ESTIMATION OF POPULATION MEAN IN STRATIFIED SAMPLING UNDER NON-RESPONSE

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ABSTRACT
In this paper we have considered the problem of estimating the population mean using auxiliary information in presence of non-response on both study as well auxiliary variables under stratified random sampling. We have studied the properties of the suggested class of estimators under large sample approximation. Expressions for the optimum sample sizes of the strata in respect to cost of the survey have been also derived. An empirical study is carried out to see the performance of the proposed class of estimators.

KEYWORDS: Study variable, Auxiliary variable, Non-response, Stratified random sampling, Exponential estimator.

MSC: 62D05.

RESUMEN
En este paper hemos considerado el problema de estimar la media de la población usando información auxiliar en presencia de no-respuesta en ambas variables de estudio y de las auxiliares bajo muestreo estratificado. Hemos estudiado las propiedades de la clase sugerida de estimadores bajo la aproximación para grandes muestras. Expresiones para el tamaño óptimo de muestral de los estratos respecto al costo del survey también han sido derivadas. Un estudio empírico se desarrolló para ver el comportamiento de la propuesta clase de estimadores.

PALABRAS CLAVE: variable de estudio , variable auxiliar, No-respuesta, Muestreo aleatorio estratificado, estimador exponencial.

1. INTRODUCTION

The problem of non-response is very common in practice. Due to non-response the estimators of the parameters yield bias outcomes. Hansen and Hurwitz (1946) was the first to tackled the problem of non-response. Later various authors including Cochran (1977), Rao (1986), Khare and Srivastava (1997), Singh and Kumar (2008, 2009), Singh et al (2010) have discussed the problem of estimating the population mean of the study variable (*SV*) using information on auxiliary variable (*AV*) in presence of non-response using Hansen and Hurwitz (1946) technique under simple random sampling without replacement (*SRSWOR*) scheme. Usually *SRSWOR* sampling scheme is used when the population units are homogeneous. However in practice heterogeneous population are also commonly encountered. In such situation stratified random sampling is recommended. Keeping this view Chaudhary et al (2009) studied the problem of non-response in stratified random sampling assuming that non-response occurs only on *SV* has been handled. Later Sanaullah et al (2015), Saleem et al (2018) and Shabbir et al (2019) have studied the problem of non-response when non-response occurs on both study and *AV*s and on the *SV* only in stratified single and two-phase sampling. For more detailed study the reader is referred to Singh et al (2010), Singh et al (2018), Singh and Sharma (2015) and Singh and Vishwakarma (2019).
In this paper we have made an effort to propose a family of estimators for estimating the population mean of the *SV* y using information on two *AV*s (x, z) in presence of non-response under stratified random sampling. The properties of suggested family of estimators have been studied under large sample approximation. It has been shown that the proposed family of estimators is better than Saleem et al (2018) class of estimators. Numerical illustration is given in support of the present study.

2. REVIEW OF EXISTING ESTIMATORS

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Let a finite population $U = (U_1, U_2, ..., U_N)$ of size N be stratified into L strata (homogeneous). Let N_h be the size of the h^{th} stratum (h=1,2,…,L) such that $\sum_{h=1}^{L} N_h = N$. Let y and (x, z) be the study and AVs taking values (y_{hi}, x_{hi}, z_{hi}) on the i^{th} unit of the h^{th} stratum respectively. Let $(\bar{y}_h, \bar{x}_h, \bar{z}_h)$ be the sample means of the h^{th} stratum corresponding to the population means $(\bar{Y}_h, \bar{X}_h, \bar{Z}_h)$ respectively. In practice it is usually not possible to gather complete information from all the units selected in the sample $n_h \left(\sum_{h=1}^{L} n_h = n \right)$. We have studied the problem in the situation, where non-response occurs on all the variables (y, x, z).
Let $n_{h(1)}$ units from a sample of size n_h respond and $n_{h(2)}$ units do not. Employing Hansen and Hurwitz (1946) method of sub-sampling the non-respondents, a sub-sample of size $r_h \left(r_h = \frac{n_{h(2)}}{f_h}, f_h > 1 \right)$ from $n_{h(2)}$ non-respondent group is selected at random and $\frac{1}{f_h}$ denotes the sampling fraction among the non-respondent group in the h^{th} stratum. In practice, r_h is generally not integer and has to be rounded. In accordance with most of the current literature on this research topic, we suppose that the followed-up $r_h (\subset n_{h(2)})$ units respond on the second call. Further, let d denotes a dummy variable taking value d_{hi} on the i^{th} population unit of stratum h and has h^{th} stratum population mean $\bar{D}_h$. Hereafter, d may stand for if, x or for a second AV z (i.e. d_h may stand for y_h, x_h and z_h in stratified sampling). Let

$$\bar{d}_{n_{h(1)}} = \frac{1}{n_{h(1)}} \sum_{i=1}^{n_{h(1)}} d_{hi(1)}, \bar{d}_{rh(2)} = \frac{1}{r_h} \sum_{i=1}^{r_h} i d_{hi(2)}$$

and

$$\bar{d}_h^* = \frac{n_{h(1)}}{n_h} \bar{d}_{nh(1)} + \frac{n_{h(2)}}{n_h} \bar{d}_{rh(2)} \tag{2.1}$$

where $\bar{d}_{nh(1)}$ is the mean of $n_{h(1)}$ respondents on first call and $d_{rh(2)}$ is the mean of r_h units respond on the second call and $\bar{d}_h^*$ denotes the unbiased Hansen and Hurwitz (1946) estimator of $\bar{D}_h$ for stratum h.

Thus, we define an unbiased estimator of the population mean $\bar{D} = \sum_{h=1}^{L} W_h \bar{D}_h$ as

$$\bar{d}_{st}^* = \sum_{h=1}^{L} W_h \bar{d}_h^* \tag{2.2}$$

and the variance/MSE of $\bar{d}_{st}^*$ is given by

$$V(\bar{d}_{st}^*) = \sum_{h=1}^{L} \delta_h W_h^2 S_{dh}^2 + \sum_{h=1}^{L} \delta_h^* W_h^2 S_{dh(2)}^2, \tag{2.3}$$

where $S_{dh}^2 = \frac{\sum_{i=1}^{N_h} (d_{hi} - \bar{D}_h)^2}{(N_h - 1)}$ and $S_{dh(2)}^2 = \frac{\sum_{i=1}^{N_{h(2)}} (d_{hi} - \bar{D}_{h(2)})^2}{(N_{h(2)} - 1)}$ are respectively mean square of entire group and non-response group of variable d in the population for the h^{th} stratum, $W_h = \frac{N_h}{N}$, $\delta_h = \left(\frac{1}{n_h} - \frac{1}{N_h} \right)$, $\delta_h^* = \frac{(f_h - 1) W_{h(2)}}{n_h}, W_{h(2)} = \frac{N_{h(2)}}{N_h}, f_h = \frac{n_{h(2)}}{r_h}$ and $N_{h(2)}$ being the size of the non-response group of the population in the h^{th} stratum.
For obtaining the bias and mean squared errors (MSEs) of the proposed estimators we below give the values of the required expectations:

We write

$$\bar{y}_{st}^{*} = \bar{Y}\left(1+e_0^{*}\right),\ \bar{x}_{st}^{*} = \bar{X}\left(1+e_1^{*}\right),\ \bar{z}_{st}^{*} = \bar{Z}\left(1+e_2^{*}\right)$$

such that

$$E\left(e_0^{*}\right) = E\left(e_1^{*}\right) = E\left(e_2^{*}\right) = 0$$

and

$$E\left(e_0^{*2}\right) = \frac{1}{\bar{Y}^2}\sum_{h=1}^{L} W_h^2\left(\delta_h S_{yh}^2 + \delta_h^{*} S_{yh(2)}^2\right) = V_{020}^{*},\ E\left(e_1^{*2}\right) = \frac{1}{\bar{X}^2}\sum_{h=1}^{L} W_h^2\left(\delta_h S_{xh}^2 + \delta_h^{*} S_{xh(2)}^2\right) = V_{200}^{*}$$

$$E\left(e_2^{*2}\right) = \frac{1}{\bar{Z}^2}\sum_{h=1}^{L} W_h^2\left(\delta_h S_{zh}^2 + \delta_h^{*} S_{zh(2)}^2\right) = V_{002}^{*},\ E\left(e_0^{*}e_1^{*}\right) = \frac{1}{\bar{X}\bar{Y}}\sum_{h=1}^{L} W_h^2\left(\delta_h S_{yxh} + \delta_h^{*} S_{yxh(2)}\right) = V_{110}^{*}$$

$$E\left(e_0^{*}e_2^{*}\right) = \frac{1}{\bar{Y}\,\bar{Z}}\sum_{h=1}^{L} W_h^2\left(\delta_h S_{yzh} + \delta_h^{*} S_{yzh(2)}\right) = V_{011}^{*},\ E\left(e_1^{*}e_2^{*}\right) = \frac{1}{\bar{X}\,\bar{Z}}\sum_{h=1}^{L} W_h^2\left(\delta_h S_{xzh} + \delta_h^{*} S_{xzh(2)}\right) = V_{101}^{*}$$

where

$$S_{yxh} = \frac{1}{\left(N_h - 1\right)}\sum_{i=1}^{N_h}\left(x_{hi} - \bar{X}_h\right)\left(y_{hi} - \bar{Y}_h\right) = \rho_{xyh} S_{xh} S_{yh},\ S_{yzh} = \frac{1}{\left(N_h - 1\right)}\sum_{i=1}^{N_h}\left(y_{hi} - \bar{Y}_h\right)\left(z_{hi} - \bar{Z}_h\right) = \rho_{yzh} S_{yh} S_{zh},$$

$$S_{xzh} = \frac{1}{\left(N_h - 1\right)}\sum_{i=1}^{N_h}\left(x_{hi} - \bar{X}_h\right)\left(z_{hi} - \bar{Z}_h\right) = \rho_{xzh} S_{xh} S_{zh},$$

$$S_{yxh(2)} = \frac{1}{\left(N_{h(2)} - 1\right)}\sum_{i=1}^{N_{h(2)}}\left(x_{hi} - \bar{X}_{h(2)}\right)\left(y_{hi} - \bar{Y}_{h(2)}\right) = \rho_{xyh(2)} S_{xh(2)} S_{yh(2)},$$

$$S_{yzh(2)} = \frac{1}{\left(N_{h(2)} - 1\right)}\sum_{i=1}^{N_{h(2)}}\left(y_{hi} - \bar{Y}_{h(2)}\right)\left(z_{hi} - \bar{Z}_{h(2)}\right) = \rho_{yzh(2)} S_{yh(2)} S_{zh(2)},$$

$$S_{xzh(2)} = \frac{1}{\left(N_{h(2)} - 1\right)}\sum_{i=1}^{N_{h(2)}}\left(x_{hi} - \bar{X}_{h(2)}\right)\left(z_{hi} - \bar{Z}_{h(2)}\right) = \rho_{xzh(2)} S_{xh(2)} S_{zh(2)},$$

$$\bar{Y}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} y_{hi},\ \bar{X}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} x_{hi},\ \bar{Z}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} z_{hi},$$

$$\bar{Y}_{h(2)} = \frac{1}{N_{h(2)}}\sum_{i=1}^{N_{h(2)}} y_{hi},\ \bar{X}_{h(2)} = \frac{1}{N_{h(2)}}\sum_{i=1}^{N_{h(2)}} x_{hi},\ \bar{Z}_{h(2)} = \frac{1}{N_{h(2)}}\sum_{i=1}^{N_{h(2)}} z_{hi},$$

