ESTIMATION FOR TWO EXPONENTIAL POPULATIONS BASED ON JOINT RANK SET SAMPLING

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ABSTRACT
In this paper, we introduce two new rank set sampling schemes namely, joint rank set sampling (JRSS) and joint modified rank set sampling (JMRSS), based on two samples come from two different production lines. We use exponential models for the two samples. This study leads to the estimation of the parameters of the models using classical and Bayesian approaches. For Bayes estimation, importance sampling method is implemented. A comparison of the proposed estimators obtained under JRSS and JMRSS is made with the estimators obtained based on joint simple random sample through simulation. A real example is cited to illustrate the procedures.

KEYWORDS: Joint modified ranked set sample, maximum likelihood estimation, Bayesian estimation, Squared error loss, inverted gamma priors, joint simple random sample

MSC: 62D05, 62F07, 62F10, 62F15, 62P30

RESUMEN
En este paper introducimos dos nuevos esquemas de muestreo por conjuntos ordenados denominados , muestreo por rangos ordenados (JMRSS), basados en dos muestras que provienen de dos diferentes líneas de producción. Usamos modelos exponenciales para las dos muestras . Este estudio lleva a la estimación de los parámetros del modelo usando enfoques clásicos y Bayesiano. Para la estimación Bayes, un método de muestreo por importancia es implementado. Una comparación de los estimadores propuestos obtenidos bajo JRSS y JMRSS es desarrollada usando muestro simple conjunto a través de simulación. Un ejemplo real es citado para ilustrar los procedimientos.

PALABRAS CLAVE: Muestreo por conjuntos ordenados conjunto**,** estimación máximo verosímil, estimación Bayesiana, pérdida cuadrática, gamma prior invertida, muestro simple conjunto.

1. INTRODUCTION

Estimation of parameters based on a sample collected from the given population is the most important research in all branches of sciences such as engineering, medical, biological, agricultural. The data collection is an important aspect to make efficient estimation of the unknown parameters of the given population. McIntyre [14] introduced a concept of rank set sampling (RSS) that utilizes additional information from individual population units providing a more representative sample from the population under consideration. RSS improves the efficiency of the estimators compared to simple random sample. Various type of RSS schemes are available in literature, like, moving rank set sampling, modified rank set sampling, median rank set sampling, moving extreme rank set sampling and so on. Recently AL-Nasser and AL-Omari [2] consideredMinimax ranked set sampling. A detail review up to 2013 on rank set sampling did by Al-Omari and Bouza [3]. Bouzaand AL-Omari [10]presented 65 years of history in improving the accuracy in data gathering using ranked set sampling.
The problem of making inferences based on ranked set sample from a particular distribution hasreceived attention of many researchers. For literature on this topic one can refer to Lam, Sinha, and Wu [12] in

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whichunbiased estimators of the parameters of two-parameter exponential distribution with their variances were derived based on RSS. They made a comparison between these estimators with the estimators obtained based on simple random sampling. Sarikavanij et al.[21] studied the location and scale estimators of two-parameter exponentialdistribution using simple random sample and ranked set sample in terms of generalized variance. Al-Saleh and Al-Hadrami[4] considered estimation of mean of exponential distribution using moving extreme rank set sampling. Chen, Bai, and Sinha[11] discussed applications of RSS. Abu-Dayyeh and Al-Sawi[1] obtained modified maximum likelihood estimator of mean of exponential distribution based on moving extreme rank set sampling. Dong et al. [12]proposed reliability estimates in case of exponential distribution using rank set sampling with unequal samples. Mahdizadeh and Zamanzade [16] used multistage ranked set sampling for estimation of a symmetric distribution function.

Most of the rank set sampling schemes reported in the literature are based on the samples from a single production line only. When the product comes from more than one product line, then the joint sampling scheme appears.

Recently, in life testing experiments recently the trend is to use joint censoring instead of single sample censoring scheme. There are situations in which one wishes to compare different populations. Suppose the products are being manufactured by two different lines under the same conditions. Two independent samples of sizes m_1 and $m_2$selected from these two lines respectively, and are put simultaneously on a life testing experiment. Such a scheme is called joint testing scheme. Recently, two sample joint censoring schemes become more popular for a life testing experiment mainly to optimize time and cost.

