ESTIMATION OF THE PARAMETERS FOR POWER FUNCTION DISTRIBUTION BASED ON TYPE-II DOUBLY CENSORED SAMPLE

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ABSTRACT
In this paper, we propose the Bayes estimators of unknown parameters of power function distribution in the context of double censoring. The maximum-likelihood estimators (MLEs) for the power function parameters are derived. Bayes estimators are derived using the squared error loss function, entropy loss function, and linex loss function using the Lindley approximation and the importance sampling procedures. We also introduced sample prediction estimates using Bayesian techniques. Finally, we perform a simulation study to compare all the proposed estimation methods and analyze a real-life data set for illustration purposes.

KEY WORDS: Estimation; Lindely approximation; Importance sampling procedure; Prediction; Doubly censored sample.

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RESUMEN
En este artículo, proponemos los estimadores de Bayes de parámetros desconocidos de la función de distribución de potencia en el contexto de la doble censura. Se derivan los estimadores de máxima verosimilitud (MLEs) para los parámetros de la función de potencia. Los estimadores bayesianos se derivan utilizando la función de pérdida de error al cuadrado, la función de pérdida de entropía y la función de pérdida de linex utilizando la aproximación de Lindley y los procedimientos de muestreo de importancia. También introdujimos estimaciones de predicción de muestra utilizando técnicas bayesianas. Finalmente, realizamos un estudio de simulación para comparar todos los métodos de estimación propuestos y analizamos un conjunto de datos de la vida real con fines ilustrativos.

PALABRAS CLAVES: Estimación; aproximación de Lindely; procedimiento de muestreo de importancia; Predicción; Muestra doblemente censurada.

1 INTRODUCTION

We consider a lifetime experiment in which n identical units are placed on test simultaneously. The first r lifetimes may be left-censored due to negligence or problems at the beginning of the investigation and, in that, terminated as a means of obtaining data in life testing experiments as soon as the s^{th} unit fails. Then consumed this information to create a type-II doubly censored sample. The resulting type II double censored model is; $x_r \leq x_{r+1} \leq .... \leq x_{s-1} \leq x_s$, $1 \leq r \leq s \leq n$. This type of censoring occurs in critical practical situations. For example, Peer et al. (1993) used doubly censored data to study the age-dependent growth rate of primary breast cancer. For more works related to doubly censored data, one can refer to papers written by Raqab et al. (1995) provided the On the maximum likelihood prediction of the exponential distribution based on doubly type-II censored samples, Estimation of the exponential-logarithmic distribution based on the doubly censored data with lifetime application introduced by Bakoban et al. (2016), El-Baset et.al (2015) show that the Estimation under burr type x distribution based on doubly type ii censored sample of dual generalized order statistics, Wang (2016) used the double type-II censored model to estimate the interval for a lower-truncated distribution, Azimi (2013) gave a Bayesian estimation of doubly censored generalised half logistic type II data, Tahir et al. (2019) presented the Bayesian Estimation of the mixture of burr type-xii distributions using doubly censored data, Fernandez (2000) investigated maximum likelihood prediction using type II doubly censored exponential data, Raqab and Madi (2002) applied Bayesian inference to doubly censored Rayleigh data prediction of the total time on the test and Wu (2008) presented estimates for the interval for a pareto distribution using a doubly type II censored sample.
Power function distribution (PFD) is a flexible lifetime distribution compatible with different failure data sets. It has many applications in finance, economics, and reliability. Besides, PFD has received much attention in the literature for modeling the reliability growth of complex systems and repairs. Meniconi and Barry (1996) proved

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that compared with exponential, lognormal, and Weibull, the PFD is the best distribution of the check electrical component's reliability using the reliability and hazard function plots. For recent work on the estimation of parameters of PFD, one can refer to Abdul-sathar et al. (2018), Abdul-sathar et al. (2019), and Abdul-sathar et al. (2021). This paper aims to develop inference procedures for the power function model parameters where the sample data are available as a doubly censored sample. We build point estimators and interval estimators for the unknown parameters of PFD. The probability density function (PDF) and the cumulative distribution function (CDF) of the PFD are given by

$$f(x) = \frac{\alpha}{\sigma}\left(\frac{\sigma}{x}\right)^{-(\alpha-1)} \quad \text{where} \quad 0 < x < \sigma, \text{ and } \alpha, \sigma > 0, \tag{1.1}$$

$$F(x) = \left(\frac{x}{\sigma}\right)^{\alpha} \quad \text{where } 0 < x < \sigma. \tag{1.2}$$

This study aims to estimate the power function distribution's parameters using a Type-II doubly censored sample. Each section of the article is organized as follows: The maximum likelihood estimate is given in Section 2. Bayes estimators of the parameters under the squared error loss function, entropy loss function, and linex loss function using the Lindley approximation method and importance sampling procedures together with HPD credible intervals discussed in Section 3. Section 4 also included a Bayes prediction. In Section 5, we conduct a performance evaluation of the estimators using simulated data. Finally, real-life data samples and conclusions are in Section 6.

2 MAXIMUM LIKELIHOOD ESTIMATION

This section describes the ML estimate of the parameters α and σ in PFD. Consider a random sample of size n, the ordered observations remaining when $(r-1)$ smallest observations and the $(n-s)$ largest observations have been censored; hence, the doubly censored sample is $x_r \ldots x_s$. The likelihood function in this situation

$$L(\alpha,\sigma|x) = \frac{n!}{(r-1)!(n-s)!}\,[F(x_r)]^{r-1}\,[1-F(x_s)]^{n-s}\prod_{i=r}^{s} f(x_i). \tag{2.1}$$

Using (1.1) and (1.2) in (2.1), the likelihood function simplifies to

$$L(\alpha,\sigma|x) \propto x_r^{\alpha(r-1)}\left[1-\left(\frac{x_s}{\sigma}\right)^{\alpha}\right]^{n-s}\alpha^{s-r+1}\,\sigma^{-\alpha s}\prod_{i=r}^{s} x_i^{(\alpha-1)}. \tag{2.2}$$

