A MODIFIED CLASS OF REGRESSION ESTIMATORS BY USING HUBER M ESTIMATION

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ABSTRACT
In this article we have proposed some new improved ratio estimators based on robust regression that are robust against outliers and provide reliable results even when outliers are present; the properties are also investigated. The proposed class of estimators has been shown to be more effective than the current classes of estimators. An empirical analysis was conducted to see how well the proposed class of estimators compared to others.

KEY WORDS: Quartiles, Deciles, Kurtosis, Non-Conventional location parameters, Median, M-Estimation, Bias, Mean Square Error, Efficiency.

MSC: 62D05

RESUMEN
En este articulo hemos propuesto un nuevo estimador razón mejorado basado en la regresión robusta para los outliers y provee de resultados confiables aun ante la presencia de outliers; las propiedades son también investigadas . Se prueba que la clase propuesta de estimadores es más efectiva que otras clases de estimadores. Un análisis empírico se desarrolló para ver cuán bien se desempeña la nueva clase de estimadores respecto a otras.

PALABRAS CLAVE: Cuartiles, Deciles, Curtosis, parámetros de posición No-Convencionales Mediana, M-Estimación, Sesgo, Error Cuadrático Medio , Eficiencia

1. INTRODUCTION

The association between the auxiliary variable x and the study variable yis exploited by ratio-type estimators. When data on an auxiliary variable that is positively correlated with the study variable is available, the ratio estimator is a good choice for estimating the population mean. In sampling theory, population knowledge of the auxiliary variable is useful for ratio estimators. The outlier problem, which occurs when data contains extreme values, reduces efficiency because classical estimators are responsive to these extreme values. As a result, we suggest using Huber M-estimates in this paper. In ratio estimators, rather than least squares (LS) calculations, to reduce the negative effects of outlier data problems. In the next section, we'll go over conventional ratio estimators for the population mean in simple random sampling, as well as their MSE equations. We propose ratio estimators and discuss their MSE equations, as well as efficiency comparisons based on the MSE equations between traditional/existing and proposed estimators. In the last part, we numerically arrive at a conclusion based on these findings.

Notations used in this paper are:

N Population size
n Sample size
$f=\frac{n}{N}$ Sampling fractuion
Y Study Variable
X Auxiliary Variable
$\bar{Y},\bar{X}$ Population means
$\bar{y},\bar{x}$ Sample means
S_y,S_x Population standard deviations
S_{yx} Population covariance between variables
C_y,C_x Population coefficient of variation
ρ Population correlation coefficient
$B(.)$ Bias of estimator
$MSE(.)$ Mean square error of estimator
φ_{ZR} Proposed estimator
M_d Population Median of auxiliary variable
$\beta_{2(x)}$ Population kurtosisof auxiliary variable
$\beta_{1(x)}$ Population skewness of auxiliary variable
$\mathrm{HL}=\mathrm{median}\left|\frac{(X_i+X_k)}{2}, 1\le j\le k\le N\right|$ Hodges − Lehman estimator
$\mathrm{MR}=\frac{\left(X_{(1)}+X_{(N)}\right)}{2}$ Population mid range

$TM = \frac{Q_1 + 2Q_2 + Q_3}{4}$

2. EXISTING ESTIMATORS IN THE LITERATURE

Using some known auxiliary information on coefficient of kurtosis and coefficient of variation, Kadilar and Cingi (2004) proposed ratio type estimators for the population mean in simple random sampling. In the estimation of the population mean, they demonstrated that their suggested estimators are more effective than conventional ratio estimators. The estimators of Kadilar&Cingi (2004) are provided by

$$\hat{\hat{Y}}_1 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}}, \quad \hat{\hat{Y}}_2 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x} + C_x}(\bar{X} + C_x), \quad \hat{\hat{Y}}_3 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x} + \beta_2}\left(\bar{X} + \beta_{2(x)}\right)$$

$$\hat{\hat{Y}}_4 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}\beta_{2(x)} + C_x}\left(\bar{X}\beta_{2(x)} + C_x\right), \quad \hat{\hat{Y}}_5 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}C_x + \beta_{2(x)}}\left(\bar{X}C_x + \beta_{2(x)}\right)$$