$\left(\rho_{yxh}, \rho_{xzh}, \rho_{yzh}\right)$ and $\left(\rho_{yxh(2)}, \rho_{xzh(2)}, \rho_{yzh(2)}\right)$ are the correlation coefficients between the subscripted variables of entire population and non-response group of population in the h^{th} group.

Saleem et al (2018) suggested the following estimators for population mean $\bar{Y}$ in stratified random sampling as

$$t_{s1} = \bar{y}_{st}^{*}\left(\frac{\bar{X}}{\bar{x}_{st}^{*}}\right)\left(\frac{\bar{Z}}{\bar{z}_{st}^{*}}\right), \tag{2.4}$$

$$t_{s2} = \bar{y}_{st}^{*}\left(\frac{\bar{x}_{st}^{*}}{\bar{X}}\right)\left(\frac{\bar{z}_{st}^{*}}{\bar{Z}}\right), \tag{2.5}$$

$$t_a = \bar{y}_{st}^{*}\left[\frac{a_x\bar{X} + b_x}{\alpha_x\left(a_x\bar{x}_{st}^{*} + b_x\right) + \left(1-\alpha_x\right)\left(a_x\bar{X} + b_x\right)}\right]^{g_x}\left[\frac{a_z\bar{Z} + b_z}{\alpha_z\left(a_z\bar{z}_{st}^{*} + b_z\right) + \left(1-\alpha_z\right)\left(a_z\bar{Z} + b_z\right)}\right]^{g_z}, \tag{2.6}$$

$$t_s = \eta_1\bar{y}_{st}^{*}\left[\frac{a_x\bar{X} + b_x}{\alpha_x\left(a_x\bar{x}_{st}^{*} + b_x\right) + \left(1-\alpha_x\right)\left(a_x\bar{X} + b_x\right)}\right]^{g_x}\left[\frac{a_z\bar{Z} + b_z}{\alpha_z\left(a_z\bar{z}_{st}^{*} + b_z\right) + \left(1-\alpha_z\right)\left(a_z\bar{Z} + b_z\right)}\right]^{g_z}, \tag{2.7}$$

where $a_x (\neq 0)$ and $a_z (\neq 0)$ are constants and b_x and b_z are either real numbers or the functions of the *AV*s in the form of coefficients of variation, standard deviations, correlation coefficients, skewness and kurtosis, whereas g_x and g_z are known scalars take the value (0, 1, -1) and $\left(\alpha_x, \alpha_z\right)$ are constants to be determined such that *MSE* of the estimator t_a is minimum and η_1 is a constant to be determined such that *MSE* of t_s is minimum.

In this paper we have suggested some classes of estimators for population mean $\bar{Y}$ of the *SV y* using information on two *AV*s (*x*, *z*) and their properties are studied up to first order of approximation. Merits of the proposed estimators are demonstrated through numerical illustration.

3. PROPOSED CLASS OF EXPONENTIAL-TYPE ESTIMATORS

When the non-response occurs on the *SV y* and on the *AV*s (*x*, *z*), we define the following exponential-type estimators for population mean $\bar{Y}$ as

$$t_{ae} = \bar{y}^*_{st} \exp\left[\frac{-\alpha_x h_x a_x\left(\bar{x}^*_{st} - \bar{X}\right)}{(2-\alpha_x)\left(a_x\bar{X}+b_x\right)+\alpha_x\left(a_x\bar{x}^*_{st}+b_x\right)}\right]\exp\left[\frac{-\alpha_z h_z a_z\left(\bar{z}^*_{st} - \bar{Z}\right)}{(2-\alpha_z)\left(a_z\bar{Z}+b_z\right)+\alpha_z\left(a_z\bar{z}^*_{st}+b_z\right)}\right] \tag{3.1}$$

where $(a_x, a_z, b_x, b_z, \alpha_x, \alpha_z)$ are same constants as defined earlier and (h_x, h_z) are known constants take value (0, 1, -1) to give unbiased, ratio-type and product-type exponential estimators.

Expressing (3.1) in terms of *e*'s we have

$$t_{ae} = \bar{Y}\left(1+e_0^*\right)\exp\left\{\frac{-h_x\alpha_x\upsilon_x e_1^*}{2}\left(1+\frac{\alpha_x\upsilon_x}{2}e_1^*\right)^{-1}\right\}\exp\left\{\frac{-h_z\alpha_z\upsilon_z e_2^*}{2}\left(1+\frac{\alpha_z\upsilon_z}{2}e_2^*\right)^{-1}\right\}, \tag{3.2}$$

where $\upsilon_x = \dfrac{a_x\bar{X}}{a_x\bar{X}+b_x}$ and $\upsilon_z = \dfrac{a_z\bar{Z}}{a_z\bar{Z}+b_z}$.

Expanding the right hand side, multiplying out and neglecting terms of *e*'s having power greater than two we have

$$t_{ae} = \bar{Y}\left[\begin{array}{l} 1+e_0^* - \dfrac{(h_x\alpha_x\upsilon_x)}{2}e_1^* - \dfrac{(h_z\alpha_z\upsilon_z)}{2}e_2^* - \dfrac{(h_x\alpha_x\upsilon_x)}{2}e_0^*e_1^* - \dfrac{(h_z\alpha_z\upsilon_z)}{2}e_0^*e_2^* \\ + \dfrac{(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)}{4}e_1^*e_2^* + \dfrac{h_x(h_x+2)(\alpha_x\upsilon_x)^2}{8}e_1^{*2} + \dfrac{h_z(h_z+2)(\alpha_z\upsilon_z)^2}{8}e_2^{*2}\end{array}\right]$$

or

$$\left(t_{ae}-\bar{Y}\right) = \bar{Y}\left[\begin{array}{l} e_0^* - \dfrac{(h_x\alpha_x\upsilon_x)}{2}e_1^* - \dfrac{(h_z\alpha_z\upsilon_z)}{2}e_2^* - \dfrac{(h_x\alpha_x\upsilon_x)}{2}e_0^*e_1^* - \dfrac{(h_z\alpha_z\upsilon_z)}{2}e_0^*e_2^* \\ + \dfrac{(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)}{4}e_1^*e_2^* + \dfrac{h_x(h_x+2)(\alpha_x\upsilon_x)^2}{8}e_1^{*2} + \dfrac{h_z(h_z+2)(\alpha_z\upsilon_z)^2}{8}e_2^{*2}\end{array}\right]. \tag{3.3}$$

Taking expectation of both sides of (3.3) we get the bias of t_{ae} to the first degree of approximation as

$$B(t_{ae}) = \frac{\bar{Y}}{8}\left[\begin{array}{l} h_x(h_x+2)(\alpha_x\upsilon_x)^2V^*_{200} + h_z(h_z+2)(\alpha_z\upsilon_z)^2V^*_{002} + 2(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)V^*_{101} \\ -4(h_x\alpha_x\upsilon_x)V^*_{110} - 4(h_z\alpha_z\upsilon_z)V^*_{011}\end{array}\right]. \tag{3.4}$$

Squaring both sides of (3.3) and neglecting terms of *e*'s having power greater than two we have

$$\left(t_{ae}-\bar{Y}\right)^2 = \bar{Y}^2\left[\begin{array}{l} e_0^{*2} + \dfrac{(h_x\alpha_x\upsilon_x)^2}{4}e_1^{*2} + \dfrac{(h_z\alpha_z\upsilon_z)^2}{4}e_2^{*2} - (h_x\alpha_x\upsilon_x)e_0^*e_1^* - (h_z\alpha_z\upsilon_z)e_0^*e_2^* \\ + \dfrac{(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)}{2}e_1^*e_2^*\end{array}\right], \tag{3.5}$$

Taking expectation of both sides of (3.5) we get the *MSE* of t_{ae} to the first degree of approximation as

$$MSE(t_{ae}) = \bar{Y}^2\left[\begin{array}{l} V^*_{020} + \dfrac{(h_x\alpha_x\upsilon_x)^2}{4}V^*_{200} + \dfrac{(h_z\alpha_z\upsilon_z)^2}{4}V^*_{002} - (h_x\alpha_x\upsilon_x)V^*_{110} - (h_z\alpha_z\upsilon_z)V^*_{011} \\ + \dfrac{(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)}{2}V^*_{101}\end{array}\right]. \tag{3.6}$$

Differentiating (3.6) partially with respect to α_x, α_z and equating to zero we have

$$\begin{bmatrix}(h_x\upsilon_x)V^*_{200} & (h_z\upsilon_z)V^*_{101} \\ (h_x\upsilon_x)V^*_{101} & (h_z\upsilon_z)V^*_{002}\end{bmatrix}\begin{bmatrix}\alpha_x \\ \alpha_z\end{bmatrix} = \begin{bmatrix}2V^*_{110} \\ 2V^*_{011}\end{bmatrix}. \tag{3.7}$$