We are differentiating (2.2) for α equates to zero, we get

$$\frac{\partial \log[L(\alpha,\sigma|x)]}{\partial\alpha} = \frac{s}{\alpha} - s\log[\sigma] + (r-1)\log[x] + \sum_{i=r}^{s}\log[x_i] + \frac{(n-s)\log[\frac{x_s}{\sigma}]\left(\frac{x_s}{\sigma}\right)^{\alpha}}{\left[1-\left(\frac{x_s}{\sigma}\right)^{\alpha}\right]} = 0.$$

The MLE of σ can be directly obtained as

$$\hat{\sigma} = x_s,\ 0 < x < \sigma \tag{2.3}$$

where $\hat{\alpha}$ is the solution of the nonlinear equation

$$\frac{s}{\alpha} + \frac{(n-s)\log[\frac{x_s}{\sigma}]\left(\frac{x_s}{\sigma}\right)^{\alpha}}{\left[1-\left(\frac{x_s}{\sigma}\right)^{\alpha}\right]} = s\log[\sigma] - (r-1)\log[x] - \sum_{i=r}^{s}\log[x_i]. \tag{2.4}$$

There are several numerical techniques for solving nonlinear equations (2.4).

2.1 Asymptotic Confidence Interval

Although the sample sizes are not very large in survival and reliability analysis, intervals based on the asymptotic normality of the MLEs are of interest when the number of censored units is large enough. In this section, we present the asymptotic and coverage probability of estimates of the parameters. We know that the observed information matrix is a consistent estimator of the Fisher information matrix. The Fisher information matrix of $\theta = (\alpha, \sigma)$ is given by

$$I(\theta) = E\begin{bmatrix} -\frac{\partial^2 \log[L(\alpha,\sigma|x)]}{\partial\alpha^2} & -\frac{\partial^2 \log[L(\alpha,\sigma|x)]}{\partial\alpha\partial\sigma} \\ -\frac{\partial^2 \log[L(\alpha,\sigma|x)]}{\partial\sigma\partial\alpha} & -\frac{\partial^2 \log[L(\alpha,\sigma|x)]}{\partial\sigma^2} \end{bmatrix}.$$

Using the delta method, we derive the asymptotic distribution of $\hat{\theta}_{mle}$. For that, we have

$$\widehat{var}\left(\hat{\theta}_{mle}\right) = \widehat{var}\left(\hat{\theta}\right) \approx T'(\hat{\theta})\,[I(\hat{\theta})^{-1}\,T(\hat{\theta}),$$

where

$$D(\hat{\alpha},\hat{\sigma}) = \begin{bmatrix} \frac{\partial\hat{\theta}}{\partial\alpha} \\ \frac{\partial\hat{\theta}}{\partial\beta} \end{bmatrix}_{(\alpha,\sigma)=(\hat{\alpha},\hat{\sigma})} = \begin{bmatrix} D_\alpha \\ D_\sigma \end{bmatrix}.$$

Hence $\frac{\hat{\theta}_{mle}-\hat{\theta}}{\sqrt{\widehat{var}\,(\hat{\theta}_{mle})}} \sim$ N (0,1). The $100(1-\delta)\%$ CI for $\hat{\theta}_{mle}$ is given as $\hat{\theta}_{mle} \pm z_{\delta/2}\sqrt{\widehat{var}\,(\hat{\theta}_{mle})}$. Also, the coverage probability for $\hat{\theta}_{mle}$ is given as

$$CP_{\hat{\theta}} = P\left[\left|\frac{\left(\hat{\theta}_{mle}-\hat{\theta}\right)}{\sqrt{\hat{v}ar\left(\hat{\theta}_{mle}\right)}}\right| \le z_{\delta/2}\right],$$

where $z_{\delta/2}$ is the doubly censored $(\delta/2)^{th}$ percentile of standard normal distribution.

3 BAYESIAN ESTIMATION

Bayesian calculations are more practical than the classical estimation method. Many adopt the Bayesian method of estimating parameters and associated functions for different distributions. In this section, we evaluate the Bayesian estimates and then provide the associated credible intervals of parameters α and σ for PFD using other loss functions. In Bayesian estimation, we treat α and σ parameters in (2.3) and (2.4) as random variables and assume prior distributions. From the prior distribution, the parameters separated from the α and σ parameters are called hyper-parameters. The preceding information combined with the likelihood function can derive the posterior information.

3.1 Prior distribution and posterior distribution

To observe the influence of priors on Bayesian estimators in this paper, we use an informative prior and a non-informative prior, respectively. Because of its flexibility, the gamma distribution using for the informative method. Moreover, they found that the gamma distribution is a conjugate before the parameters α and σ, making it easy to implement the sample simulation algorithm later. Here we use the piecewise independent gamma priors for the parameters α and σ, which obey Γ (c, d) and Γ (e, f), respectively. One may refer to Kundu and Pradhan (2009) for further details. Nadar et al. (2013) have also considered gamma priors for the Kumaraswamy distribution. Therefore, the prior gamma distribution is given by

$$g_1(\alpha) = \frac{d^c}{\Gamma(c)}\alpha^{c-1}e^{-d\alpha}, \alpha > 0, c, d > 0$$

and

$$g_2(\sigma) = \frac{a^b}{\Gamma(b)}\sigma^{b-1}e^{-a\sigma}, \sigma > 0, a, b > 0.$$

The prior joint distribution corresponds to α and σ can be obtained as

$$G^*(\alpha,\sigma)\ \alpha^{c-1}\,\sigma^{b-1}\,e^{-(d\alpha+a\sigma)}. \tag{3.1}$$

One can obtain the joint posterior distribution using (2.2) and (3.1).

$$\Pi(\alpha,\sigma)=\frac{1}{K}\,\alpha^{(c+s-r+1)-1}\,e^{-\alpha(d-(r-1)\log(x)-Z)}\,\sigma^{(b-as)-1}\,e^{-\sigma(a)}\left(1-\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{n-s}e^{-Z} \tag{3.2}$$

where K is the normalizing constant and is given by

$$K^{-1} = \int_0^\infty\int_0^\infty \alpha^{(c+s-r+1)-1}\,e^{-\alpha(d-(r-1)\log(x)-Z)}\,\sigma^{(b-as)-1}\,e^{-\sigma(a)}\left(1-\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{n-s}e^{-Z}\,d\alpha\,d\sigma,$$

where $Z = \sum_{i=r}^{s}\log[x_i]$, $A(\sigma) = a$.