The following modified ratio estimators were developed by Kadilar and Cingi (2006) using known values of coefficient of correlation, kurtosis, and coefficient of variation.

$$\hat{\hat{Y}}_6 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x} + \rho}(\bar{X} + \rho), \quad \hat{\hat{Y}}_7 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}C_x + \rho}(\bar{X}C_x + \rho),$$

$$\hat{\hat{Y}}_8 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}\rho + C_x}(\bar{X}\rho + C_x) \quad \hat{\hat{Y}}_9 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}\beta_{2(x)} + \rho}\left(\bar{X}\beta_{2(x)} + \rho\right),$$

$$\hat{\hat{Y}}_{10} = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}\rho + \beta_{2(x)}}\left(\bar{X}\rho + \beta_{2(x)}\right)$$

The MSE of the estimators is given by

$$MSE\left(\hat{\hat{Y}}_i\right) = \lambda\left(R_{kci}^2 s_x^2 + 2BR_{kci}s_x^2 + B^2 s_x^2 - 2R_{kci}s_{xy} - 2Bs_{xy} + s_y^2\right), i = 1\ to\ 10$$

Where

$R_{kc1} = \frac{\bar{Y}}{\bar{X}}$, $\quad R_{kc2} = \frac{\bar{Y}}{\bar{X} + C_x}$, $\quad R_{kc3} = \frac{\bar{Y}}{\bar{X} + \beta_{2(x)}}$

$R_{kc4} = \frac{\bar{Y}\beta_{2(x)}}{\bar{X}\beta_{2(x)} + C_x}$, $\quad R_{kc5} = \frac{\bar{Y}C_x}{\bar{X}C_x + \beta_{2(x)}}$, $\quad R_{kc6} = \frac{\bar{Y}}{\bar{X} + \rho}$

$R_{kc7} = \frac{\bar{Y}C_x}{\bar{X}C_x + \rho}$, $\quad R_{kc8} = \frac{\bar{Y}\rho}{\bar{X}\rho + C_x}$, $\quad R_{kc9} = \frac{\bar{Y}\beta_{2(x)}}{\bar{X}\beta_{2(x)} + \rho}$

$R_{kc10} = \frac{\bar{Y}\rho}{\bar{X}\rho + \beta_{2(x)}}$

Yan and Tian (2010) suggested the following two modified ratio estimators based on kurtosis and coefficient of skewness.

$$\hat{\hat{Y}}_{11} = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x} + \beta_{1(x)}}\left(\bar{X} + \beta_{1(x)}\right), \quad \hat{\hat{Y}}_{12} = \frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}\beta_{1(x)} + \beta_{2(x)}}\left(\bar{X}\beta_{1(x)} + \beta_{2(x)}\right)$$

The MSE of the estimators is given by

$$MSE\left(\hat{\hat{Y}}_i\right) = \lambda\left(R_{YTi}^2 s_x^2 + 2BR_{YTi}s_x^2 + B^2 s_x^2 - 2R_{YTi}s_{xy} - 2Bs_{xy} + s_y^2\right), i = 11\ to\ 12$$

Where

$$R_{YT11} = \frac{\bar{Y}}{\bar{X} + \beta_{1(x)}}, \quad R_{YT12} = \frac{\bar{Y}\beta_{1(x)}}{\bar{X}\beta_{1(x)} + \beta_{2(x)}}$$