After solving (3.7) for α_x and α_z we get the optimum values of α_x and α_z that minimizes the *MSE* of t_{ae} as

$$\left.\begin{aligned}\alpha_x &= \frac{2\left(V_{110}^{*}V_{002}^{*}-V_{011}^{*}V_{101}^{*}\right)}{\left(h_x\upsilon_x\right)\left(V_{200}^{*}V_{002}^{*}-V_{101}^{*2}\right)}=\alpha_{x0}(say)\\ \alpha_z &= \frac{2\left(V_{011}^{*}V_{200}^{*}-V_{101}^{*}V_{110}^{*}\right)}{\left(h_z\upsilon_z\right)\left(V_{200}^{*}V_{002}^{*}-V_{101}^{*2}\right)}=\alpha_{z0}(say)\end{aligned}\right\}. \tag{3.8}$$

Thus the resulting minimum *MSE* of t_{ae} is given by

$$MSE_{\min}\left(t_{ae}\right)=\bar{Y}^2\left[V_{020}^{*}-\frac{\left(V_{110}^{*2}V_{002}^{*}-2V_{110}^{*}V_{011}^{*}V_{101}^{*}+V_{011}^{*2}V_{200}^{*}\right)}{\left(V_{200}^{*}V_{002}^{*}-V_{101}^{*2}\right)}\right]. \tag{3.9}$$

Now we established the following theorem.

Theorem 3.1: To the first degree of approximation,

$$MSE\left(t_{ae}\right)\geq\bar{Y}^2\left[V_{020}^{*}-\frac{\left(V_{110}^{*2}V_{002}^{*}-2V_{110}^{*}V_{011}^{*}V_{101}^{*}+V_{011}^{*2}V_{200}^{*}\right)}{\left(V_{200}^{*}V_{002}^{*}-V_{101}^{*2}\right)}\right]$$

with equality holding if $\alpha_x=\alpha_{x0}$ and $\alpha_z=\alpha_{z0}$.

For different values of constants $\left(\alpha_x,\alpha_z,a_x,a_z,b_x,b_z,h_x,h_z\right)$ a large number of estimators can be generated from the proposed class of exponential type estimators t_{ae} at (3.1).

The biases and *MSE*s of the members of the proposed class of exponential type estimators t_{ae} can be easily obtained from (3.4) and (3.6) just by putting the suitable values of the constants $\left(\alpha_x,\alpha_z,a_x,a_z,b_x,b_z,h_x,h_z\right)$.

Some members of the class of exponential-type estimators t_{ae} are shown in Table 3.1.

Table 3.1: Some members of the proposed class of exponential-type estimators t_{ae}.

Ratio-cum ratio-type exponential estimators $h_x=h_z=1$	Product-cum product-type exponential estimators $h_x=h_z=-1$	a_x	b_x	a_z	b_z	α_x	α_z
$t_{ae(1)}=\bar{y}_{st}^{*}\exp\left(\frac{\bar{X}-\bar{x}_{st}^{*}}{\bar{X}+\bar{x}_{st}^{*}}\right)\exp\left(\frac{\bar{Z}-\bar{z}_{st}^{*}}{\bar{Z}+\bar{z}_{st}^{*}}\right)$	$t_{ae(2)}=\bar{y}_{st}^{*}\exp\left(\frac{\bar{x}_{st}^{*}-\bar{X}}{\bar{x}_{st}^{*}+\bar{X}}\right)\exp\left(\frac{\bar{z}_{st}^{*}-\bar{Z}}{\bar{z}_{st}^{*}+\bar{Z}}\right)$	1	0	1	0	1	1
$t_{ae(3)}=\bar{y}_{st}^{*}\exp\left(\frac{\bar{X}-\bar{x}_{st}^{*}}{\bar{X}+\bar{x}_{st}^{*}+2\rho_{xy}}\right)\exp\left(\frac{\bar{Z}-\bar{z}_{st}^{*}}{\bar{Z}+\bar{z}_{st}^{*}+2\rho_{yz}}\right)$	$t_{ae(4)}=\bar{y}_{st}^{*}\exp\left(\frac{\bar{x}_{st}^{*}-\bar{X}}{\bar{X}+\bar{x}_{st}^{*}+2\rho_{xy}}\right)\exp\left(\frac{\bar{z}_{st}^{*}-\bar{Z}}{\bar{Z}+\bar{z}_{st}^{*}+2\rho_{yz}}\right)$	1	ρ_{xy}	1	ρ_{yz}	1	1
$t_{ae(5)}=\bar{y}_{st}^{*}\exp\left\{\frac{\sigma_x\left(\bar{X}-\bar{x}_{st}^{*}\right)}{\sigma_x\left(\bar{X}+\bar{x}_{st}^{*}\right)+2}\right\}\exp\left\{\frac{\sigma_z\left(\bar{Z}-\bar{z}_{st}^{*}\right)}{\sigma_z\left(\bar{Z}+\bar{z}_{st}^{*}\right)+2}\right\}$	$t_{ae(6)}=\bar{y}_{st}^{*}\exp\left\{\frac{\sigma_x\left(\bar{x}_{st}^{*}-\bar{X}\right)}{\sigma_x\left(\bar{X}+\bar{x}_{st}^{*}\right)+2}\right\}\exp\left\{\frac{\sigma_z\left(\bar{z}_{st}^{*}-\bar{Z}\right)}{\sigma_z\left(\bar{Z}+\bar{z}_{st}^{*}\right)+2}\right\}$	σ_x	1	σ_z	1	1	1
$t_{ae(7)}=\bar{y}_{st}^{*}\exp\left\{\frac{\rho_{xy}\left(\bar{X}-\bar{x}_{st}^{*}\right)}{\rho_{xy}\left(\bar{X}+\bar{x}_{st}^{*}\right)+2}\right\}\exp\left\{\frac{\rho_{yz}\left(\bar{Z}-\bar{z}_{st}^{*}\right)}{\rho_{yz}\left(\bar{Z}+\bar{z}_{st}^{*}\right)+2}\right\}$	$t_{ae(8)}=\bar{y}_{st}^{*}\exp\left\{\frac{\rho_{xy}\left(\bar{x}_{st}^{*}-\bar{X}\right)}{\rho_{xy}\left(\bar{X}+\bar{x}_{st}^{*}\right)+2}\right\}\exp\left\{\frac{\rho_{yz}\left(\bar{z}_{st}^{*}-\bar{Z}\right)}{\rho_{yz}\left(\bar{Z}+\bar{z}_{st}^{*}\right)+2}\right\}$	ρ_{xy}	1	ρ_{yz}	1	1	1

$t_{ae(9)} = \bar{y}_{st}^{*} \exp\left\{\frac{\left(\bar{X}-\bar{x}_{st}^{*}\right)}{\left(\bar{X}+\bar{x}_{st}^{*}\right)+2C_x}\right\} \exp\left\{\frac{\left(\bar{Z}-\bar{z}_{st}^{*}\right)}{\left(\bar{Z}+\bar{z}_{st}^{*}\right)+2C_z}\right\}$	$t_{ae(10)} = \bar{y}_{st}^{*} \exp\left\{\frac{\left(\bar{x}_{st}^{*}-\bar{X}\right)}{\left(\bar{X}+\bar{x}_{st}^{*}\right)+2C_x}\right\} \exp\left\{\frac{\left(\bar{z}_{st}^{*}-\bar{Z}\right)}{\left(\bar{Z}+\bar{z}_{st}^{*}\right)+2C_z}\right\}$	1	C_x	1	C_z	1	1

For generalized family of ratio-cum-ratio-type exponential estimators shown in Table 3.1, we can give the *MSE* expression in equation (3.6) as

$$MSE\left(t_{ae(i)}\right) = \begin{bmatrix} \bar{Y}^2\left(V_{020}^{*}+\frac{1}{4}V_{200}^{*}+\frac{1}{4}V_{002}^{*}-V_{110}^{*}-V_{011}^{*}+\frac{1}{2}V_{101}^{*}\right) = MSE\left(t_{ae(1)}\right); i=1 \\ \bar{Y}^2\begin{pmatrix} V_{020}^{*}+\frac{1}{4}\upsilon_{x\left(\frac{i-1}{2}\right)}^2 V_{200}^{*}+\frac{1}{4}\upsilon_{z\left(\frac{i-1}{2}\right)}^2 V_{002}^{*}-\upsilon_{x\left(\frac{i-1}{2}\right)}V_{110}^{*}-\upsilon_{z\left(\frac{i-1}{2}\right)}V_{011}^{*} \\ +\frac{1}{2}\upsilon_{x\left(\frac{i-1}{2}\right)}\upsilon_{z\left(\frac{i-1}{2}\right)}V_{101}^{*} \end{pmatrix}; i=3,5,7,9 \end{bmatrix}, \tag{3.10}$$

and for generalized class of product-cum-product-type exponential estimators depicted in Table 3.1 ; the *MSE* expression can be given as

$$MSE\left(t_{ae(j)}\right) = \begin{bmatrix} \bar{Y}^2\left(V_{020}^{*}+\frac{1}{4}V_{200}^{*}+\frac{1}{4}V_{002}^{*}+V_{110}^{*}+V_{011}^{*}+\frac{1}{2}V_{101}^{*}\right); j=2 \\ \bar{Y}^2\begin{pmatrix} V_{020}^{*}+\frac{1}{4}\upsilon_{x\left(\frac{j}{2}-1\right)}^2 V_{200}^{*}+\frac{1}{4}\upsilon_{z\left(\frac{j}{2}-1\right)}^2 V_{002}^{*}+\upsilon_{x\left(\frac{j}{2}-1\right)}V_{110}^{*}+\upsilon_{z\left(\frac{j}{2}-1\right)}V_{011}^{*} \\ +\frac{1}{2}\upsilon_{x\left(\frac{j}{2}-1\right)}\upsilon_{z\left(\frac{j}{2}-1\right)}V_{101}^{*} \end{pmatrix}; j=4,6,8,10 \end{bmatrix} \tag{3.11}$$

where

$$\upsilon_{x(1)} = \frac{\bar{X}}{\bar{X}+\rho_{xy}} \text{ and } \upsilon_{z(1)} = \frac{\bar{Z}}{\bar{Z}+\rho_{yz}},$$

$$\upsilon_{x(2)} = \frac{\sigma_x\bar{X}}{\sigma_x\bar{X}+1} \text{ and } \upsilon_{z(2)} = \frac{\sigma_z\bar{Z}}{\sigma_z\bar{Z}+1},$$

$$\upsilon_{x(3)} = \frac{\rho_{xy}\bar{X}}{\rho_{xy}\bar{X}+1} \text{ and } \upsilon_{z(3)} = \frac{\rho_{yz}\bar{Z}}{\rho_{yz}\bar{Z}+1},$$

$$\upsilon_{x(4)} = \frac{\bar{X}}{\bar{X}+C_x} \text{ and } \upsilon_{z(4)} = \frac{\bar{Z}}{\bar{Z}+C_z}.$$