Bhattacharya [9] reviewed the developments in parametric and non-parametric methods based on joint type-II censoring scheme. Balakrishnan and Rasouli [8] first considered the likelihood inference for two exponential populations under a joint type-II censoring scheme and compared the performance of estimators with those based on approximate Bayesian method.

Rasouli and Balakrishnan [19] derived exact likelihood inference for two exponential distributions based on joint progressive type- II censoring scheme. Shafay, Balakrishnan, and Abdel-Aty [22] considered Bayesian inference based on joint type-II censored sample from two exponential populations with the use of squared-error, linear-exponential and general entropy loss functions. Balakrishnan and Feng [7] generalized the work done by Balakrishnan and Rasouli[8] by considering joint type-II censoring scheme in case of k independent exponential distributions. Ashour and Abo-Kasem [5] employed classical and Bayesian estimation in case of two generalized exponential distributions based on joint type-II censoring scheme.

To the best of our knowledge, inference based on joint rank set sampling scheme is not available in the literature. Because of the importance of joint sample in making inference for the products from different production lines, we propose two new rank set sampling schemes based on joint sample so called joint rank set sampling (JRSS) and joint modified rank set sampling (JMRSS) schemes.We have considered perfect ranking, i.e. no error in ranking. We use two independent samples from two exponential distributions and these samples are jointly used to obtain a rank set sample.We propose maximum likelihood estimators and Bayes estimators to estimate the parameters of the two joint exponential populations.

The rest of the paper is organized as follows. In Section 2, joint rank set sampling scheme is developed. The maximum likelihood (ML) estimation and Bayes estimation of the parameters of the two exponential distributions are considered. Joint modified rank set sampling scheme is proposed and the parameters of the joint model are estimated using the method of ML and Bayes estimation in Section 3. Section 4 deals with ML estimation and Bayes estimation of the parameters using joint simple random samples. To investigate and compare the performance of the propose estimators presented in this paper, a simulation study is carried out. A real life example is also considered to illustrate the estimation methods discussed in the previous sections. The outcomes of the study are presented in Section 5. Finally, some conclusions are provided in Section 6.

2. JOINT RANK SET SAMPLING SCHEME

In this section we develop the joint rank set sampling scheme and estimation of the parameters of the model based on JRSS.

Suppose there are two lines of similar products and it is important to compare the relative merits of these two products. A sample of size m_1 is drawn from one product line (say A) and another sample of size m_2 is drawn from the other product line (say B).

Let X and Y be random variables under study for the product type A and type B respectively.

We assume that random variables X and Y follow exponential distribution with mean $\theta_1, \theta_1>0$ and $\theta_2, \theta_2>0$ respectively. Their pdf and cdf are respectively given below,

$$f_X(x) = \frac{1}{\theta_1} e^{\frac{-x}{\theta_1}}; \quad F_X(x) = 1 - e^{\frac{-x}{\theta_1}}, \quad x > 0 \tag{1}$$

$$f_Y(y) = \frac{1}{\theta_2} e^{\frac{-y}{\theta_2}}; \quad F_Y(y) = 1 - e^{\frac{-y}{\theta_2}}, \quad y > 0 \tag{2}$$

We use the following process to obtain joint ranked set sample of size $n = mk$.

Algorithm to obtain joint ranked set sample:

Step 1: Randomly select m_1 elements $(X_1, X_2, \dots, X_{m_1})$from product type A and m_2 elements$(Y_1, Y_2, \dots, Y_{m_2})$ from product type Band combine them to create a joint sample of $m = m_1 + m_2$ elements.

Step 2: Arrange all m elements of the joint sample in ascending order, visually or by based on actual measurements.

Step 3: Repeat the above two steps m times, so that we have m sets each of with m ordered elements.

Step 4: Select and quantify the i-th minimum from the i-th set having m elements, $i = 1, 2, \dots, m$ to get a new set of size m, which is called the joint ranked set sample.

Step 5: Repeat steps 1 – 4, for k times (cycles) to increase the size of joint ranked set sample to $n = mk$. Thus, we get k joint ranked set samples each of size m.

Denote w_{ij} as the i-th element of i-th joint ranked set sample in j-th cycle, $i = 1,2, \dots, m; j = 1,2, \dots, k$.