Subzar et al (2019) proposed the following robust regression estimator as

$$\hat{\hat{Y}}_{13} = \frac{\bar{y} + b_{rob}(\bar{X} - \bar{x})}{\bar{x}Q_1 + M_d}(\bar{X}Q_1 + M_d), \quad \hat{\hat{Y}}_{14} = \frac{\bar{y} + b_{rob}(\bar{X} - \bar{x})}{\bar{x}Q_2 + M_d}(\bar{X}Q_2 + M_d)$$

$$\hat{\hat{Y}}_{15} = \frac{\bar{y} + b_{rob}(\bar{X} - \bar{x})}{\bar{x}Q_3 + M_d}(\bar{X}Q_3 + M_d), \quad \hat{\hat{Y}}_{16} = \frac{\bar{y} + b_{rob}(\bar{X} - \bar{x})}{\bar{x}TM + M_d}(\bar{X}TM + M_d)$$

$$\hat{\hat{Y}}_{17} = \frac{\bar{y} + b_{rob}(\bar{X} - \bar{x})}{\bar{x}MR + M_d}(\bar{X}MR + M_d), \quad \hat{\hat{Y}}_{18} = \frac{\bar{y} + b_{rob}(\bar{X} - \bar{x})}{\bar{x}HL + M_d}(\bar{X}HL + M_d)$$

The MSE of the estimators is given by

$$MSE\left(\hat{\bar{Y}}_i\right) = \lambda\left(R_{SBi}^2 s_x^2 + 2B_{rob}R_{SBi}s_x^2 + B_{rob}^2 s_x^2 - 2R_{SBi}s_{xy} - 2B_{rob}s_{xy} + s_y^2\right)$$

Where,

$$R_{SB13} = \frac{\bar{Y}Q_1}{\bar{X}Q_1 + M_d}, \qquad R_{SB14} = \frac{\bar{Y}Q_2}{\bar{X}Q_2 + M_d}, \qquad R_{SB15} = \frac{\bar{Y}Q_3}{\bar{X}Q_3 + M_d}$$

$$R_{SB16} = \frac{\bar{Y}TM}{\bar{X}TM + M_d}, \qquad R_{SB17} = \frac{\bar{Y}MR}{\bar{X}MR + M_d}, \qquad R_{SB18} = \frac{\bar{Y}HL}{\bar{X}HL + M_d}$$

Where,C_x,$\beta_{2(x)}$, HL,TM,MR, M_d, $\beta_{1(x)}$, Q_1,Q_2, and$Q_3$are the population coefficient of variation, population coefficient of the kurtosis, Hodges and Lehman, Trimmed mean, Mid-range, Median, Skewness, first quartile, second quartile, and third quartile respectively, of the auxiliary variable; $\bar{y}$ and $\bar{x}$ are the sample means of the study variable and auxiliary variable respectively and it is assumed that the population mean $\bar{X}$ of the auxiliary variable x is known. Here $\frac{s_{xy}}{s_x^2}$ is obtained by least square method, where s_x^2 and s_y^2 are the sample variances of the auxiliary and the study variable, respectively and s_{xy} is the sample covariance between the study and the auxiliary variable

The main advantage of the Huber M-estimates over LS estimates is that they are not sensitive to outliers. Thus, when there are outliers in the data, M-estimation is more accurate than LS estimation. Huber M-estimates use a function $\rho(e)$ that is a compromise between $e^2|e|$, where e is the error term of the regression model $y = a + bx + e$, a being the constant of the model. The Huber $\rho(e)$ function has the form

$$\rho(e) = \begin{cases} e^2 - k \leq e \leq k \\ 2\mathrm{k}|e| - k^2 e < -kork < e \end{cases}$$

Where k is a tuning constant that controls the robustness of the estimators. Huber (1964) suggested $k = 1.5\hat{\sigma}$, where $\hat{\sigma}$, is an estimate of the standard deviation, σ of the population random errors. Details about constant k and M-estimators can be found in Candan(1995), Rousseeuw and Leroy(1987)

The value of the regression coefficient,b_{rob} is obtained by minimizing

$$\sum_{i=1}^{n} \rho(y_i - a - bx_i)$$

With respect to a and b. The details of the minimization procedure can be found in Birkes and Dodge (1993).