3.2 Loss Functions

In Bayesian estimation, the loss function using to evaluate estimators' performances. The loss function measures the gap between the estimates and actual values. Therefore, we give Bayes estimators of different loss functions for H in Table 1. To obtain the best estimator in the Bayesian approach, one has to choose a loss function corresponding to each possible estimate. In this study, we consider symmetric as well as asymmetric loss functions, such as the squared error loss function (SELF), entropy loss function (ELF), and linex loss function (LLF) respectively, for estimating the parameters.

Table 1: Bayesian estimators of the different loss functions

Loss Functions	Bayesian Estimator
SELF: $(\widehat{H}-H)$	$E\,(H\,\vert x)$
ELF: $[[\widehat{H}/\mathrm{H}]]^{-q}$ - q (log([$\widehat{H}$/H]))-1	$[E(H^{-q}/x)]\text{^}(-1/q)$
LLF: $exp\,[h\,(H-\widehat{H})]-h\,(H-\widehat{H})-1$	$\left(-\frac{1}{h}\ \mathrm{Log}\,(E\ (exp\ (-h\,H)\mid x))\right)$

Using Table 1, the Bayes estimator of parameters $\theta = (\alpha,\ \sigma)$ under SELF, LLF and ELF, then given

$$\hat{\theta}_{self}=\frac{\int_0^\infty\int_0^\infty \theta\ \Pi(\alpha,\sigma)\ d\alpha\ d\sigma}{\int_0^\infty\int_0^\infty \Pi(\alpha,\sigma)\ d\alpha\ d\sigma}, \tag{3.3}$$

$$\hat{\theta}_{elf}=\left[\frac{\int_0^\infty\int_0^\infty \theta^{-q}\ \Pi(\alpha,\sigma)\ d\alpha\ d\sigma}{\int_0^\infty\int_0^\infty \Pi(\alpha,\sigma)\ d\alpha\ d\sigma}\right]^{-1/q}, \tag{3.4}$$

and

$$\hat{\theta}_{llf}=-\frac{1}{h}\ log\left[\frac{\int_0^\infty\int_0^\infty e^{-h\theta}\ \Pi(\alpha,\sigma)\ d\alpha\ d\sigma}{\int_0^\infty\int_0^\infty \Pi(\alpha,\sigma)\ d\alpha\ d\sigma}\right]. \tag{3.5}$$

where $\Pi\ (\alpha,\ \sigma)$ is given by (3.2), We can see that Bayes estimators are in the form of a ratio of integrals, which Couldn't be more straightforward than closed forms. Here, we use two approximation methods, namely the Lindley approximation and importance sampling methods, to solve the above ratio of integrals discussed in the following sections.

3.3 Approximation Techniques

We can use various techniques in the literature to compute approximate Bayes estimates. This section employs the approaches to evaluate the Bayes estimates obtained in the preceding description. First, consider Lindley's method, which expresses how to approximate a ratio of two particular integrations. For details, please refer to Lindley (1980).

3.3.1 Lindely Approximation Method

This subsection deals with the computation of the Bayes estimates of α and σ of PFD under the loss functions when a doubly censored sample is available. Several approximate methods are available to solve the ratio of integrals. One of the simplest methods is Lindley's approximation method proposed by Lindley (1980). The Lindley approximation for obtaining the Bayes estimates of α and σ by considering the function $I(x)$ is defined as follows;

$$I(x)=\frac{\int \Delta(\alpha,\sigma)e^{L(\alpha,\sigma)+G^*(\alpha,\sigma)}d(\alpha,\sigma)}{\int e^{L(\alpha,\sigma)+G^*(\alpha,\sigma)}d(\alpha,\sigma)},$$

We can approximate the ratio of the integral as

$$\begin{aligned} I(x) \approx\ & \Delta(\alpha,\sigma)+\frac{1}{2}[(\Delta_{\sigma\sigma}+2\Delta_\sigma\Upsilon_\sigma)\Psi_{\sigma\sigma}+(\Delta_{\alpha\sigma}+2\Delta_\alpha\Upsilon_\sigma)\Psi_{\alpha\sigma}+(\Delta_{\sigma\alpha}+2\Delta_\sigma\Upsilon_\alpha)\Psi_{\sigma\alpha} \\ & +\ (\Delta_{\alpha\alpha}+2\Delta_\alpha\Upsilon_\alpha)\Psi_{\alpha\alpha}]+\frac{1}{2}[(\Delta_\sigma\Psi_{\sigma\sigma}+\Delta_\alpha\Psi_{\sigma\alpha})\Theta_1+(\Delta_\sigma\Psi_{\alpha\sigma}+\Delta_\alpha\Psi_{\alpha\alpha})\Theta_2], \end{aligned} \tag{3.6}$$

Where,

$$\Theta_1 = L_{\sigma\sigma\sigma}\Psi_{\sigma\sigma} + L_{\sigma\alpha\sigma}\Psi_{\sigma\alpha} + L_{\alpha\sigma\sigma}\Psi_{\alpha\sigma} + L_{\alpha\alpha\sigma}\Psi_{\alpha\alpha}$$

$$\Theta_2 = L_{\alpha\sigma\sigma}\Psi_{\sigma\sigma} + L_{\sigma\alpha\alpha}\Psi_{\sigma\alpha} + L_{\alpha\sigma\alpha}\Psi_{\alpha\sigma} + L_{\alpha\alpha\alpha}\Psi_{\alpha\alpha}$$