Maximum Likelihood Estimation under JRSS

The likelihood function based on the observations w_{ij} ; $i = 1,2, \dots, m$; $j = 1,2, \dots, k$,obtained in joint ranked set sample, is constructed as

$$L = L(\theta_1, \theta_2, \underline{x}, \underline{y})$$

$$= \prod_{j=1}^{k} \prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!} \left\{ \begin{array}{c} [F_x(w_{ij})]^{a_{ij}} [G_y(w_{ij})]^{c_{ij}} [f_x(w_{ij})]^{z_{ij}} [g_y(w_{ij})]^{1-z_{ij}} \\ [1 - F_x(w_{ij})]^{b_{ij}} [1 - G_y(w_{ij})]^{d_{ij}} \end{array} \right\}, \tag{3}$$

where $z_{ij} = \begin{cases} 1, & \text{if } w_{ij} \text{is from X} - \text{sample} \\ 0, & \text{if } w_{ij} \text{is from Y} - \text{sample} \end{cases}, i = 1,2, \dots, m; j = 1, 2, \dots, k.$

a_{ij} = Number of X observations less than or equal to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

b_{ij} = Number of X observations greater thanw_{ij} in i^{th} set of combined samples in j^{th}cycle.

c_{ij} = $m - 1 - a_{ij}$= Number of Y observations less than or equal to w_{ij} in i^{th} set of combined samples in j^{th}cycle

d_{ij} = $m - 1 - b_{ij}$ = Number of Y observations greater than or equal tow_{ij} in i^{th} set of combined samples inj^{th}cycle

From (3), with (1) and (2), the log-likelihood function is given by

$$l = \log L = logL(\theta_1, \theta_2, \underline{x}, \underline{y})$$

$$\propto \sum_{j=1}^{k} \sum_{i=1}^{m} a_{ij} log\left(1 - e^{\frac{-w_{ij}}{\theta_1}}\right) + \sum_{j=1}^{k} \sum_{i=1}^{m} c_{ij} \log\left(1 - \frac{1}{\theta_2} e^{\frac{-w_{ij}}{\theta_2}}\right) - \sum_{j=1}^{k} \sum_{i=1}^{m} z_{ij}\, log\theta_1 - \sum_{j=1}^{k} \sum_{i=1}^{m} z_{ij} \log \theta_2 - \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} w_{ij}(b_{ij} + z_{ij})}{\theta_1} - \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} w_{ij}(d_{ij} + 1 - z_{ij})}{\theta_2}. \tag{4}$$

Now, by taking partial derivatives of the log-likelihood function with respect to θ_1 and θ_2, the likelihood equations can be obtained as follows,

$$\frac{\partial l}{\partial \theta_1} = \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{a_{ij} w_{ij}}{{\theta_1}^2} \left(\frac{e^{-\frac{w_{ij}}{\theta_1}}}{1 - e^{-\frac{w_{ij}}{\theta_1}}} \right) - \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} z_{ij}}{\theta_1} + \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} w_{ij}(b_{ij}+z_{ij})}{{\theta_1}^2} = 0.$$

(5)

$$\frac{\partial l}{\partial \theta_2} = \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{c_{ij} w_{ij}}{{\theta_2}^2} \left(\frac{e^{-\frac{w_{ij}}{\theta_2}}}{1 - e^{-\frac{w_{ij}}{\theta_2}}} \right) - \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} (1-z_{ij})}{\theta_2} + \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} w_{ij}(d_{ij}+1-z_{ij})}{{\theta_2}^2} = 0.$$

(6)

By simplifying, (5) and (6) we get the equations

$$\theta_1 = \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} w_{ij}(b_{ij}+z_{ij}) - \sum_{j=1}^{k} \sum_{i=1}^{m} \left(\frac{a_{ij} w_{ij} e^{-\frac{w_{ij}}{\theta_1}}}{1 - e^{-\frac{w_{ij}}{\theta_1}}} \right)}{\sum_{j=1}^{k} \sum_{i=1}^{m} z_{ij}},$$

(7)

and

$$\theta_2 = \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} w_{ij}(d_{ij}+1-z_{ij}) - \sum_{j=1}^{k} \sum_{i=1}^{m} \left(\frac{c_{ij} w_{ij} e^{-\frac{w_{ij}}{\theta_2}}}{1 - e^{-\frac{w_{ij}}{\theta_2}}} \right)}{\sum_{j=1}^{k} \sum_{i=1}^{m} (1-z_{ij})}.$$