3. PROPOSED ESTIMATOR

Motivated from Sisodia and Dwevedi (1981) and classical regression estimator, we have proposed a modified class of regression estimator using Huber M estimation.

$$t_{ZRi} = \bar{y}\left(\frac{\bar{X}'}{\bar{x}'}\right) + b_{rob}(\bar{X} - \bar{x}) \tag{1}$$

Where $\bar{X}' = \alpha\bar{X} + \beta$ and $\bar{x}' = \alpha\bar{x} + \beta$

To obtain the bias and MSE of the modified estimator φ_{ZR} given by (1), we write

$\varepsilon_y = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$ and $\varepsilon_x = \frac{\bar{x}-\bar{X}}{\bar{X}}$

Such that $E(\varepsilon_y) = E(\varepsilon_x) = 0$ and$E(\varepsilon_y^2) = \lambda C_y^2 E(\varepsilon_x^2) = \lambda C_x^2 E(\varepsilon_y \varepsilon_x) = \lambda C_{yx}$

Expressing (1) in terms of ε_y and ε_x, neglecting the terms of $\varepsilon's$ having power greater than two, we have

$t_{ZRi} = \bar{Y}(1+\varepsilon_y)(1+\varphi_i\varepsilon_x)^{-1} + b_{rob}\bar{X}\varepsilon_x$; $\varphi_i = \frac{\alpha\bar{X}}{\alpha\bar{X}+\beta}$, $\varphi_1 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_1}$ $\varphi_2 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_2}$, $\varphi_3 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_3}$, $\varphi_4 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_4}$ $\varphi_5 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_5}$, $\varphi_6 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_6}$, $\varphi_7 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_7}$, $\varphi_8 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_8}$, $\varphi_9 = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_9}$, $\varphi_{10} = \frac{\beta_2\bar{X}}{\beta_2\bar{X}+D_{10}}$

The proposed families of estimators are

Proposed estimators	α	β
$t_{ZR1} = \bar{y}\left(\frac{\beta_2\bar{X} + D_1}{\beta_2\bar{x} + D_1}\right) + b_{rob}(\bar{X} - \bar{x})$	β_2	D_1
$t_{ZR2} = \bar{y}\left(\frac{\beta_2\bar{X} + D_2}{\beta_2\bar{x} + D_2}\right) + b_{rob}(\bar{X} - \bar{x})$	β_2	D_2
$t_{ZR3} = \bar{y}\left(\frac{\beta_2\bar{X} + D_3}{\beta_2\bar{x} + D_3}\right) + b_{rob}(\bar{X} - \bar{x})$	β_2	D_3
$t_{ZR4} = \bar{y}\left(\frac{\beta_2\bar{X} + D_4}{\beta_2\bar{x} + D_4}\right) + b_{rob}(\bar{X} - \bar{x})$	β_2	D_4

$t_{ZR5} = \bar{y}\left(\frac{\beta_2\bar{X}+D_5}{\beta_2\bar{x}+D_5}\right)+b_{rob}(\bar{X}-\bar{x})$	β_2	D_5
$t_{ZR6} = \bar{y}\left(\frac{\beta_2\bar{X}+D_6}{\beta_2\bar{x}+D_6}\right)+b_{rob}(\bar{X}-\bar{x})$	β_2	D_6
$t_{ZR7} = \bar{y}\left(\frac{\beta_2\bar{X}+D_7}{\beta_2\bar{x}+D_7}\right)+b_{rob}(\bar{X}-\bar{x})$	β_2	D_7
$t_{ZR8} = \bar{y}\left(\frac{\beta_2\bar{X}+D_8}{\beta_2\bar{x}+D_8}\right)+b_{rob}(\bar{X}-\bar{x})$	β_2	D_8
$t_{ZR9} = \bar{y}\left(\frac{\beta_2\bar{X}+D_9}{\beta_2\bar{x}+D_9}\right)+b_{rob}(\bar{X}-\bar{x})$	β_2	D_9
$t_{ZR10} = \bar{y}\left(\frac{\beta_2\bar{X}+D_{10}}{\beta_2\bar{x}+D_{10}}\right)+b_{rob}(\bar{X}-\bar{x})$	β_2	D_{10}