$\Delta_\alpha = \frac{\partial\Delta(\alpha,\sigma)}{\partial\alpha}$, $\Delta_\sigma = \frac{\partial\Delta(\alpha,\sigma)}{\partial\sigma}$, $\Delta_{\alpha\sigma} = \frac{\partial\Delta(\alpha,\sigma)}{\partial\alpha\partial\sigma}$, $\Delta_{\sigma\alpha} = \frac{\partial\Delta(\alpha,\sigma)}{\partial\sigma\partial\alpha}$, $\Delta_{\alpha\alpha} = \frac{\partial\Delta(\alpha,\sigma)}{\partial\alpha^2}$, $\Delta_{\sigma\sigma} = \frac{\partial\Delta(\alpha,\sigma)}{\partial\sigma^2}$ $\Upsilon_\alpha = \frac{\partial G^*(\alpha,\sigma)}{\partial\alpha}$, $\Upsilon_\sigma = \frac{\partial G^*(\alpha,\sigma)}{\partial\sigma}$, $L_{\alpha\alpha} = \frac{\partial^2 L(\alpha,\sigma)}{\partial\alpha^2}$, $L_{\sigma\sigma} = \frac{\partial^2 L(\alpha,\sigma)}{\partial\sigma^2}$, $L_{\alpha\alpha\alpha} = \frac{\partial^3 L(\alpha,\sigma)}{\partial\alpha^3}$, $L_{\alpha\alpha\sigma} = \frac{\partial^3 L(\alpha,\sigma)}{\partial\alpha\partial\alpha\partial\sigma}$, $L_{\sigma\sigma\alpha} = \frac{\partial^3 L(\alpha,\sigma)}{\partial\sigma\partial\sigma\partial\alpha}$, $L_{\sigma\alpha\sigma} = \frac{\partial^3 L(\alpha,\sigma)}{\partial\sigma\partial\alpha\partial\sigma}$, $L_{\alpha\alpha\sigma} = \frac{\partial^3 L(\alpha,\sigma)}{\partial\alpha\partial\alpha\partial\sigma}$, $L_{\alpha\sigma\sigma} = \frac{\partial^3 L(\alpha,\sigma)}{\partial\alpha\partial\sigma\partial\sigma}$, $L_{\sigma\alpha\alpha} = \frac{\partial^3 L(\alpha,\sigma)}{\partial\sigma\partial\alpha\partial\alpha}$ and $\Psi_{ij} = \left(-\frac{1}{L_{ij}}\right)$.

First, let us write the Bayes estimate of σ for the SELF. Under this loss function, $\Delta(\alpha, \sigma) = \sigma$, $\Delta_\sigma = 1$, $\Delta_{\alpha\alpha} = \Delta_{\sigma\sigma} = \Delta_{\alpha\sigma} = \Delta_{\sigma\alpha} = 0$. In the following, Bayes estimate σ under SELF is given by

$$\hat{\sigma}_{self} = \sigma + \Upsilon_\sigma\Psi_{\alpha\sigma} + \Upsilon_\sigma\Psi_{\sigma\sigma} + \frac{1}{2}[\Psi_{\sigma\alpha}\Theta_1 + \Psi_{\sigma\sigma}\Theta_2].$$

If so, then the α_{self} estimate is

$$\hat{\alpha}_{self} = \alpha + \Upsilon_\alpha\Psi_{\alpha\sigma} + \Upsilon_\alpha\Psi_{\alpha\alpha} + \frac{1}{2}[\Psi_{\sigma\alpha}\Theta_1 + \Psi_{\alpha\alpha}\Theta_2].$$

Next, then the ELF of σ is given as

$$\Delta(\alpha,\sigma) = \sigma^{-q},\ \Delta_\sigma = -q\sigma^{-(q+1)},\ \Delta_{\sigma\sigma} = q(q+1)\sigma^{-(q+2)},\ \Delta_{\alpha\alpha} = \Delta_{\alpha\sigma} = \Delta_{\sigma\alpha} = \Delta_\alpha = 0.$$

$$\hat{\sigma}_{elf} = \sigma^{-q} + \frac{1}{2}[2\Upsilon_\sigma\Psi_{\alpha\sigma} + (\Delta_{\sigma\sigma} + 2\Upsilon_\sigma)\Psi_{\sigma\sigma}] + \frac{1}{2}[\Delta_\sigma\Psi_{\sigma\alpha}\Theta_1 + \Delta_\sigma\Psi_{\sigma\sigma}\Theta_2].$$

Then the Bayes estimate of α_{elf} is given as

$$\hat{\alpha}_{elf} = \alpha^{-q} + \frac{1}{2}[2\Upsilon_\alpha\Psi_{\alpha\sigma} + (\Delta_{\alpha\alpha} + 2\Upsilon_\alpha)\Psi_{\alpha\alpha}] + \frac{1}{2}[\Delta_\alpha\Psi_{\sigma\alpha}\Theta_1 + \Delta_\alpha\Psi_{\alpha\alpha}\Theta_2].$$

Next, then the LLF of σ is given as

$$\Delta(\alpha, \sigma) = e^{-p\sigma},\ \Delta\sigma = -p\, e^{-p\sigma},\ \Delta\sigma\sigma = 2pe^{-p\sigma},\ \Delta\sigma\sigma = \Delta\alpha\sigma = \Delta\sigma\alpha = \Delta\sigma = 0.$$

$$\hat{\sigma}_{llf} = e^{-p\sigma} + \frac{1}{2}[2\Upsilon_\sigma\Psi_{\alpha\sigma} + (\Delta_{\sigma\sigma} + 2\Upsilon_\sigma)\Psi_{\sigma\sigma}] + \frac{1}{2}[\Delta_\sigma\Psi_{\sigma\alpha}\Theta_1 + \Delta_\sigma\Psi_{\sigma\sigma}\Theta_2].$$

Similarly, the Bayes estimate of α_{llf} is

$$\hat{\alpha}_{llf} = e^{-p\alpha} + \frac{1}{2}[2\Upsilon_\alpha\Psi_{\alpha\sigma} + (\Delta_{\alpha\alpha} + 2\Upsilon_\alpha)\Psi_{\alpha\alpha}] + \frac{1}{2}[\Delta_\alpha\Psi_{\sigma\alpha}\Theta_1 + \Delta_\alpha\Psi_{\alpha\alpha}\Theta_2].$$

3.3.2 Importance sampling procedure

In this subsection, we discuss the importance sampling procedure to derive the ratio of integrals for finding the Bayes estimator of parameters α and σ. We also derive the HPD credible intervals of the unknown parameters. Here we use the importance sampling method to obtain Bayes estimates of the parameters α and σ as considered by Kundu and Pradhan (2009a). We can write the joint posterior distribution given in (3.2).