(8)

Hence, the maximum likelihood estimates of the parameters θ_1 and θ_2 can be obtained by solving the equations (7) and (8) by the numerical method of iteration such as the Newton-Raphson method.
Now, to obtain asymptotic variance of the ML estimators, we consider second derivatives with respect to θ_1 and $\theta_2$of log likelihood function given in (4) as

$$\frac{\partial^2 l}{\partial \theta_1^2} = \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{2 a_{ij} w_{ij}}{\theta_1^3} \left(\frac{e^{-\frac{w_{ij}}{\theta_1}}}{1 - e^{-\frac{w_{ij}}{\theta_1}}} \right) - \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{a_{ij} w_{ij}^2}{\theta_1^4} \left(\frac{e^{-\frac{w_{ij}}{\theta_1}}}{\left(1 - e^{-\frac{w_{ij}}{\theta_1}}\right)^2} \right) + \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} z_{ij}}{\theta_1^2} - \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} 2 w_{ij}(b_{ij}+z_{ij})}{\theta_1^3},$$

(9)

$$\frac{\partial^2 l}{\partial \theta_2^2} = \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{2 c_{ij} w_{ij}}{\theta_2^3} \left(\frac{e^{-\frac{w_{ij}}{\theta_2}}}{1 - e^{-\frac{w_{ij}}{\theta_2}}} \right) - \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{c_{ij} w_{ij}^2}{\theta_2^4} \left(\frac{e^{-\frac{w_{ij}}{\theta_2}}}{\left(1 - e^{-\frac{w_{ij}}{\theta_2}}\right)^2} \right) + \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} (1-z_{ij})}{\theta_2^2} - \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} 2 w_{ij}(d_{ij}+1-z_{ij})}{\theta_2^3}.$$

(10)

Hence the observed asymptotic variance of ML estimators of θ_1 and θ_2 are given by

$$V(\hat{\theta}_1) = -\frac{1}{\left.\frac{\partial^2 l}{\partial \theta_1^2}\right]_{\theta_1=\hat{\theta}_1}},$$

(11)

and

$$V(\hat{\theta}_2) = -\frac{1}{\left.\frac{\partial^2 l}{\partial \theta_2^2}\right]_{\theta_2=\hat{\theta}_2}}.$$

(12)

2.1. Bayes Estimation Using JRSS

One of the widely used method of estimation of the parameters of a distribution is the Bayes estimation.In this method, for estimating the parameter θ, by the estimator $\hat{\theta}$,the researcher needs a prior distribution, $\pi(\theta)$ for θ, as well as a loss function,$L(\theta, \hat{\theta})$ for estimation process. The Bayes estimate of θ is obtained by minimizing the posterior risk $E[L(\theta, \hat{\theta})|X]$with respect to θ. In literature, for simplicity, the squared error loss function(SELF) $L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$ is used for estimation problems. Under the SE loss function, the Bayes estimatorof θ is given by the mean of the posterior distribution. This loss function is convex and symmetric, and assigns equal losses to the overestimation and underestimation.

In this section, we provide the Bayes estimation under squared error loss function (SELF) for unknown parameters θ_1 and θ_2 . To compute the Bayes estimators of the unknown parameters, we need to consider some prior distributions of parameters θ_1 and θ_2. We assume inverted gamma priors ($igamma(\alpha_i, \beta_i)$) for two parameters having pdf is given by

$$\pi(\theta_i) = \frac{\beta_i{}^{\alpha_i}}{\Gamma\alpha_i}\theta_i{}^{-(\alpha_i+1)}e^{-\frac{\beta_i}{\theta_i}}, \quad \theta_i > 0,\ \alpha_i > 0, \beta_i > 0; i = 1,2 \tag{13}$$