$$t_{ZRi}-\bar{Y} = \bar{Y}(\varepsilon_y+\varphi_i\varepsilon_x+\varphi_i^2\varepsilon_x^2-\varphi_i\varepsilon_y\varepsilon_x-b_{rob}L\varepsilon_x)$$

Taking expectation of both sides, we have the bias of t_{ZRi} up to the first degree of approximation as:

$$Bias(t_{ZRi}) = \lambda\bar{Y}(C_x^2-\varphi_i C_{yx})$$

Squaring both sides of and neglecting the terms of $e's$ having power greater than two and then taking expectation of both sides, we get the MSE of the estimator t_{ZRi} to the first degree of approximation as

$$E(t_{ZRi}-\bar{Y})^2 = E\left[\bar{Y}^2(\varepsilon_y+\varphi_i\varepsilon_x+\varphi_i^2\varepsilon_x^2-\varphi_i\varepsilon_y\varepsilon_x-b_{rob}L\varepsilon_x)^2\right]$$

$$MSE(t_{ZRi}) = \lambda\left[S_y^2+\mathrm{B}_{\mathrm{rob}}^2S_x^2+2B_{rob}\left(\varphi_iRS_x^2-S_{yx}\right)+\varphi_i\left(\varphi_i\mathrm{R}^2S_x^2-2RS_{yx}\right)\right]$$

Where $R=\frac{\bar{Y}}{\bar{X}},\ L=\frac{\bar{X}}{\bar{Y}}$.

4. EFFICIENCY COMPARISONS

Comparison with existing estimators: This section deals with the derivation of algebraic situation, under which the proposed estimators will have minimum MSE as compared to estimators in literature, t_{ZRi} Perform better than Kadilar and Cingi (2004,2006) estimators if

$$MSE(t_{ZRi}) > MSE\left(\hat{\bar{Y}}_i\right), \quad i = 1\ to\ 10$$

$$\lambda\left[S_y^2+\mathrm{B}_{\mathrm{rob}}^2S_x^2+2B_{rob}\left(\varphi_iRS_x^2-S_{yx}\right)+\varphi_i\left(\varphi_i\mathrm{R}^2S_x^2-2RS_{yx}\right)\right] < \lambda\left[R_{kci}^2s_x^2+2BR_{kci}s_x^2+B^2s_x^2-2R_{kci}s_{xy}-2Bs_{xy}+s_y^2\right]$$

$$[(\mathrm{B}_{\mathrm{rob}}^2-B^2)+2(\varphi_iRB_{rob}-BR_{kci})+(\varphi_i^2\mathrm{R}^2-R_{kci}^2)]S_x^2-2[(B_{rob}-B)+(R\varphi_i-R_{kci})]s_{xy}<0$$

1. t_{ZRi} Perform better than Yan and Tian (2010) estimators if

$$MSE(t_{ZRi}) > MSE\left(\hat{\bar{Y}}_i\right), \quad i = 10\ to\ 12$$

$$\lambda\left[S_y^2+\mathrm{B}_{\mathrm{rob}}^2S_x^2+2B_{rob}\left(\varphi_iRS_x^2-S_{yx}\right)+\varphi_i\left(\varphi_i\mathrm{R}^2S_x^2-2RS_{yx}\right)\right] < \lambda\left[R_{YTi}^2s_x^2+2BR_{YTi}s_x^2+B^2s_x^2-2R_{YTi}s_{xy}-2Bs_{xy}+s_y^2\right]$$

$$[(\mathrm{B}_{\mathrm{rob}}^2-B^2)+2(\varphi_iRB_{rob}-BR_{YTi})+(\varphi_i^2\mathrm{R}^2-R_{YTi}^2)]S_x^2-2[(B_{rob}-B)+(R\varphi_i-R_{YTi})]s_{xy}<0$$