$\Pi(\alpha,\sigma \mid x) \propto f_\alpha(c+s-r+1, d-(r-1)\,log[x_r]-Z)\ f_{\sigma|\alpha}(b-as,\ A(\sigma)),\ h(\alpha,\sigma|x)$

where

$$h(\alpha,\sigma|x) = \frac{e^{-Z}\left[1-\left(\frac{x_s}{\sigma}\right)^{\alpha}\right]^{n-s}\Gamma(b-as)}{\exp\{(b-as)\,log(A(\sigma))\}} \tag{3.7}$$

$$f_\alpha(c+s-r+1, d-(r-1)\,log[x_r]-Z) = \alpha^{c+s-r+1}\, e^{-\alpha(d-(r-1)\,log[x_r]-Z)}\,\frac{(d-(r-1)\,log[x_r]-Z)^{c+s-r+1}}{\Gamma(c+s-r+1)} \tag{3.8}$$

which is gamma distribution with scale parameter $d-(r-1)\,log[x_r]-Z$ and shape parameter $(c+s-r=+1)$ and

$$f_{\sigma|\alpha}(b) \tag{3.9}$$

is gamma distribution with scale parameter $A(\sigma)$ and shape parameter b $-\alpha s$is gamma distribution with scale parameter $A(\sigma)$ and shape parameter b $-\alpha s$

Thus, the Bayes estimations of the parameters under the importance sampling procedure can approximate using the following:

Algorithm 1

I. The initial values are n, r, and s.

II. Based on the method of Type-II double censored method generate the data and derive Bayes estimators of α and σ.

III. Using the joint posterior distribution in (3.2).

a Generate N values of α say $(\alpha_{1,}\ \alpha_{2,}\ \ \ldots, \alpha_N)$ from gamma $(c + s - r + 1, d - (r-1)\ log[x_r] - Z)$ distribution.

b Generate N values of σ say $(\sigma_{1,}\ \sigma_{2,}\ \ \ldots, \sigma_N)$ from gamma $(b - as,\ A(\sigma))$ distribution based on the N value of α say $(\alpha_{1,}\ \alpha_{2,}\ \ \ldots, \alpha_N)$ respectively obtain in the earlier step (a).

IV. Based on the N value of α and σ compute N value of $h\ (\alpha, \sigma|x)$.

V. Then, the Bayes estimators of the parameter $\theta = (\alpha, \sigma)$ under SELF, ELF, and LLF simplify.

$$\hat{\theta}_{self} = \frac{\sum_{k_1=1}^{N} \theta_{k_1}\ h(\alpha_{k_1}, \sigma_{k_1})}{\sum_{k_1=1}^{N}\ h(\alpha_{k_1}, \sigma_{k_1})},$$

$$\hat{\theta}_{elf} = \left[\frac{\sum_{k_1=1}^{N} (\theta_{k_1})^{-q}\ h(\alpha_{k_1}, \sigma_{k_1})}{\sum_{k_1=1}^{N}\ h(\alpha_{k_1}, \sigma_{k_1})}\right]^{-1/q},$$

and

$$\hat{\theta}_{llf} = -\frac{1}{h}\ log\left[\frac{\sum_{k_1=1}^{N} e^{-h\theta_{k_1}}\ h(\alpha_{k_1}, \sigma_{k_1})}{\sum_{k_1=1}^{N}\ h(\alpha_{k_1}, \sigma_{k_1})}\right],$$

Where $h(\alpha_{k_1}, \sigma_{k_1}),\ k_1 = 1{,}2$ are given by (3.7).

3.3.3 HPD credible interval

In this section, we suggest the HPD credible intervals of α and σ using the procedure discussed by Chen and Sao (1999). Define $(\alpha, \sigma) = (\hat{\alpha}_{(\mu)}, \hat{\sigma}_{(\mu)})$, where $\alpha^{(\mu)}$ and $\sigma^{(\mu)}$ for $\mu = 1, 2 \ldots, M$ are posterior samples generated respectively from (3.8) to (3.9) for α and σ. Let $\hat{\alpha}_\mu$ and $\hat{\sigma}_\mu$ be the ordered values of $\hat{\alpha}_{(\mu)}$ and $\hat{\sigma}_{(\mu)}$. Define

$$\omega_\mu = \frac{h(\alpha^\mu, \sigma^\mu)}{\sum_{\mu=1}^{M} h(\alpha^\mu, \sigma^\mu)}, \mu = 1, 2, .., M.$$

The t^{th} quantile of $\hat{\alpha}$ can be estimated as

$$\hat{\alpha}^t = \begin{cases} \hat{\alpha}_{(1)} & if\ \ t = 0 \\ \hat{\alpha}_{(i)} & if\ \ \sum_{j=1}^{\mu-1} \omega_j < t < \sum_{j=1}^{\mu} \omega_j \end{cases}.$$

The $100(1-\delta)$ %, where $0 < \delta < 1$, confidence interval for $\hat{\alpha}$ is given by $(\hat{\alpha}^{j/M}, \hat{\alpha}^{(j+[(1-\mu)M]/M)}\ )$, $j = 1,2, \ldots, M$, where [.] is the greatest integer function. Similarly, we can construct the $100(1-\delta)$ % HPD credible interval for σ.