3.1 Optimum n_h With Respect to Cost of The Survey

Let C_{h0} be the cost per unit of selecting n_h units at first attempt, C_{h1} be the cost per unit in enumerating $n_{h(1)}$ units and C_{h2} be the cost per unit of enumerating r_h units. Then the total cost for the n_h stratum is given by

$$C_h = C_{h0}n_h + C_{h1}n_{h(1)} + C_{h2}r_h \quad \forall h = 1,2,...,L \tag{3.12}$$

Now, the total expected cost per stratum of the survey is given by

$$E\left(C_h\right) = n_h\left[C_{h0} + C_{h1}W_{h(1)} + C_{h2}\frac{W_{h(2)}}{f_h}\right], \tag{3.13}$$

where $W_{h(1)} = \frac{N_{h(1)}}{N_h}$ and $W_{h(2)} = \frac{N_{h(2)}}{N_h}$ such that $N_{h(1)} + N_{h(2)} = N_h$.

Thus the total cost over all the strata is given by

$$C_0=\sum_{h=1}^{L}E(C_h)=\sum_{h=1}^{L}n_h\left[C_{h0}+C_{h1}W_{h(1)}+C_{h2}\frac{W_{h(2)}}{f_h}\right] \tag{3.14}$$

Let us consider the function for the estimator t_a

$$\phi=MSE(t_{ae})+\mu C_0 \tag{3.15}$$

where μ is Lagrangian multiplier.

We write

$$\phi=\left[\bar{Y}^2\left(V_{020}^*+H_x^2V_{200}^*+H_z^2V_{002}^*-2H_xV_{110}^*-2H_zV_{011}^*+2H_xH_zV_{101}^*\right)+\mu\sum_{h=1}^{L}n_h\left(C_{h0}+C_{h1}W_{h(1)}+C_{h2}\frac{W_{h(2)}}{f_h}\right)\right], \tag{3.16}$$

where $H_x=\dfrac{(h_x\alpha_x\upsilon_x)}{2}$ and $H_z=\dfrac{(h_z\alpha_z\upsilon_z)}{2}$.

We further let

$$S_h=\left[\frac{S_{yh}^2}{\bar{Y}^2}+H_x^2\frac{S_{xh}^2}{\bar{X}^2}+H_z^2\frac{S_{zh}^2}{\bar{Z}^2}-2H_x\frac{S_{xyh}}{\bar{X}\bar{Y}}-2H_z\frac{S_{yzh}}{\bar{Y}\bar{Z}}+2H_xH_z\frac{S_{xzh}}{\bar{X}\bar{Z}}\right], \tag{3.17}$$

$$S_{h(2)}=\left[\frac{S_{yh(2)}^2}{\bar{Y}^2}+H_x^2\frac{S_{xh(2)}^2}{\bar{X}^2}+H_z^2\frac{S_{zh(2)}^2}{\bar{Z}^2}-2H_x\frac{S_{xyh(2)}}{\bar{X}\bar{Y}}-2H_z\frac{S_{yzh(2)}}{\bar{Y}\bar{Z}}+2H_xH_z\frac{S_{xzh(2)}}{\bar{X}\bar{Z}}\right]. \tag{3.18}$$

Then we express the function ϕ as

$$\phi=\bar{Y}^2\sum_{h=1}^{L}W_h^2\left[\left(\frac{1}{n_h}-\frac{1}{N_h}\right)S_h+\frac{(f_h-1)W_{h(2)}}{n_h}S_{h(2)}\right]+\mu\sum_{h=1}^{L}n_h\left(C_{h0}+C_{h1}W_{h(1)}+C_{h2}\frac{W_{h(2)}}{f_h}\right). \tag{3.19}$$

Differentiating (3.19) partially with respect to n_h and f_h and equating to zero, we have

$$\frac{\partial\phi}{\partial n_h}=-\bar{Y}^2\frac{W_h^2}{n_h^2}\left\{S_h+(f_h-1)W_{h(2)}S_{h(2)}\right\}+\mu\left(C_{h0}+C_{h1}W_{h(1)}+C_{h2}\frac{W_{h(2)}}{f_h}\right)=0, \tag{3.20}$$

$$\frac{\partial\phi}{\partial f_h}=\bar{Y}^2\frac{W_h^2W_{h(2)}}{n_h}S_{h(2)}-\mu n_hC_{h2}\frac{W_{h(2)}}{f_h^2}=0. \tag{3.21}$$

From equations (3.20) and (3.21), we have

$$n_h=\bar{Y}\frac{W_h\sqrt{S_h+(f_h-1)W_{h(2)}S_{h(2)}}}{\sqrt{\mu}\sqrt{C_{h0}+C_{h1}W_{h(1)}+C_{h2}\dfrac{W_{h(2)}}{f_h}}} \tag{3.22}$$

and

$$\sqrt{\mu}=\bar{Y}\frac{W_hf_h\sqrt{S_{h(2)}}}{n_h\sqrt{C_{h(2)}}} \tag{3.23}$$

Inserting the value of $\sqrt{\mu}$ from equation (3.23) into the equation (3.22), we get

$$f_{h(opt)}=\frac{\sqrt{C_{h(2)}\left(S_h-W_{h(2)}S_{h(2)}\right)}}{\sqrt{S_{h(2)}\left(C_{h0}+C_{h1}W_{h(1)}\right)}} \tag{3.24}$$

Putting from equation (3.24) into equation (3.22), we get the optimum value of n_h as

$$n_h=\frac{\bar{Y}W_h\sqrt{B_h^2+\dfrac{B_hW_{h(2)}\sqrt{C_{h2}S_{h(2)}}}{A_h}}}{\sqrt{\mu}\sqrt{A_h^2+\dfrac{A_hW_{h(2)}\sqrt{C_{h2}S_{h(2)}}}{B_h}}} \tag{3.25}$$

where $A_h=\sqrt{\left(C_{h0}+C_{h1}W_{h(1)}\right)}$ and $B_h=\sqrt{\left(S_h-W_{h(2)}S_{h(2)}\right)}$.

The $\sqrt{\mu}$ in terms of total cost C_0 can be obtained by putting the values of $f_{h(opt)}$ and n_h from equations (3.24) and (3.25) respectively into equation (3.14) as

$$\sqrt{\mu} = \frac{\bar{Y}}{C_0} \sum_{h=1}^{L} W_h \left(A_h B_h + W_{h(2)} \sqrt{C_{h2} S_{h(2)}} \right) \tag{3.26}$$

Substitution of (3.26) into (3.25) yields the optimum value of n_h as

$$n_{h(opt)} = \frac{C_0}{\sum_{h=1}^{L} W_h \left(A_h B_h + W_{h(2)} \sqrt{C_{h2} S_{h(2)}} \right)} \frac{W_h \sqrt{B_h^2 + \frac{B_h W_{h(2)} \sqrt{C_{h2} S_{h(2)}}}{A_h}}}{\sqrt{A_h^2 + \frac{A_h W_{h(2)} \sqrt{C_{h2} S_{h(2)}}}{B_h}}} \tag{3.27}$$

The values of $f_{h(opt)}$ and $n_{h(opt)}$ can be computed from the expressions (3.24) and (3.27) for different values of $\left(W_{h(1)}, W_{h(2)}, C_{h0}, C_{h1} \text{ and } C_{h2}\right)$.

3.2. Improved Class of Exponential Type Estimators on the Line of Searls (1964)

Following Searls (1964) we define the following class of estimators for the population mean $\bar{Y}$ as

$$t_{se} = \eta_2 \bar{y}_{st}^* \exp\left[\frac{-\alpha_x h_x a_x \left(\bar{x}_{st}^* - \bar{X} \right)}{(2-\alpha_x)\left(a_x \bar{X} + b_x\right) + \alpha_x \left(a_x \bar{x}_{st}^* + b_x\right)} \right] \exp\left[\frac{-\alpha_z h_z a_z \left(\bar{z}_{st}^* - \bar{Z} \right)}{(2-\alpha_z)\left(a_z \bar{Z} + b_z\right) + \alpha_z \left(a_z \bar{z}_{st}^* + b_z\right)} \right], \tag{3.28}$$

where η_2 is a suitably chosen constants that *MSE* of t_{se} is minimum.

To the first degree of approximation the bias and *MSE* of t_{se} are respectively given by

$$B(t_{se}) = \bar{Y}(\eta_2 - 1) + Bias(t_{ae}) \tag{3.29}$$

and

$$MSE(t_{se}) = \bar{Y}^2 (\eta_2 - 1)^2 + \eta_2^2 MSE(t_{ae}) + 2\bar{Y}\eta_2(\eta_2 - 1) Bias(t_{ae}), \tag{3.30}$$

where $Bias(t_{ae})$ and $MSE(t_{ae})$ are respectively given by (3.4) and (3.6).