2. t_{ZRi} Perform better than Subzar (2019)estimators if

$$MSE(t_{ZRi}) > MSE\left(\hat{\bar{Y}}_i\right), \quad i = 13\ to\ 18$$

$$\lambda\left[S_y^2+\mathrm{B}_{\mathrm{rob}}^2S_x^2+2B_{rob}\left(\varphi_iRS_x^2-S_{yx}\right)+\varphi_i\left(\varphi_i\mathrm{R}^2S_x^2-2RS_{yx}\right)\right] < \lambda\left(R_{SBi}^2s_x^2+2B_{rob}R_{SBi}s_x^2+B_{rob}^2s_x^2-2R_{SBi}s_{xy}-2B_{rob}s_{xy}+s_y^2\right)$$

$$[2B_{rob}(\varphi_iR-R_{SBi})+(\varphi_i^2\mathrm{R}^2-R_{SBi}^2)]S_x^2-2[\varphi_iR-R_{SBi}]S_{yx}<0$$

5. EMPIRICAL STUDY

We obtained the data from Singh, D., and Chaudhary, F. S. (1986), page 177 of their book Theory and Analysis of Sample Survey Designs, in which the data for wheat in 1971 and 1973 are given, and in which the area under wheat in the region was to be estimated during 1974 is denoted by Y (study variable) by using the data for cultivated area under wheat in 1971 is denoted by X. (auxiliary variable).

Population Characteristics (Table I)

Parameter	Population	Parameter	Population	Parameter	Population
N	34	S_x	150.5059	TM	162.25
N	20	C_x	0.7205	MR	284.5
$\bar{Y}$	856.4117	$\beta_{2(x)}$	0.0978	HL	190
$\bar{X}$	208.8823	$\beta_{1(x)}$	0.9782	Q_1	94.25
ρ	0.4491	QD	80.85	Q_2	150
S_y	733.1407	B_{rob}	1.57	Q_3	254.75
C_y	0.8561	B	2.19	M_d	150

The MSE of the existing estimators (TABLE II). The MSE of the Proposed estimators (Table III)

Estimators	MSE	Estimators	MSE		Estimators	MSE	Estimators	MSE
$\hat{Y}_1$	16673.45	$\hat{Y}_{10}$	16657.19		t_{ZR1}	8878.43	t_{ZR6}	8871.41
$\hat{Y}_2$	16619.64	$\hat{Y}_{11}$	16600.54		t_{ZR2}	8862.63	t_{ZR7}	8879.36
$\hat{Y}_3$	16666.14	$\hat{Y}_{12}$	16665.98		t_{ZR3}	8835.47	t_{ZR8}	8903.21
$\hat{Y}_4$	16146..61	$\hat{Y}_{13}$	14373.69		t_{ZR4}	8836.57	t_{ZR9}	8922.94
$\hat{Y}_5$	16663.31	$\hat{Y}_{14}$	14410.62		t_{ZR5}	8842.41	t_{ZR10}	8939.96
$\hat{Y}_6$	16639.85	$\hat{Y}_{15}$	14436.48					
$\hat{Y}_7$	16626.87	$\hat{Y}_{16}$	14415.36					
$\hat{Y}_8$	16554.4	$\hat{Y}_{17}$	14445.47					
$\hat{Y}_9$	16338.65	$\hat{Y}_{18}$	14445.48					

Percent Relative Efficiencies of proposed estimators with Kadilar and Cingi (2004, 2006)