3.3.4 Markov Chain Monte Carlo Method

This subsection discusses the Gibbs sampling procedure to generate a sample from the posterior distribution. We have used the concept of Metropolis-Hastings (M-H) under the Gibbs sampling procedure to create models from the posterior density function (3.2). The assumption is that parameters α and σ have independent gamma density functions with hyper-parameters c, d, b, and a, respectively. The conditional posterior densities of α and σ are, respectively;

$$\Pi(\alpha|\sigma, \underline{x}) \propto \alpha^{\phi_1 - 1} e^{-\psi_1 \alpha} \left[1 - \left(\frac{x}{\sigma}\right)^\alpha\right]^{n-s} e^{-[Z]}, \tag{3.10}$$

and

$$\Pi(\sigma|\alpha, \underline{x}) \propto \sigma^{\phi_2 - 1} e^{-\psi_2 \sigma} \left[1 - \left(\frac{x}{\sigma}\right)^\alpha\right]^{n-s} e^{-[Z]}. \tag{3.11}$$

We shall use the following steps for the generation of the required sample:

Algorithm 2:

a) Start with initial guess $(\alpha^{(0)} = \ \hat{\alpha}\ and\ \sigma^{(0)} = \ \hat{\sigma})$
b) Based on M-H create α^I using (3.10) with the $N(\alpha^{(I-1)}, \beta_1)$ proposal distribution, where β_1 is from variance-covariance matrix.
c) Based on M-H create σ^I using (3.11) with the $N(\sigma^{(I-1)}, \beta_2)$ proposal distribution, where β_2 is from variance-covariance matrix.
d) Calculate α^I and σ^I
e) Put I=I+1.Repeat steps (2-5) M times.

f) We get the point estimation by Bayes estimates of the parameters $\theta = (\alpha, \sigma)$ for the SEL, ELF, and LLF are given respectively by

$$\hat{\theta}_{self} = \left(\frac{1}{M-T_0}\left[\sum_{I=1}^{M-T_0}(\theta^I)\right]\right),$$

$$\hat{\theta}_{elf} = \left(\frac{1}{M-T_0}\left[\sum_{I=1}^{M-T_0}(\theta^I)^{-q}\right]\right)^{-\frac{1}{q}}$$

and

$$\hat{\theta}_{llf} = -\frac{1}{p}\log\left(\frac{1}{M-T_0}\left[\sum_{I=1}^{M-T_0}e^{-p\theta^I}\right]\right).$$

g) To compute the credible intervals of order $\theta_{T_0+1}, \ldots, \theta_M$ respectively. Hence, the symmetric credible interval with $100(1-\delta)\%$, $0 < \delta < 1$ is

$$\theta_{((M-T_0)(\delta/2))}, \theta_{((M-T_0)(1-\delta/2))},$$

where T_0 is the number of iterations (burn-in-period).

4 PREDICTION

In many business and engineering problems, the behaviour of future samples is predicted based on information from existing ones. For example, past data indicate the number of random failures for the coming year. Jeffrey (1961) and many others have studied this problem from a Bayesian perspective. In this section, based on the type-II doubly censored sample $\underline{X} = x_r, \ldots, x_s$, the issue under consideration is predicting the failure times of the unfailed things, $Y_{k:n}$, $k = s+1,\ldots,n$ in a future sample of n items independently drawn from the same population. We consider the problem of predicting the unfailed items $Y_{k:n} = X_{s+1},\ldots, X_n$. For predicting $Y_{k:n}$, we first obtain the posterior predictive density of $Y_{k:n}$ given X. Then, the posterior predictive density of $Y_{k:n}$ given X is given by

$$\Pi^*(\alpha, \sigma|\underline{x}) = \int_\alpha\int_\sigma f(Y_{k:n}|\alpha,\sigma)\Pi(\alpha,\sigma|\underline{X})\, d\alpha\, d\sigma. \tag{4.1}$$

We are often interested in predicting an order statistic, $Y_{k:n}$ of an unfailed item of size n. Thus, we need the predictive density of the future order statistics $Y_{k:n}$ given informative sample X. The pdf $g(Y_{k:n}|\alpha,\sigma)$ of n^{th} order statistics is given by;

$$g(Y_{k:n}|\alpha,\sigma) = \frac{n!}{(r-1)!(n-s)!}\{F(Y_{k:n}|\alpha,\sigma)\}^{r-1}\{1-F(Y_{k:n}|\alpha,\sigma)\}^{n-s}f(Y_{k:n}|\alpha,\sigma), \tag{4.2}$$

where $f(.|\alpha, \sigma)$ and $F(.|\alpha, \sigma)$ are the corresponding density and cumulative distribution function. If we denote the predictive density of $Y_{k:n}$ as $g^*(.|\underline{X})$, then

$$g^*(Y_{k:n}|\underline{X}) = \iint g(Y_{k:n}|\alpha,\sigma)\,\Pi(\alpha,\sigma|X), d\alpha\, d\sigma, \tag{4.3}$$

where $\Pi(\alpha, \sigma|\underline{X})$ is the joint posterior density of α and σ. The above shows that can not explicitly solve the integral in equation (4.3). Thus, we propose using the MCMC technique to estimate the predictive density. For any fixed value of $Y_{k:n}$, we simulate a value of α, σ. Therefore, we can write a consistent simulation estimator of $g^*(.|\underline{X})$ as,

$$\hat{g}^*(Y_{k:n}|\underline{X}) = \frac{1}{N}\sum_{k=s+1}^{n} g(Y_{k:n}|\alpha_k,\sigma_k). \tag{4.4}$$

It should note that the evaluation of the expression (4.4), which was briefly discuses in the previous section, may be done using the same MCMC approach. Therefore, the Bayes estimators of the $Y_{k:n}$ given X using predictive density are offered by and where $g(Y_{k:n}|\alpha_k, \sigma_k)$ is provided by (4.3).