The $MSE(t_{se})$ at (3.30) is minimum when

$$\eta_2 = \frac{\bar{Y}\left(\bar{Y} + Bias(t_{ae})\right)}{\left\{\bar{Y}^2 + MSE(t_{ae}) + 2\bar{Y}\, Bias(t_{ae})\right\}} = \eta_{2(opt)}. \tag{3.31}$$

Thus the resulting minimum *MSE* of t_{se} is given by

$$MSE_{\min}(t_{se}) = \bar{Y}^2 - \frac{\bar{Y}^2 \left(\bar{Y} + Bias(t_{ae})\right)^2}{\left\{\bar{Y}^2 + MSE(t_{ae}) + 2\bar{Y}\, Bias(t_{ae})\right\}}$$

or

$$MSE_{\min}(t_{se}) = \bar{Y}^2 \left[1 - \frac{\left(1 + \frac{Bias(t_{ae})}{\bar{Y}}\right)^2}{\left\{1 + \frac{MSE(t_{ae})}{\bar{Y}^2} + \frac{2\,Bias(t_{ae})}{\bar{Y}}\right\}} \right] \tag{3.32}$$

Thus we state the following theorem.

Theorem 3.2- The *MSE* of the class of estimators t_{se} is greater than equal to $MSE_{\min}(t_{se})$ at (3.32) i.e.

$$MSE(t_{se}) \geq \bar{Y}^2 \left[1 - \frac{\left(1 + \frac{Bias(t_{ae})}{\bar{Y}}\right)^2}{\left\{1 + \frac{MSE(t_{ae})}{\bar{Y}^2} + \frac{2\,Bias(t_{ae})}{\bar{Y}}\right\}} \right] \tag{3.33}$$

with equality of holding

$\eta_2 = \eta_{2(opt)}$ where $\eta_{2(opt)}$ is given by (3.31).

A large number of improved estimators can be generated from the proposed class of exponential type estimators t_{se} for appropriate values of the scalars $(\alpha_x, \alpha_z, h_x, h_z, a_x, a_z, b_x, b_z)$. For example some members of the class of exponential type estimators t_{se} are shown in Table 3.2.

Table 3.2: Some members of the class of exponential type estimators t_{se}.

Ratio-cum-ratio-type exponential estimators $h_x = h_z = 1$	Product-cum-product-type exponential estimators $h_x = h_z = -1$	a_x	b_x	a_z	b_z	α_x	α_z	η_2
$t_{se(1)}$	$t_{se(2)}$	1	0	1	0	1	1	η_2
$t_{se(3)}$	$t_{se(4)}$	1	ρ_{xy}	1	ρ_{yz}	1	1	η_2
$t_{se(5)}$	$t_{se(6)}$	σ_x	1	σ_z	1	1	1	η_2
$t_{se(7)}$	$t_{se(8)}$	ρ_{xy}	1	ρ_{yz}	1	1	1	η_2
$t_{se(9)}$	$t_{se(10)}$	1	C_x	1	C_z	1	1	η_2

The minimum *MSE* of the ratio-cum-ratio-type exponential and product-cum-product-type exponential estimators given in Table 3.2 can be given using equation (3.32) as

$$MSE_{\min}\left(t_{se(i)}\right) = \bar{Y}^2\left[1 - \frac{\left(1 + \frac{Bias\left(t_{ae(i)}\right)}{\bar{Y}}\right)^2}{\left\{1 + \frac{MSE\left(t_{ae(i)}\right)}{\bar{Y}^2} + \frac{2\,Bias\left(t_{ae(i)}\right)}{\bar{Y}}\right\}}\right] \tag{3.34}$$

for i=1,3,5,7,9;

$$MSE_{\min}\left(t_{se(j)}\right) = \bar{Y}^2\left[1 - \frac{\left(1 + \frac{Bias\left(t_{ae(j)}\right)}{\bar{Y}}\right)^2}{\left\{1 + \frac{MSE\left(t_{ae(j)}\right)}{\bar{Y}^2} + \frac{2\,Bias\left(t_{ae(j)}\right)}{\bar{Y}}\right\}}\right] \tag{3.35}$$

for j=2,4,6,8,10.

3.3. Mathematical Efficiency Comparison

For the purpose of comparison of the suggested estimators with stratified sample mean estimator $\bar{y}_{st}^* = \sum_{h=1}^{L} W_h \bar{y}_h^*$, ratio-cum-ratio-type exponential estimator $t_{ae(1)}$, we consider the following notations:

$$\theta_i = \left[\frac{1}{4}\upsilon^2_{x\left(\frac{i-1}{2}\right)}V^*_{200} + \frac{1}{4}\upsilon^2_{z\left(\frac{i-1}{2}\right)}V^*_{002} - \upsilon_{x\left(\frac{i-1}{2}\right)}V^*_{110} - \upsilon_{z\left(\frac{i-1}{2}\right)}V^*_{011} + \frac{1}{2}\upsilon_{x\left(\frac{i-1}{2}\right)}\upsilon_{z\left(\frac{i-1}{2}\right)}V^*_{101}\right], i = 3,5,7,9$$

$$\theta_j = \left[\frac{1}{4}\upsilon^2_{x\left(\frac{j}{2}-1\right)}V^*_{200} + \frac{1}{4}\upsilon^2_{z\left(\frac{j}{2}-1\right)}V^*_{002} - \upsilon_{x\left(\frac{j}{2}-1\right)}V^*_{110} - \upsilon_{z\left(\frac{j}{2}-1\right)}V^*_{011} + \frac{1}{2}\upsilon_{x\left(\frac{j}{2}-1\right)}\upsilon_{z\left(\frac{j}{2}-1\right)}V^*_{101}\right], j = 2,4,6,8,10$$

$$B_{1e} = \frac{1}{4}V^*_{200} + \frac{1}{4}V^*_{002} - V^*_{110} - V^*_{011} + \frac{1}{2}V^*_{101}.$$

- From (3.10) we have that

$MSE\left(t_{ae(i)}\right) < MSE\left(t_{ae(1)}\right)$ if

$$\theta_i < B_{1e} \tag{3.36}$$

Further let us designate

$$B_2 = \frac{\left(V^*_{002}V^{*2}_{110} + V^*_{200}V^{*2}_{011} - 2V^*_{101}V^*_{011}V^*_{110}\right)}{\left(V^*_{200}V^*_{002} - V^{*2}_{101}\right)}$$

$$C_i = \frac{\left\{1+\frac{Bias(t_{ae(i)})}{\bar{Y}}\right\}^2}{\left[1+\frac{MSE(t_{ae(i)})}{\bar{Y}^2}+\frac{2Bias(t_{ae(i)})}{\bar{Y}}\right]}$$

- Also from (3.10) we have that

$MSE(t_{ae(i)}) < MSE(\bar{y}^*_{st})$ if

$$\theta_i < 0 \tag{3.37}$$

- From (3.9) we have that

$MSE_{\min}(t_{ae}) < MSE(\bar{y}^*_{st})$ if

$$B_2 > 0 \tag{3.38}$$

which is always true

- From (3.9) we have that

$MSE_{\min}(t_{ae}) < MSE(t_{ae(1)})$ i.e. if

$$B_2 > \left(\frac{1}{4}V^*_{200} + \frac{1}{4}V^*_{002} - V^*_{110} - V^*_{011} + \frac{1}{2}V^*_{101}\right) \tag{3.39}$$

- From (3.34) we have that

$MSE_{\min}(t_{se(i)}) < MSE(\bar{y}^*_{st})$ if

$$(1 - V^*_{020}) < C_i \tag{3.40}$$

- From (3.34) we have that

$MSE_{\min}(t_{se(i)}) < MSE(t_{ae(1)})$ if

$$C_i > \left[1 - \left(V^*_{020} + \frac{1}{4}V^*_{200} + \frac{1}{4}V^*_{002} - V^*_{110} - V^*_{011} + \frac{1}{2}V^*_{101}\right)\right] \tag{3.41}$$

- From (3.9) and (3.34) we have that

$MSE_{\min}(t_{se(i)}) < MSE_{\min}(t_{ae})$ if

$$C_i > (1 + B_2 - V^*_{020}) \tag{3.42}$$

The MSEs of t_{s1} and t_{s2} to the first degree of approximation is given by

$$MSE(t_{s1}) = \bar{Y}^2\left[V^*_{020} + V^*_{200} + V^*_{002} - 2V^*_{110} - 2V^*_{011} + 2V^*_{101}\right] \tag{3.43}$$

$$MSE(t_{s2}) = \bar{Y}^2\left[V^*_{020} + V^*_{200} + V^*_{002} + 2V^*_{110} + 2V^*_{011} + 2V^*_{101}\right] \tag{3.44}$$

To the first degree of approximation, the minimum MSEs of the estimators t_a and t_s are respectively given by

$$MSE_{\min}(t_a) = \bar{Y}^2\left[V^*_{020} - \frac{\left(V^*_{002}V^{*2}_{110} + V^*_{200}V^{*2}_{011} - 2V^*_{101}V^*_{011}V^*_{110}\right)}{\left(V^*_{200}V^*_{002} - V^{*2}_{101}\right)}\right] \tag{3.45}$$

$$= MSE_{\min}(t_{ae})$$

$$MSE_{\min}(t_s) = \bar{Y}^2\left[1 - \frac{\left(1+\frac{Bias(t_a)}{\bar{Y}}\right)^2}{\left\{1+\frac{MSE(t_a)}{\bar{Y}^2}+\frac{2Bias(t_a)}{\bar{Y}}\right\}}\right] \tag{3.46}$$

where bias and MSE of t_a to the first degree of approximation is

$$B(t_a) = \left[(g_x\alpha_x\upsilon_x)V^*_{110} + \frac{g_x(g_x+1)}{2}\alpha_x^2\upsilon_x^2V^*_{200} - (g_z\alpha_z\upsilon_z)V^*_{011} + (g_x\alpha_x\upsilon_x)(g_z\alpha_z\upsilon_z)V^*_{101} + \frac{g_z(g_z+1)}{2}\alpha_z^2\upsilon_z^2V^*_{002}\right],$$

$$MSE(t_a) = \bar{Y}^2\left[\begin{matrix} V^*_{020} + (g_x\alpha_x\upsilon_x)^2V^*_{200} + (g_z\alpha_z\upsilon_z)^2V^*_{002} - 2(g_x\alpha_x\upsilon_x)V^*_{110} - 2(g_z\alpha_z\upsilon_z)V^*_{011} \\ + 2(g_x\alpha_x\upsilon_x)(g_z\alpha_z\upsilon_z)V^*_{101} \end{matrix}\right].$$