From the likelihood function of joint rank set sampling given in (3) and using (1), (2) and (13), the joint posterior distribution of θ_1 and θ_2 given $\underline{w}$ can be derived as,

$$h(\theta_1, \theta_2|\underline{w}) \propto \prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_1}}\right)^{a_{ij}} \theta_1{}^{-\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}} e^{-\frac{\sum_{j=1}^{k}\sum_{i=1}^{m} w_{ij}(b_{ij}+z_{ij})}{\theta_1}} \theta_1{}^{-(\alpha_1+1)} e^{-\frac{\beta_1}{\theta_1}}$$

$$\prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_2}}\right)^{c_{ij}} \theta_2{}^{-\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})} e^{-\frac{\sum_{j=1}^{k}\sum_{i=1}^{m} w_{ij}(d_{ij}+1-z_{ij})}{\theta_2}} \theta_2{}^{-(\alpha_2+1)} e^{-\frac{\beta_2}{\theta_2}}$$

$$\propto \theta_1{}^{-\left(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}+\alpha_1+1\right)} e^{-\frac{(A_1+\beta_1)}{\theta_1}} \theta_2{}^{-\left(\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})+\alpha_2+1\right)} e^{-\frac{(A_2+\beta_2)}{\theta_2}}$$

$$\prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_1}}\right)^{a_{ij}} \prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_2}}\right)^{c_{ij}}. \tag{14}$$

$$= h_1(\theta_1|\underline{w})h_2(\theta_2|\underline{w}), \tag{15}$$

where,

$A_1 = \sum_{j=1}^{k}\sum_{i=1}^{m} w_{ij}(b_{ij} + z_{ij})$ and $A_2 = \sum_{j=1}^{k}\sum_{i=1}^{m} w_{ij}(d_{ij} + 1 - z_{ij})$

$$h_1(\theta_1|\underline{w}) \propto \theta_1{}^{-\left(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}+\alpha_1+1\right)} e^{-\frac{(A_1+\beta_1)}{\theta_1}} \prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_1}}\right)^{a_{ij}}. \tag{16}$$

$$h_2(\theta_2|\underline{w}) \propto \theta_2{}^{-\left(\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})+\alpha_2+1\right)} e^{-\frac{(A_2+\beta_2)}{\theta_2}} \prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_2}}\right)^{c_{ij}}. \tag{17}$$

Here, $h_1(\theta_1|\underline{w})$ and $h_2(\theta_2|\underline{w})$ are considered as marginal posterior distributions of parameters θ_1 and θ_2 respectively. They can be represented as,

$$h_1(\theta_1|\underline{w}) = igamma\left(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij} + \alpha_1,\ A_1 + \beta_1\right)h_{13}(\theta_1), \tag{18}$$

and

$$h_2(\theta_2|\underline{w}) = igamma\left(\sum_{j=1}^{k}\sum_{i=1}^{m}(1 - z_{ij}) + \alpha_2, A_2 + \beta_2\right)h_{23}(\theta_2), \tag{19}$$

where,

$$h_{13}(\theta_1) = \prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_1}}\right)^{a_{ij}}, \tag{20}$$

and

$$h_{23}(\theta_2) = \prod_{j=1}^{k}\prod_{i=1}^{m}\left(1 - e^{-\frac{w_{ij}}{\theta_2}}\right)^{c_{ij}}. \tag{21}$$

Hence, the Bayes estimates of θ_1 and θ_2 under SELF will be

$$\hat{\theta}_{1B} = \int_0^\infty \theta_1\, h_1(\theta_1|\underline{w})d\theta_1, \tag{22}$$

and

$$\hat{\theta}_{2B} = \int_0^\infty \theta_2\, h_2(\theta_2|\underline{w})d\theta_2. \tag{23}$$

As, $h_1(\theta_1|\underline{w})$ and $h_2(\theta_2|\underline{w})$ are complex functions, Bayes estimates of θ_1 and θ_2 cannot be obtained in a closed form. Many approximation methods are available in the literature. We use the importance sampling method to obtain Bayes estimates of $\theta_1$and θ_2. Among others, some further applications of this method can also be found in Kundu and Pradhan [13], Rastogi and Tripathi [20], Sultana et al. [23]. The algorithm of the method is as follows:

Step 1: Decide the values of $m(= m_1 + m_2)$ and k.

Step 2: Generate two independent samples of sizes $m_1$and m_2 respectively from exponential distribution with parameter $\theta_1$and θ_2.

Step 3: Repeat the steps 1 and 2 for m times.