$$Y_{k:n}^{\ self} = \left[\frac{1}{N}\sum_{k=s+1}^{n} Y_{k:n}\, g(Y_{k:n} \mid \alpha_k, \sigma_k)\right],$$

$$Y_{k:n}^{\ elf} = \left[\frac{1}{N}\sum_{k=s+1}^{n} Y_{k:n}^{\ -q}\, g(Y_{k:n} \mid \alpha_k, \sigma_k)\right]^{-\frac{1}{q}},$$

and

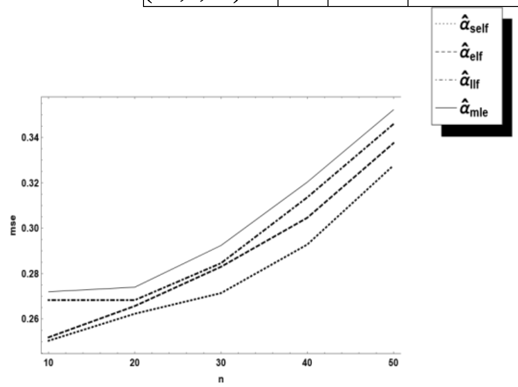
$$Y_{k:n}{}^{llf} = -\frac{1}{p}\, log\left[\frac{1}{N}\sum_{k=s+1}^{n} e^{-pY_{k:n}}\, g(Y_{k:n} \mid \alpha_k, \sigma_k)\right]$$

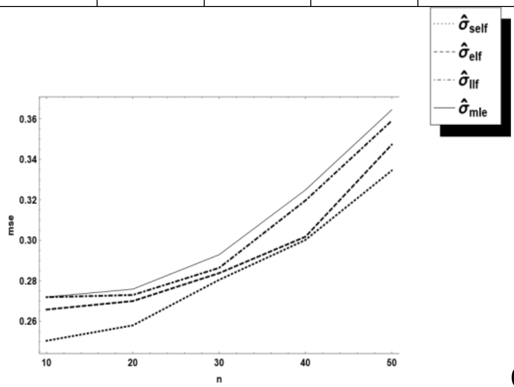
Table 2: Bias and MSE for MLE and AIL and CP for CI of the α

(n, r, s)	α	σ	$\hat{\alpha}_{mle}$		confidence interval			
					asymptotic		HPD	
			Bias	MSE	AIL	CP	AIL	CP
(10,3,5)	1	0.5	-0.4818	0.2811	0.1007	0.9129	0.4323	0.9572
(10,3,5)	3	0.75	-0.2827	0.8025	0.1825	0.9091	0.1544	0.9398
(15,3,10)	1	0.5	-0.8426	0.7113	0.1019	0.9499	0.3670	0.9335
(15,3,10)	3	0.75	-0.2924	0.8555	0.1552	0.9399	0.2704	0.9313
(20,3,15)	1	0.5	-0.6930	0.4829	0.0929	0.9104	0.1687	0.9299
(20,3,15)	3	0.75	-0.2886	0.8333	0.0925	0.9085	0.0181	0.9157

Table 3: Bias and MSE for MLE and AIL and CP for CI of the σ

(n, r, s)	α	σ	$\hat{\sigma}_{\mathrm{mle}}$		confidence interval			
					asymptotic		HPD	
			Bias	MSE	AIL	CP	AIL	CP
(10,3,5)	1	0.5	-0.2727	0.1013	0.5083	0.9881	0.4953	0.9867
(10,3,5)	3	0.75	-0.2686	0.1108	0.4995	0.9782	0.5580	0.9917
(15,3,10)	1	0.5	-0.1849	0.0598	0.4688	0.9805	0.5131	0.9891
(15,3,10)	3	0.75	-0.2007	0.0733	0.4818	0.9716	0.5021	0.9786
(20,3,15)	1	0.5	-0.3130	0.1326	0.4955	0.9863	0.4268	0.9663
(20,3,15)	3	0.75	-0.3547	0.1906	0.5422	0.9886	0.6520	0.9982





(a) (b)

Figure 1: MLE and Bayes estimators of the α and σ under different loss functions based on the M-H Algorithm

Table 4: MSE value for Bayes estimators of α and σ under Lindley approximation method

(n, r, s)	α	σ	$\hat{\alpha}_{self}$	$\hat{\alpha}_{elf}$	$\hat{\alpha}_{llf}$	$\hat{\sigma}_{self}$	$\hat{\sigma}_{elf}$	$\hat{\sigma}_{llf}$
			MSE	MSE	MSE	MSE	MSE	MSE
(10,3,5)	1	0.5	0.3233	0.1811	0.5093	0.2627	0.3433	0.3178
(10,3,5)	3	0.75	0.2977	0.1577	0.1712	0.2297	0.2077	0.2577
(15,3,10)	1	0.5	0.1897	0.1087	0.4111	0.1853	0.1805	0.2558
(15,3,10)	3	0.75	0.0290	0.0939	0.1203	0.0262	0.1412	0.1004
(20,3,15)	1	0.5	0.0258	0.0658	0.0883	0.0099	0.1291	0.0296
(20,3,15)	3	0.75	0.0054	0.0534	0.0296	0.0045	0.0597	0.0242

Table 5: MSE value for Bayes estimators of α and σ under importance sampling procedure

(n, r, s)	α	σ	$\hat{\alpha}_{self}$	$\hat{\alpha}_{elf}$	$\hat{\alpha}_{llf}$	$\hat{\sigma}_{self}$	$\hat{\sigma}_{elf}$	$\hat{\sigma}_{elf}$
			MSE	MSE	MSE	MSE	MSE	MSE
(10,3,5)	1	0.5	0.2805	0.3527	0.3920	0.4536	0.3003	0.2528