From (3.10) and (3.43) we have

$MSE\left(t_{ae(1)}\right) < MSE\left(t_{s1}\right)$ if

$$\frac{3}{4}\left(V_{200}^{*}+V_{002}^{*}+2V_{101}^{*}\right)<\left(V_{110}^{*}+V_{011}^{*}\right)$$

i.e. if

$$\frac{\left(V_{110}^{*}+V_{011}^{*}\right)}{\left(V_{200}^{*}+V_{002}^{*}+2V_{101}^{*}\right)}>\frac{3}{4} \tag{3.47}$$

Also $MSE\left(t_{ae(2)}\right) < MSE\left(t_{s2}\right)$ if

$$\left(V_{020}^{*}+\frac{1}{4}V_{200}^{*}+\frac{1}{4}V_{002}^{*}+V_{110}^{*}+V_{011}^{*}+\frac{1}{2}V_{101}^{*}\right)<\left(V_{020}^{*}+V_{200}^{*}+V_{002}^{*}+2V_{110}^{*}+2V_{011}^{*}+2V_{101}^{*}\right)$$

i.e. if

$$\frac{\left(V_{110}^{*}+V_{011}^{*}\right)}{\left(V_{200}^{*}+V_{002}^{*}+2V_{101}^{*}\right)}>-\frac{3}{4} \tag{3.48}$$

From (3.32) and (3.46) we have

$MSE_{\min}\left(t_{se}\right) < MSE_{\min}\left(t_{s}\right)$ if

$$\frac{\left(1+\frac{Bias\left(t_a\right)}{\bar{Y}}\right)^2}{\left\{1+\frac{MSE\left(t_a\right)}{\bar{Y}^2}+\frac{2Bias\left(t_a\right)}{\bar{Y}}\right\}}<\frac{\left(1+\frac{Bias\left(t_{ae}\right)}{\bar{Y}}\right)^2}{\left\{1+\frac{MSE\left(t_{ae}\right)}{\bar{Y}^2}+\frac{2Bias\left(t_{ae}\right)}{\bar{Y}}\right\}} \tag{3.49}$$

3.4. A More General Class of Estimators

We define a class of estimators for population mean $\bar{Y}$ of the *SV y* using auxiliary information on (*x*, *z*) in presence of non-response on all the variables (*y*, *x*, *z*) under stratified random sampling as

$$t_{(\eta_1,\eta_2)}=\bar{y}_{st}^{*}\left[\begin{array}{l}\eta_1\left\{\frac{a_x\bar{X}+b_x}{\alpha_x\left(a_x\bar{x}_{st}^{*}+b_x\right)+\left(1-\alpha_x\right)\left(a_x\bar{X}+b_x\right)}\right\}^{g_x}\left\{\frac{a_z\bar{Z}+b_z}{\alpha_z\left(a_z\bar{z}_{st}^{*}+b_z\right)+\left(1-\alpha_z\right)\left(a_z\bar{Z}+b_z\right)}\right\}^{g_z}\\+\eta_2\exp\left\{\frac{-\alpha_x h_x a_x\left(\bar{x}_{st}^{*}-\bar{X}\right)}{\alpha_x\left(a_x\bar{x}_{st}^{*}+b_x\right)+\left(2-\alpha_x\right)\left(a_x\bar{X}+b_x\right)}\right\}\exp\left\{\frac{-\alpha_z h_z a_z\left(\bar{z}_{st}^{*}-\bar{Z}\right)}{\alpha_z\left(a_z\bar{z}_{st}^{*}+b_z\right)+\left(2-\alpha_z\right)\left(a_z\bar{Z}+b_z\right)}\right\}\end{array}\right], \tag{3.50}$$

where $\left(g_x, g_z, \alpha_x, \alpha_z, h_x, h_z, a_x, a_z, b_x, b_z\right)$ are same constants defined earlier and $\left(\eta_1, \eta_2\right)$ are scalars such that *MSE* of $t_{(\eta_1,\eta_2)}$ is minimum.

Expressing (3.50) in terms of *e*'s we have

$$t_{(\eta_1,\eta_2)}=\bar{Y}\left(1+e_0^{*}\right)\left[\begin{array}{l}\eta_1\left(1+\alpha_x\upsilon_x e_1^{*}\right)^{-g_x}\left(1+\alpha_z\upsilon_z e_2^{*}\right)^{-g_z}+\eta_2\exp\left\{\frac{-h_x\alpha_x\upsilon_x e_1^{*}}{2}\left(1+\frac{\alpha_x\upsilon_x}{2}e_1^{*}\right)^{-1}\right\}\\\exp\left\{\frac{-h_z\alpha_z\upsilon_z e_2^{*}}{2}\left(1+\frac{\alpha_z\upsilon_z}{2}e_2^{*}\right)^{-1}\right\}\end{array}\right]. \tag{3.51}$$

We assume that $\left|\alpha_x\upsilon_x e_1^{*}\right|<1$ and $\left|\alpha_z\upsilon_z e_2^{*}\right|<1$, so that we may expand the series, $\left(1+\alpha_x\upsilon_x e_1^{*}\right)^{-g_x}$ and $\left(1+\alpha_z\upsilon_z e_2^{*}\right)^{-g_z}$. Now expanding right hand side of (3.51), multiplying out and neglecting terms of *e*'s having power greater than two we have

$$\left(t_{(\eta_1,\eta_2)}-\bar{Y}\right)\cong\bar{Y}\left[\begin{array}{l}\eta_1\left\{\begin{array}{l}1+e_0^{*}-\left(g_x\alpha_x\upsilon_x\right)e_1^{*}-\left(g_z\alpha_z\upsilon_z\right)e_2^{*}-\left(g_x\alpha_x\upsilon_x\right)e_0^{*}e_1^{*}-\left(g_z\alpha_z\upsilon_z\right)e_0^{*}e_2^{*}+\\\left(g_x\alpha_x\upsilon_x\right)\left(g_z\alpha_z\upsilon_z\right)e_1^{*}e_2^{*}+\frac{g_x\left(g_x+1\right)\left(\alpha_x\upsilon_x\right)^2}{2}e_1^{*2}+\frac{g_z\left(g_z+1\right)\left(\alpha_z\upsilon_z\right)^2}{2}e_2^{*2}\end{array}\right\}\\+\eta_2\left\{\begin{array}{l}1+e_0^{*}-\frac{\left(h_x\alpha_x\upsilon_x\right)}{2}e_1^{*}-\frac{\left(h_z\alpha_z\upsilon_z\right)}{2}e_2^{*}-\frac{\left(h_x\alpha_x\upsilon_x\right)}{2}e_0^{*}e_1^{*}-\frac{\left(h_z\alpha_z\upsilon_z\right)}{2}e_0^{*}e_2^{*}+\\\frac{\left(h_x\alpha_x\upsilon_x\right)\left(h_z\alpha_z\upsilon_z\right)}{4}e_1^{*}e_2^{*}+\frac{h_x\left(h_x+2\right)\left(\alpha_x\upsilon_x\right)^2}{8}e_1^{*2}+\frac{h_z\left(h_z+2\right)\left(\alpha_z\upsilon_z\right)^2}{8}e_2^{*2}\end{array}\right\}-1\end{array}\right], \tag{3.52}$$

Taking expectation of both sides of (3.52) we get the bias of $t_{(\eta_1,\eta_2)}$ to the first degree of approximation as

$$B\left(t_{(\eta_1,\eta_2)}\right)=\bar{Y}\left[\begin{array}{l}\eta_1\left\{\begin{array}{l}1-(g_x\alpha_x\upsilon_x)V_{110}^*-(g_z\alpha_z\upsilon_z)V_{011}^*+(g_x\alpha_x\upsilon_x)(g_z\alpha_z\upsilon_z)V_{101}^*\\+\dfrac{g_x(g_x+1)(\alpha_x\upsilon_x)^2}{2}V_{200}^*+\dfrac{g_z(g_z+1)(\alpha_z\upsilon_z)^2}{2}V_{002}^*\end{array}\right\}\\+\eta_2\left\{\begin{array}{l}1-\dfrac{(h_x\alpha_x\upsilon_x)}{2}V_{110}^*-\dfrac{(h_z\alpha_z\upsilon_z)}{2}V_{011}^*+\dfrac{(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)}{4}V_{101}^*\\+\dfrac{h_x(h_x+2)(\alpha_x\upsilon_x)^2}{8}V_{200}^*+\dfrac{h_z(h_z+2)(\alpha_z\upsilon_z)^2}{8}V_{002}^*\end{array}\right\}-1\end{array}\right]. \tag{3.53}$$