Step 4: Determine the ranked set sample $w_{11}, w_{21}, \dots, w_{m1}$.

Step 5: Repeat the steps 1 to 4 for k times to get joint ranked set samples $w_{1j}, w_{2j}, \dots, w_{mj}; j = 1,2, \dots, k$ for k cycles.

Step 6: Generate Nvalues, say $\theta_{11}, \theta_{12}, \dots, \theta_{1N}$ of θ_1 from inverted gamma distribution $igamma\left(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij} + \alpha_1, A_1 + \beta_1\right)$.

Step 7: Generate N values,say$\theta_{21}, \theta_{22}, \dots, \theta_{2N}$of θ_2 from inverted gamma distribution $igamma\left(\sum_{j=1}^{k}\sum_{i=1}^{m}(1 - z_{ij}) + \alpha_2, A_2 + \beta_2\right)$.

Step 8: Based on N values of θ_1 and θ_2, compute N values of functions $h_{13}(\theta_1)$ and $h_{23}(\theta_2)$ respectively from (20) and (21).

Step 9: Under SELF, the Bayes estimate of θ_1 and θ_2 can be obtained as

$$\hat{\theta}_{1B} = \frac{\sum_{i=1}^{N}\theta_{1i}h_{13}(\theta_{1i})}{\sum_{i=1}^{N}h_{13}(\theta_{1i})}, \tag{24}$$

and

$$\hat{\theta}_{2B} = \frac{\sum_{i=1}^{N}\theta_{2i}h_{23}(\theta_{2i})}{\sum_{i=1}^{N}h_{23}(\theta_{2i})}. \tag{25}$$

3. JOINT MODIFIED RANK SET SAMPLING

In this section we develop the joint modified rank set sampling and estimation of the parameters of the model based on ML estimation and Bayes estimation. The detailed process of generating JMRSS is described as follows.

Step 1: Randomly select m_1 elements ($X_1, X_2, \dots, X_{m_1}$) from product type A and m_2 elements ($Y_1, Y_2, \dots, Y_{m_2}$) from product type B and combine them to create a joint sample of $m = m_1 + m_2$ elements.

Step 2: Arrange all m elements of the joint sample in ascending order, visually or by based on actual measurements.

Step 3: Repeat the above two steps m times so that we have m sets each of with m ordered elements.

Step 4: Select and quantify the smallest element from all $m$sets, each having m elements to get a new set of size m, which is called the jointmodified ranked set sample.

Step 5. Repeat steps 1– 4, for k times (cycles) to increase the size of joint rank set sample to $n = mk$. Thus, we have k joint modified ranked set samples each of size m.

Denote $w_{(1)ij}$ as the smallest element of i-th joint ranked set sample in j-th cycle, $i = 1,2, \dots, m$; $j = 1,2, \dots, k$.

3.1. Maximum Likelihood Estimation under JMRSS

The likelihood function based on observations $w_{(1)ij}$; $i = 1,2, \dots, m$; $j = 1,2, \dots, k$, can be constructed as

$$L = L(\theta_1, \theta_2, \underline{x}, \underline{y}) = \prod_{j=1}^{k}\prod_{i=1}^{m}\frac{m!}{(m-1)!}\left\{\begin{matrix}\left[f_x\left(w_{(1)ij}\right)\right]^{z_{ij}}\left[g_y\left(w_{(1)ij}\right)\right]^{1-z_{ij}} \\ \left[1 - F_x\left(w_{(1)ij}\right)\right]^{b_{ij}}\left[1 - G_y\left(w_{(1)ij}\right)\right]^{d_{ij}}\end{matrix}\right\}. \tag{26}$$

Note that the above likelihood function will be a particular case of the likelihood function in (3) for $a_{ij} = c_{ij} = 0$ for all $i = 1,2, \dots, m$; $j = 1,2, \dots, k$.

