , 2.049)	(- , 0.126)	196.416	198.417	198.607	200.607	0.227	0.002
Burr X	(2.349, 0.438)	(0.431,0.028)	177.272	181.273	181.463	185.653	0.120	0.293
Power Lindley	(2.510, 0.124)	(0.208,0.031)	171.610	175.611	175.801	179.990	0.079	0.806

```
1.68  1.73  1.81  2.00  0.74  1.04  1.27  1.39  1.49
1.59  1.61  1.66  1.68  1.76  1.82  2.01  0.77  1.11
1.42  1.50  1.54  1.60  1.62  1.66  1.69  1.76  1.84
0.81  1.13  1.29  1.48  1.50  1.55  1.61  1.62  1.66
1.77  1.84  0.84  1.24  1.30  1.48  1.51  1.55  1.61
1.70  1.78  1.89  1.67  1.53  1.28  2.24  1.70  1.63
```

The MLEs of the parameters with standard errors and the values of the statistics ($-2ln(L)$, AIC, $AICC$, BIC, $K-S$ and the p-value) are given in Table 3. The starting points of the iterative processes for the data set for the RL distribution are $(0.52, 0.058)$. From the values of the statistics for the glass fibre data, we note that the RL model is better than Weibull, Lindley, Rayeigh, Burr X and power Lindley distributions in terms of fitting this set of data.

Table 3: Goodness of fit tests for the real data set in Example 2

	$(\hat{\alpha}, \hat{\theta})$	Std. Error	-2 ln L	AIC	AICc	BIC	K-S	p-value
RL	(8.057, 1.975)	(3.114, 0.211)	14.560	33.121	33.321	37.406	0.124	0.596
Weibull	(5.781, 1.628)	(0.576, 0.037)	15.207	34.414	34.614	38.700	0.152	0.108
Lindley	(- , 0.996)	(- , 0.095)	81.278	164.55	164.75	166.7	0.386	0.0
Rayleigh	(- , 1.089)	(- , 0.069)	49.791	101.582	101.782	103.725	0.334	0.0
Burr X	(5.487, 0.987)	(1.185,0.054)	23.929	51.858	52.058	56.144	0.215	0.006
Power Lindley	(4.457, 0.223)	(0.387,0.047)	14.690	33.380	33.58	37.666	0.386	0.146

10. CONCLUSION

We have introduced the Rayleigh-Lindley distribution by replacing x with the odds ratio $G(x)/(1 - G(x))$ in the cdf formula of Rayleigh distribution using Lindley distribution as a baseline distribution. Many statistical properties of the proposed Rayleigh-Lindley distribution like monotonicity, hazard rate, moments, reliability, quantile function, generating algorithm, order statistics and Rényi entropy have been discussed. The estimation of the model parameters has been carried out using the maximum likelihood estimation. The privilege of the proposed model over some related other models has been discussed in consideration of $-2\ln(L)$, AIC, $AICC$, BIC, $K-S$ test and the p-value in two different real life examples. Based on our finding, we recommend to use the RL distribution over the Rayleigh and Lindley distributions when the numerical values of the data under study are small. Finally, a future research direction may include the study of the proposed model under some types of censored data, mainly the estimation of the model parameters and the prediction of the unobserved or future data. Work in this direction is currently under progress and we hope to report these findings in a future paper.

CONFLICTS OF INTEREST

The authors declare that there is no conflicts of interest regarding the publication of this paper.

RECEIVED: AUGUST, 2020.
REVISED: JUNE, 2021.

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