Squaring both sides of (3.52) and neglecting terms of *e*’s having power greater than two we have

$$\left(t_{(\eta_1,\eta_2)}-\bar{Y}\right)^2\cong\bar{Y}^2\left[\begin{array}{l}1+\eta_1^2\left\{\begin{array}{l}1+2e_0^*-2(g_x\alpha_x\upsilon_x)e_1^*-2(g_z\alpha_z\upsilon_z)e_2^*+e_0^{*2}+g_x(2g_x+1)(\alpha_x\upsilon_x)^2e_1^{*2}\\+g_z(2g_z+1)(\alpha_z\upsilon_z)^2e_2^{*2}-4(g_x\alpha_x\upsilon_x)e_0^*e_1^*-4(g_z\alpha_z\upsilon_z)e_0^*e_2^*+\\4(g_x\alpha_x\upsilon_x)(g_z\alpha_z\upsilon_z)e_1^*e_2^*\end{array}\right\}\\+\eta_2^2\left\{\begin{array}{l}1+2e_0^*-(h_x\alpha_x\upsilon_x)e_1^*-(h_z\alpha_z\upsilon_z)e_2^*+e_0^{*2}+\dfrac{h_x(h_x+1)(\alpha_x\upsilon_x)^2}{2}e_1^{*2}\\+\dfrac{h_z(h_z+1)(\alpha_z\upsilon_z)^2}{2}e_2^{*2}-2(h_x\alpha_x\upsilon_x)e_0^*e_1^*-2(h_z\alpha_z\upsilon_z)e_0^*e_2^*+\\(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)e_1^*e_2^*\end{array}\right\}\\+2\eta_1\eta_2\left\{\begin{array}{l}1+2e_0^*-\dfrac{(2g_x+h_x)(\alpha_x\upsilon_x)}{2}e_1^*-\dfrac{(2g_z+h_z)(\alpha_z\upsilon_z)}{2}e_2^*+e_0^{*2}-\dfrac{(2g_x+h_x)(\alpha_x\upsilon_x)}{2}e_0^*e_1^*\\-\dfrac{(2g_z+h_z)(\alpha_z\upsilon_z)}{2}e_0^*e_2^*+\dfrac{(2g_x+h_x)(2g_z+h_z)(\alpha_x\upsilon_x)(\alpha_z\upsilon_z)}{4}e_1^*e_2^*\\+\dfrac{(2g_x+h_x)(2g_x+h_x+2)(\alpha_x\upsilon_x)^2}{8}e_1^{*2}+\dfrac{(2g_z+h_z)(2g_z+h_z+2)(\alpha_x\upsilon_x)^2}{8}e_2^{*2}\end{array}\right\}\\-2\eta_1\left\{\begin{array}{l}1+e_0^*-(g_x\alpha_x\upsilon_x)e_1^*-(g_z\alpha_z\upsilon_z)e_2^*-(g_x\alpha_x\upsilon_x)e_0^*e_1^*-(g_z\alpha_z\upsilon_z)e_0^*e_2^*+\\(g_x\alpha_x\upsilon_x)(g_z\alpha_z\upsilon_z)e_1^*e_2^*+\dfrac{g_x(g_x+1)}{2}(\alpha_x\upsilon_x)^2e_1^{*2}+\dfrac{g_z(g_z+1)}{2}(\alpha_z\upsilon_z)^2e_2^{*2}\end{array}\right\}\\-2\eta_2\left\{\begin{array}{l}1+e_0^*-\dfrac{(h_x\alpha_x\upsilon_x)}{2}e_1^*-\dfrac{(h_z\alpha_z\upsilon_z)}{2}e_2^*-\dfrac{(h_x\alpha_x\upsilon_x)}{2}e_0^*e_1^*-\dfrac{(h_z\alpha_z\upsilon_z)}{2}e_0^*e_2^*+\\\dfrac{(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)}{4}e_1^*e_2^*+\dfrac{h_x(h_x+2)}{8}(\alpha_x\upsilon_x)^2e_1^{*2}+\dfrac{h_z(h_z+2)}{8}(\alpha_z\upsilon_z)^2e_2^{*2}\end{array}\right\}\end{array}\right] \tag{3.54}$$

Taking expectation of both sides of (3.54), we get the *MSE* of $t_{(\eta_1,\eta_2)}$ to the first degree of approximation as

$$MSE\left(t_{(\eta_1,\eta_2)}\right)=\bar{Y}^2\left[1+\eta_1^2A_1+\eta_2^2A_2+2\eta_1\eta_2A_3-2\eta_1A_4-2\eta_2A_5\right], \tag{3.55}$$

where

$$A_1=\left[\begin{array}{l}1+V_{020}^*+(\alpha_x\upsilon_x)^2g_x(2g_x+1)V_{200}^*+(\alpha_z\upsilon_z)^2g_z(2g_z+1)V_{002}^*-4(g_x\alpha_x\upsilon_x)V_{110}^*-4(g_z\alpha_z\upsilon_z)V_{011}^*\\+4(g_x\alpha_x\upsilon_x)(g_z\alpha_z\upsilon_z)V_{101}^*\end{array}\right],$$

$$A_2=\left[\begin{array}{l}1+V_{020}^*+\dfrac{(\alpha_x\upsilon_x)^2h_x(h_x+1)}{2}V_{200}^*+\dfrac{(\alpha_z\upsilon_z)^2h_z(h_z+1)}{2}V_{002}^*-2(h_x\alpha_x\upsilon_x)V_{110}^*-2(h_z\alpha_z\upsilon_z)V_{011}^*\\+(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)V_{101}^*\end{array}\right],$$

$$A_3=\left[\begin{array}{l}1+V_{020}^*+\dfrac{(2g_x+h_x)(2g_x+h_x+2)(\upsilon_x\alpha_x)^2}{8}V_{200}^*+\dfrac{(2g_z+h_z)(2g_z+h_z+2)(\upsilon_z\alpha_z)^2}{8}V_{002}^*\\-\dfrac{(2g_x+h_x)(\upsilon_x\alpha_x)}{2}V_{110}^*-\dfrac{(2g_z+h_z)(\upsilon_z\alpha_z)}{2}V_{011}^*+\dfrac{(2g_x+h_x)(2g_z+h_z)(\upsilon_x\alpha_x)(\upsilon_z\alpha_z)}{4}V_{101}^*\end{array}\right],$$

$$A_4=\left[\begin{array}{l}1-(g_x\alpha_x\upsilon_x)V_{110}^*-(g_z\alpha_z\upsilon_z)V_{011}^*+(g_x\alpha_x\upsilon_x)(g_z\alpha_z\upsilon_z)V_{101}^*+\dfrac{(\alpha_x\upsilon_x)^2g_x(g_x+1)}{2}V_{200}^*\\+\dfrac{(\alpha_z\upsilon_z)^2g_z(g_z+1)}{2}V_{002}^*\end{array}\right],$$

$$A_5 = \begin{bmatrix} 1 - \frac{(h_x\alpha_x\upsilon_x)}{2}V_{110}^* - \frac{(h_z\alpha_z\upsilon_z)}{2}V_{011}^* + \frac{(h_x\alpha_x\upsilon_x)(h_z\alpha_z\upsilon_z)}{4}V_{101}^* + \frac{(\alpha_x\upsilon_x)^2 h_x(h_x+2)}{8}V_{200}^* \\ + \frac{(\alpha_z\upsilon_z)^2 h_z(h_z+2)}{8}V_{002}^* \end{bmatrix}.$$

The $MSE(t_{(\eta_1,\eta_2)})$ at (3.55) is minimized for

$$\left.\begin{aligned} \eta_1 &= \frac{(A_2A_4 - A_3A_5)}{(A_1A_2 - A_3^2)} = \eta_{10} \text{ (say)} \\ \eta_2 &= \frac{(A_1A_5 - A_3A_4)}{(A_1A_2 - A_3^2)} = \eta_{20} \text{ (say)} \end{aligned}\right\}, \tag{3.56}$$

Thus the resulting minimum MSE of $t_{(\eta_1,\eta_2)}$ is given by

$$MSE_{\min}(t_{(\eta_1,\eta_2)}) = \bar{Y}^2\left[1 - \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right] \tag{3.57}$$

Putting (3.56) in (3.53) we get the resulting bias of $t_{(\eta_1,\eta_2)}$ as

$$B_0(t_{(\eta_1,\eta_2)}) = -\bar{Y}\left[1 - \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right] \tag{3.58}$$

From (3.57) and (3.58) it follows that

$$RMSE_{\min}(t_{(\eta_1,\eta_2)}) = \frac{MSE_{\min}(t_{(\eta_1,\eta_2)})}{\bar{Y}^2} = ARB_0(t_{(\eta_1,\eta_2)}) = \left|\frac{B_0(t_{(\eta_1,\eta_2)})}{\bar{Y}}\right| = \left[1 - \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right] \tag{3.59}$$

where $RMSE(.)$ and $ARB_0(.)$ stand for minimum relative MSE of (.) and absolute relative bias (.) at optimum conditions respectively.

Now we state the following theorem.

Theorem 3.3- The MSE of $(t_{(\eta_1,\eta_2)})$ is greater than equal to the $MSE_{\min}(t_{(\eta_1,\eta_2)})$ i.e.

$$MSE(t_{(\eta_1,\eta_2)}) \geq \left[1 - \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right]$$

with equality $\eta_1 = \eta_{10}$, $\eta_2 = \eta_{20}$.

where (η_{10}, η_{20}) are optimum values of (η_1, η_2) given by (3.56).

3.5. Efficiency Comparison

From (3.9) and (3.32) we have

$$MSE(t_{ae}) - MSE_{\min}(t_{se}) = \frac{[MSE(t_{ae}) + \bar{Y}\,B(t_{ae})]^2}{[\bar{Y}^2 + MSE(t_{ae}) + 2\bar{Y}\,B(t_{ae})]} \geq 0 \tag{3.60}$$

It follows from (3.60) that the t_{se}-family of estimators is more efficient than the t_{ae}-family of estimators.

From (3.32) and (3.57) we have

$$MSE_{\min}(t_{se}) - MSE_{\min}(t_{(\eta_1,\eta_2)}) = \frac{\bar{Y}^2(A_2A_4 - A_3A_5)^2}{A_2(A_1A_2 - A_3^2)} \geq 0 \tag{3.61}$$

It follows that the proposed $t_{(\eta_1,\eta_2)}$-family of estimators is more efficient than t_{se}-family of estimators.

From (3.46) we have

$$MSE(t_a) - MSE_{\min}(t_s) = \frac{[MSE(t_a) + \bar{Y}\,B(t_a)]^2}{[\bar{Y}^2 + MSE(t_a) + 2\bar{Y}\,B(t_a)]} \geq 0 \tag{3.62}$$

It follows from (3.62) that the proposed t_s-family is better than the t_a-family of estimators due to Saleem et al (2018).

From (3.46) and (3.57)

$$MSE_{\min}(t_s) - MSE_{\min}\left(t_{(\eta_1,\eta_2)}\right) = \frac{\bar{Y}^2 (A_1 A_5 - A_3 A_4)^2}{A_1 (A_1 A_2 - A_3^2)} \geq 0 \tag{3.63}$$

It follows that the suggested $t_{(\eta_1,\eta_2)}$-family of estimators is better than the t_s-family of estimators due to Saleem et al (2018).