Using (1) and (2) in (26), the log likelihood function can be obtained as

$l = log(L) = -\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}\log\theta_1 - \frac{\sum_{j=1}^{k}\sum_{i=1}^{m} w_{(1)ij}(b_{ij}+z_{ij})}{\theta_1} - \sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}\log\theta_2 - \frac{\sum_{j=1}^{k}\sum_{i=1}^{m} w_{(1)ij}(d_{ij}+1-z_{ij})}{\theta_2}$

(27)

Differentiating (27) with respect to θ_1 and θ_2 and comparing them with zero, maximum likelihood estimates of the parameters θ_1 and $\theta_2$can be derived as

$\hat{\theta}_{1M} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m} w_{(1)ij}(b_{ij}+z_{ij})}{\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}},$

(28)

and

$\hat{\theta}_{2M} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m} w_{(1)ij}(d_{ij}+1-z_{ij})}{\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})}.$

(29)

To obtain observed asymptotic variances of the estimators we differentiae log likelihood function in (27) two times with respect θ_1 and θ_2 separately,the results are

$\frac{\partial^2 l}{\partial\theta_1^2} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}}{\theta_1^2} - \frac{\sum_{j=1}^{k}\sum_{i=1}^{m} 2w_{(1)ij}(b_{ij}+z_{ij})}{\theta_1^3},$

(30)

$\frac{\partial^2 l}{\partial\theta_2^2} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})}{\theta_2^2} - \frac{\sum_{j=1}^{k}\sum_{i=1}^{m} 2w_{(1)ij}(d_{ij}+1-z_{ij})}{\theta_2^3}.$

(31)

Hence the observed asymptotic variances of MLEs of θ_1 and $\theta_2$in case of JMRSS can be computed as

$V(\hat{\theta}_{1M}) = -\frac{1}{\left.\frac{\partial^2 l}{\partial\theta_1^2}\right]_{\theta_1=\hat{\theta}_{1M}}},$

(32)

and

$V(\hat{\theta}_{2M}) = -\frac{1}{\left.\frac{\partial^2 l}{\partial\theta_2^2}\right]_{\theta_2=\hat{\theta}_{2M}}}.$

(33)

3.2. Bayes Estimation under JMRSS

As discussed in Section3.1, the likelihood function under the joint MRSS becomes a particular case of the likelihood function under the joint RSS; the marginal posterior distributions of θ_1 and θ_2 can be directly obtained by substituting $a_{ij} = 0$ and $c_{ij} = 0$ in (18) and (19) as

$h_1(\theta_1|\underline{w}) = igamma\left(\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij} + \alpha_1,\ A_1 + \beta_1\right),$

(34)

and

$h_2(\theta_2|\underline{w}) = igamma\left(\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij}) + \alpha_2, A_2 + \beta_2\right).$

(35)

As the Bayes estimates$\hat{\theta}_{1BM}$ *and* $\hat{\theta}_{2BM}$of θ_1 and θ_2 under SELF are nothing but their posterior means, the estimators are given by

$\hat{\theta}_{1BM} = E_{h1}(\theta_1)$ = Mean of inverted gamma distribution in (34)

$= \frac{A_1+\beta_1}{\sum_{j=1}^{k}\sum_{i=1}^{m} z_{ij}+\alpha_1-1},$

(36)

and

$\hat{\theta}_{2BM} = E_{h2}(\theta_2)$ = Mean of inverted gamma distribution in (35)

$= \frac{A_2+\beta_2}{\sum_{j=1}^{k}\sum_{i=1}^{m}(1-z_{ij})+\alpha_2-1}.$

(37)

4. JOINT SIMPLE RANDOM SAMPLING(JSRS)

To compare the performance of proposed estimators obtained under joint RSS and joint MRSS with the estimators obtained under joint simple random sampling, we derive MLE and Bayes estimates of the parameters in case of joint simple random sampling.

4.1. MaximumLikelihood Estimationunder JSRS

Consider a joint simple random sample of size $n = mk$ where $m = m_1 + m_2$ is the size of a set and k is number of cycles. Thus, we have two independent random samples of sizes $n_1 = m_1 k$ and $n_2 = m_2 k$ respectively from, product type A (X- Sample) and product type B (Y-sample).
The likelihood function under JSRS becomes

$$L = \prod_{i=1}^{n_1} f(x_i;\theta_1) \prod_{i=1}^{n_2} f(y_i;\theta_2)$$

Using (1) and (2) it reduces to

$$L = \frac{1}{\theta_1^{n_1}} e^{-\frac{\sum_{i=1}^{n_1} x_i}{\theta_1}} \frac{1}{\theta_2^{n_2}} e^{-\frac{\sum_{i=1}^{n_2} y_i}{\theta_2}} . \qquad (38)$$