	$\hat{Y}_1$	$\hat{Y}_2$	$\hat{Y}_3$	$\hat{Y}_4$	$\hat{Y}_5$	$\hat{Y}_6$	$\hat{Y}_7$	$\hat{Y}_8$	$\hat{Y}_9$	$\hat{Y}_{10}$
t_{ZR1}	187.7973	187.1912	187.7149	181.8633	187.6831	187.4188	187.2726	186.4564	184.0263	187.6141
t_{ZR2}	188.1321	187.5249	188.0496	182.1876	188.0177	187.753	187.6065	186.7888	184.3544	187.9486
t_{ZR3}	188.7104	188.1014	188.6277	182.7476	188.5956	188.3301	188.1832	187.363	184.9211	188.5264
t_{ZR4}	188.6869	188.078	188.6042	182.7249	188.5721	188.3067	188.1598	187.3397	184.8981	188.5029
t_{ZR5}	188.5623	187.9537	188.4796	182.6042	188.4476	188.1823	188.0355	187.2159	184.776	188.3784
t_{ZR6}	187.9459	187.3393	187.8635	182.0073	187.8316	187.5671	187.4208	186.6039	184.172	187.7626
t_{ZR7}	187.7776	187.1716	187.6953	181.8443	187.6634	187.3992	187.253	186.4369	184.0071	187.5945
t_{ZR8}	187.2746	186.6702	187.1925	181.3572	187.1607	186.8972	186.7514	185.9374	183.5141	187.092
t_{ZR9}	186.8605	186.2574	186.7786	180.9562	186.7469	186.4839	186.3385	185.5263	183.1084	186.6783
t_{ZR10}	186.5047	185.9028	186.423	180.6117	186.3913	186.1289	185.9837	185.1731	182.7598	186.3229

Percent Relative Efficiencies of proposed estimators with Yan and Tian (2010) & Subzar (2019)

	$\hat{Y}_{11}$	$\hat{Y}_{12}$	$\hat{Y}_{13}$	$\hat{Y}_{14}$	$\hat{Y}_{15}$	$\hat{Y}_{16}$	$\hat{Y}_{17}$	$\hat{Y}_{18}$
t_{ZR1}	186.9761	187.7131	161.8945	162.3105	162.6017	162.3638	162.703	162.7031
t_{ZR2}	187.3094	188.0478	162.1831	162.5998	162.8916	162.6533	162.993	162.9932
t_{ZR3}	187.8852	188.6258	162.6817	163.0996	163.3923	163.1533	163.4941	163.4942
t_{ZR4}	187.8618	188.6024	162.6614	163.0793	163.372	163.133	163.4737	163.4738
t_{ZR5}	187.7377	188.4778	162.554	162.9716	163.2641	163.0252	163.3658	163.3659
t_{ZR6}	187.124	187.8617	162.0226	162.4389	162.7304	162.4923	162.8317	162.8318
t_{ZR7}	186.9565	187.6935	161.8775	162.2935	162.5847	162.3468	162.6859	162.686
t_{ZR8}	186.4557	187.1907	161.4439	161.8587	162.1492	161.9119	162.2501	162.2502
t_{ZR9}	186.0434	186.7768	161.0869	161.5008	161.7906	161.5539	161.8914	161.8915
t_{ZR10}	185.6892	186.4212	160.7802	161.1933	161.4826	161.2464	161.5832	161.5833

The percent relative efficiencies (PRE) of the proposed estimators, with respective to the existing estimators is computed by

$$PRE = \frac{MSE\ of\ Existing\ Estimator}{MSE\ of\ proposed\ estimator} \times 100$$

6. CONCLUSION

As a result, the auxiliary knowledge of Kurtosis and Deciles may be used. Our proposed estimators outperform classical and current estimators in terms of mean square error and bias, as compared to the literature's classical and existing estimators. We strongly suggest that our proposed estimators be used in the future for realistic applications over the estimators in the literature and even over classical estimators.

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