(10,3,5)	3	0.75	0.2559	0.3278	0.3380	0.3460	0.3246	0.2188
(15,3,10)	1	0.5	0.2103	0.2260	0.2843	0.2400	0.2573	0.1807
(15,3,10)	3	0.75	0.1660	0.1871	0.1597	0.1694	0.2336	0.1190
(20,3,15)	1	0.5	0.1260	0.0778	0.1109	0.1431	0.1805	0.1160
(20,3,15)	3	0.75	0.0885	0.0986	0.0877	0.1277	0.1070	0.0485

5 SIMULATION STUDY

This section conducts the simulation study to investigate the performance of the proposed estimators. This study's MLE and Bayes estimators may calculate the difference numerically using Monte Carlo Simulations. A Monte Carlo simulation study compares the Bayes estimators under various loss functions, sample sizes, left and right test termination times, and parametric values. Take random samples of sizes n = 10;15;20 from PFD with α = 1;3 and σ = 0.5;0.75 respectively. Consider the values of hyperparameters as c = 2.2, d = 2, b = 2.4 and a = 2.5 respectively. We use the following algorithm, the generated samples under a doubly censored scheme for N=1000 times.

Algorithm 3:

i Start with initial values n, α, σ, a, b, c, d, r, s, q, and h.
ii Create the Power function distribution with parameters (α, σ).
iii Determine the test terminations points on left and right, that is, the values of x_r and x_s.
iv The observations x_r is less, and x_s is more excellent have been considered censored from each sample.
v Calculate the MLE and Bayes estimators using SELF, ELF, and LLF with the censored samples.
vi Compute the average bias and MSE.
vii Repeat step (1-6) 1000 times for the desired sample size.

Table 6: The Bayes estimators of the parameters correspond to the electrical insulator's actual data.

α = 0.0137193 and σ = 0.399591									
(n, r, s)		estimation						Prediction	
		MLE	AIL	CP	LAM	ISP	M-H	k=10	k=18
(19,3,5)	$\hat{\alpha}_{mle}$	0.3629	-	-	-	-	-	-	-
	$\hat{\sigma}_{mle}$	0.1323	-	-	-	-	-	-	-
	$\hat{\alpha}_{ACI}$	-	0.5924	0.9680	-	-	-	-	-
	$\hat{\sigma}_{ACI}$	-	0.6004	0.9969	-	-	-	-	-
	$\hat{\alpha}_{self}$	-	-	-	0.2675	0.3005	0.2614	-	-
	$\hat{\sigma}_{self}$	-	-	-	0.1856	0.1228	0.4021	-	-
	$\hat{\alpha}_{elf}$	-	-	-	0.1478	0.1623	0.2213	-	-
	$\hat{\sigma}_{elf}$	-	-	-	0.1462	0.1005	0.3210	-	-
	$\hat{\alpha}_{llf}$	-	-	-	0.2845	0.2680	0.1956	-	-
	$\hat{\sigma}_{llf}$	-	-	-	0.2036	0.2215	0.2732	-	-
	$Y_{k:n}^{self}$	-	-	-	-	-	-	0.3527	0.6542
	$Y_{k:n}^{elf}$	-	-	-	-	-	-	0.3637	0.8324
	$Y_{k:n}^{llf}$	-	-	-	-	-	-	0.3689	0.7954

Tables 2 and 3 present the average bias and MSE of the MLE together with the asymptotic confidence intervals of AIL and CP. MSE value of Bayes estimates using Lindley approximation and importance sampling procedures are given in Tables 4 and 5, respectively. We can report the following points from the numerical results presented in Tables 2-5.

1. We can see that the Bayes estimator is better than MLE in bias and MSE. It is because Bayes estimators have more information in the form of prior knowledge than MLE.
2. The average length of the approximate confidence intervals also increases as the sample size increases, while the coverage probability is around 0.95.
3. From Table 4-5, the MSE of the estimators decreases with increasing sample size.
4. The HPD credible intervals' width is smaller than the width of the asymptotic confidence intervals in all the cases. However, the width of the confidence / HPD intervals decreases as the sample size increases.
5. In Figure 1, using MSE, Bayes estimators and MLE perform the better increasing

6 NUMERICAL EXAMPLE

In this example, we analyze an actual data set, considered by Lawless (1982), representing the lifetime of a type of electrical insulator subject to constant voltage stress. It shows that the Power function distribution fits the data set well. We have checked the validity of PFD using the Kolmogorov Smirnov (KS) test and observed that the KS distance corresponding to the p-value = 0.265107 is 0.114393. Suppose that the first $r = 3$ smallest and the largest $s = 5$ observations censored. They represent the lifetime (in minutes) to fail: 0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89. Then the doubly Type-II censored sample is observed as

$(x_4, x_5, ..., x_{15}) = (0.78,8.01,31.75,7.35,6.50,8.27,33.91,32.52,3.16,4.85,2.78)$.

For the Bayes estimation, we choose hyper-parameters values as a=1.6, b=3.2, c=2, and d=2, respectively. Bayesian estimates using LL and EL functions are evaluated for h = 2 and q = 2. The results of the analysis are in Table 6. The amounts of estimation associated with each estimator have given in the table. The table presents the estimated value of MLE and Bayes estimator as the best when looking at the actual data results.

7 CONCLUSION

This paper proposed the estimating parameters of the PFD for the Type-II doubly censored data using MLE and Bayesian techniques. Bayes estimates are simplified using the Lindley approximation method and the Importance sampling procedure. The predictive posterior density for the future ordered statistics is difficult to obtain; using Bayesian computational techniques can be easily implemented to estimate the predictive densities. In our simulation study, we found that the bias and MSE values are satisfactory based on the estimated value of the MLE and Bayes estimators of the parameters under different loss functions. In data analysis, we study the performance of the proposed MLE and Bayes estimators under different loss functions. Large Bayesian samples are better than small classical samples in interpreting the actual study. The simulation shows that Bayes estimates perform better than the confidence interval and MLE. Also, the Bayes estimation of SELF is better in all cases. We can get the forecast interval for future observation and the ratio of subsequent failure times. An example shows the use of all the methods included in this paper.

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