From (3.60), (3.61), (3.62) and (3.63) we have the inequalities:

$$MSE(t_{ae}) \leq MSE_{\min}(t_{se}) \leq MSE\left(t_{(\eta_1,\eta_2)}\right) \tag{3.64}$$

and

$$MSE(t_a) \leq MSE_{\min}(t_s) \leq MSE\left(t_{(\eta_1,\eta_2)}\right) \tag{3.65}$$

It follows from (3.64) and (3.65) that the proposed $t_{(\eta_1,\eta_2)}$-family of estimators is more efficient than the suggested (t_{ae}, t_{se})-families of estimators and the (t_a, t_s)-families of estimators due to Saleem et al (2018).

4. NUMERICAL ILLUSTRATION

For numerical illustration we consider a data set [Source: Koyuncu and Kadilar (2009)], in which
y: Number of teachers; *x*: number of students and
z: number of classes in primary and secondary schools for 923 districts and 6 regions in Turkey in 2007.

Stratum (*h*)		1	2	3	4	5	6
Stratified mean, Standard deviations and Correlation coefficients	N_h	127	117	103	170	205	201
	n_h	31	21	29	38	22	39
	n'_h	70	50	75	95	70	90
	S_{yh}	883.84	644.92	1033.40	810.58	403.65	711.72
	S_{xh}	30486.70	15180.77	27549.69	18218.93	8497.77	23094.14
	S_{zh}	555.58	365.46	612.95	458.03	260.85	397.05
	$\bar{Y}_h$	703.74	413.00	573.17	424.66	267.03	393.84
	$\bar{X}_h$	20804.59	9211.79	14309.30	9478.85	5569.95	12997.59
	$\bar{Z}_h$	498.28	318.33	431.36	311.32	227.20	313.71
	ρ_{yxh}	0.94	1.00	0.99	0.98	0.99	0.97
	ρ_{xzh}	0.94	0.97	0.98	0.96	0.97	1.00
	ρ_{yzh}	0.98	0.98	0.98	0.98	0.96	0.98
W_h=10% Non-response	$S_{yh(2)}$	510.57	386.77	1872.88	1603.30	264.19	497.84
	$S_{xh(2)}$	9446.93	9198.29	52429.99	34794.90	4972.56	12485.10
	$S_{zh(2)}$	303.92	278.51	960.71	821.29	190.85	287.99
	$\rho_{yxh(2)}$	1.00	1.00	1.00	0.97	1.00	0.93
	$\rho_{xzh(2)}$	0.99	0.99	1.00	0.96	0.99	0.98
	$\rho_{yzh(2)}$	0.99	0.99	1.00	0.99	0.99	0.96
W_h=20% Non-response	$S_{yh(2)}$	396.77	406.15	1654.40	1333.35	335.83	903.91
	$S_{xh(2)}$	7439.16	8880.46	45784.78	29219.30	6540.43	28411.44
	$S_{zh(2)}$	244.56	274.42	965.42	680.28	214.49	469.86
	$\rho_{yxh(2)}$	1.00	0.99	1.00	0.98	1.00	0.99
	$\rho_{xzh(2)}$	0.99	0.99	0.98	0.96	0.98	0.98
	$\rho_{yzh(2)}$	0.99	0.98	0.98	0.99	0.98	0.99
W_h=30% Non-response	$S_{yh(2)}$	500.26	356.95	1383.70	1193.47	289.41	825.24
	$S_{xh(2)}$	14017.99	7812.00	38379.77	26090.60	5611.32	24571.95

	$S_{zh(2)}$	284.44	247.63	811.21	631.28	188.30	437.90
	$\rho_{yxh(2)}$	0.96	0.99	1.00	0.98	1.00	0.97
	$\rho_{xzh(2)}$	0.91	0.98	0.98	0.97	0.98	0.96
	$\rho_{yzh(2)}$	0.97	0.98	0.98	0.99	0.98	0.98

Table 4.1: *PRE* of estimators for three different cases

Estimators	$W_h = 10\%$ non-response	$W_h = 20\%$ non-response	$W_h = 30\%$ non-response
t_{ae}	2585.74	2797.02	2754.44
$t_{ae(1)}$	1868.39	1930.19	1984.82
$t_{ae(2)}$	28.40	28.33	28.41

Table 4.2: *PRE* of the proposed estimator when $W_h = 10\%$ non-response for different values of the constants $(g_x, g_z, \alpha_x, \alpha_z, h_x, h_z, a_x, a_z, b_x, b_z)$.

g_x	g_z	α_x	α_z	h_x	h_z	a_x	a_z	b_x	b_z	***PRE***
-0.5	-0.5	0.4	1	1	1	1	1	1	1	2765.53
-0.5	-0.5	0.2	1	1	1	1	1	1	1	2780.23
-0.5	-0.5	0.25	1	1	1	1	1	1	1	2811.40
-0.5	-0.5	0.3	1	1	1	1	1	1	1	2815.24
-0.6	-0.6	1.25	1	1	1	1	1	1	1	2906.22
-0.6	-0.6	1.3	1	1	1	1	1	1	1	3266.09
-0.6	-0.6	1.4	1	1	1	1	1	1	1	4341.51
-0.7	-0.7	1.2	1	1	1	1	1	1	1	4874.14
-0.8	-0.8	1.1	1	1	1	1	1	1	1	5354.84
-0.6	-0.6	1.5	1	1	1	1	1	1	1	6497.11
-0.6	-0.6	1.6	1	1	1	1	1	1	1	13105.78
-0.7	-0.7	1.3	1	1	1	1	1	1	1	15251.2

Table 4.3: *PRE* of the suggested estimator $t_{(\eta_1,\eta_2)}$ when $W_h = 20\%$ non-response for different values of the constants $(g_x, g_z, \alpha_x, \alpha_z, h_x, h_z, a_x, a_z, b_x, b_z)$.

g_x	g_z	α_x	α_z	h_x	h_z	a_x	a_z	b_x	b_z	***PRE***
-0.5	-0.5	0.4	1	1	1	1	1	1	1	2822.13
-0.5	-0.5	0.2	1	1	1	1	1	1	1	2832.26
-0.5	-0.5	0.25	1	1	1	1	1	1	1	2863.72
-0.5	-0.5	0.3	1	1	1	1	1	1	1	2868.6
-0.6	-0.6	1.25	1	1	1	1	1	1	1	3115.53
-0.6	-0.6	1.3	1	1	1	1	1	1	1	3549.77
-0.6	-0.6	1.4	1	1	1	1	1	1	1	4927.21
-0.7	-0.7	1.2	1	1	1	1	1	1	1	5522.1
-0.8	-0.8	1.1	1	1	1	1	1	1	1	6110.36
-0.6	-0.6	1.5	1	1	1	1	1	1	1	8110.65
-0.6	-0.6	1.6	1	1	1	1	1	1	1	23771.7
-0.7	-0.7	1.3	1	1	1	1	1	1	1	25936.9

Table 4.4: *PRE* of the suggested estimator $t_{(\eta_1,\eta_2)}$ when $W_h = 30\%$ non-response for different values of the constants $(g_x, g_z, \alpha_x, \alpha_z, h_x, h_z, a_x, a_z, b_x, b_z)$.

g_x	g_z	α_x	α_z	h_x	h_z	a_x	a_z	b_x	b_z	***PRE***
-0.5	-0.5	0.2	1	1	1	1	1	1	1	2990.23
-0.5	-0.5	0.4	1	1	1	1	1	1	1	3036.49
-0.5	-0.5	0.25	1	1	1	1	1	1	1	3047.62
-0.5	-0.5	0.3	1	1	1	1	1	1	1	3069.89
-0.6	-0.6	1.25	1	1	1	1	1	1	1	3236.24
-0.6	-0.6	1.3	1	1	1	1	1	1	1	3701.08
-0.6	-0.6	1.4	1	1	1	1	1	1	1	5200.57
-0.7	-0.7	1.2	1	1	1	1	1	1	1	5735.12
-0.8	-0.8	1.1	1	1	1	1	1	1	1	6241.62
-0.6	-0.6	1.5	1	1	1	1	1	1	1	8820.07

-0.7	-0.7	1.3	1	1	1	1	1	1	1	30438.6
-0.6	-0.6	1.6	1	1	1	1	1	1	1	30502.33

Table 4.1 gives *PRE* of $t_{ae}, t_{ae(1)}$ and $t_{ae(2)}$ under three different cases (non-response 10%, 20%, 30%) respectively.
Table 4.2 – 4.4 gives *PRE*s of $t_{(\eta_1,\eta_2)}$ under different cases of non-response that are 10%, 20%, 30% respectively for different values of the constants $(g_x, g_z, \alpha_x, \alpha_z, h_x, h_z, a_x, a_z, b_x, b_z)$.
From Tables 4.1 to 4.4, it is clearly seen that the proposed class of estimators performs very well and gives too high *PRE*s as compared with $t_{ae}, t_{ae(1)}$ and $t_{ae(2)}$ under all the three cases which are 10%, 20%, 30% non-response respectively for different values of the constants $(g_x, g_z, \alpha_x, \alpha_z, h_x, h_z, a_x, a_z, b_x, b_z)$. It is also advantageous that there are 10 constants and each constant has a wide range itself. So one can choose too many different combinations and attain higher values of *PRE*. These estimators are reliable in practice so we recommend them for their practical use.

5. CONCLUSION

In this article, we have addressed the problem of estimating the population mean using auxiliary information in presence of non-response on both study variable as well auxiliary variables under stratified random sampling. We have proposed class of exponential-type estimators, improved class of exponential-type estimators and a more general class of estimators for population mean in the presence of non-response in stratified random sampling. Properties of the estimators were studied up to first degree of approximation. Some members of the proposed class of exponential-type estimators t_{ae} and t_{se} are also given. Expressions for the optimum sample sizes of the strata in respect to cost of the survey have been also derived. We have proved that the proposed $t_{(\eta_1,\eta_2)}$-family of estimators is more adequate than the suggested (t_{ae}, t_{se})-families of estimators and the (t_a, t_s)-families of estimators due to Saleem et al (2018). Numerical illustration is also carried out to support the theoretical findings. From the entire discussions we conclude that these estimators are reliable in practice so we recommend them for their practical use.

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