Hence the log likelihood equation will be

$$l = logL = -n_1 \log(\theta_1) - \frac{\sum_{i=1}^{n_1} x_i}{\theta_1} - n_2 \log(\theta_2) - \frac{\sum_{i=1}^{n_2} y_i}{\theta_2}$$

The maximum likelihood estimates of the parameters can be easily derived as

$$\hat{\theta}_{1SRS} = \frac{\sum_{i=1}^{n_1} x_i}{n_1}, \qquad (39)$$

and

$$\hat{\theta}_{2SRS} = \frac{\sum_{i=1}^{n_2} y_i}{n_2}. \qquad (40)$$

The asymptotic variances of the estimators are given by

$$V\left(\hat{\theta}_{1SRS}\right) = -\frac{1}{\left.\frac{\partial^2 l}{\partial \theta_1^2}\right|_{\theta_1 = \hat{\theta}_{1SRS}}}, \qquad (41)$$

and

$$V\left(\hat{\theta}_{2SRS}\right) = -\frac{1}{\left.\frac{\partial^2 l}{\partial \theta_2^2}\right|_{\theta_2 = \hat{\theta}_{2SRS}}}. \qquad (42)$$

where

$$\frac{\partial^2 l}{\partial \theta_1^2} = \frac{n_1}{\theta_1^2} - \frac{2\sum_{i=1}^{n_1} x_i}{\theta_1^3}. \qquad (43)$$

and

$$\frac{\partial^2 l}{\partial \theta_2^2} = \frac{n_2}{\theta_2^2} - \frac{2\sum_{i=1}^{n_1} y_i}{\theta_2^3}. \qquad (44)$$

4.2. Bayes Estimation

Considering the same inverted gamma priors used in (13) for the parameters of the exponential distributions and the likelihood function in (38),the joint posterior distribution of $\theta_1$and θ_2 can be constructed as

$$h_1\left(\theta_1, \theta_2 | \underline{x}, \underline{y}\right) \propto L\, \pi_1(\theta_1)\pi_2(\theta_2) \propto \frac{1}{\theta_1^{n_1}} e^{-\frac{\sum_{i=1}^{n_1} x_i}{\theta_1}} \frac{{\beta_1}^{\alpha_1}}{\Gamma\alpha_1} {\theta_1}^{-(\alpha_1+1)} e^{-\frac{\beta_1}{\theta_1}} \frac{1}{\theta_2^{n_2}} e^{-\frac{\sum_{i=1}^{n_2} y_i}{\theta_2}} \frac{{\beta_2}^{\alpha_2}}{\Gamma\alpha_2} {\theta_2}^{-(\alpha_2+1)} e^{-\frac{\beta_2}{\theta_2}}$$
$$= \frac{{\beta_1}^{\alpha_1}}{\Gamma\alpha_1} {\theta_1}^{-(n_1+\alpha_1+1)} e^{-\frac{\left(\sum_{i=1}^{n_1} x_i + \beta_1\right)}{\theta_1}} \frac{{\beta_2}^{\alpha_2}}{\Gamma\alpha_2} {\theta_2}^{-(n_2+\alpha_2+1)} e^{-\frac{\left(\sum_{i=1}^{n_2} y_i + \beta_2\right)}{\theta_2}} = h_1(\theta_1|\underline{x}) h_2\left(\theta_2|\underline{y}\right)$$

which is the product of two marginal posterior distributions of θ_1 and θ_2 where,

$h_1(\theta_1|\underline{x}) = igamma(n_1 + \alpha_1, \sum_{i=1}^{n_1} x_i + \beta_1)$,

(45)

and

$h_2\left(\theta_2|\underline{y}\right) = igamma(n_2 + \alpha_2, \sum_{i=1}^{n_2} y_i + \beta_2)$.

(46)

HenceBayes estimates of parameters$\theta_1$and $\theta_2$under SELF,are respectively given by the mean of the inverted gamma distributions in (45) and (46) as

$\hat{\theta}_{1B} = \frac{(\sum_{i=1}^{n_1} x_i + \beta_1)}{n_1 + \alpha_1 - 1}$,

(47)

and

$$\hat{\theta}_{2B} = \frac{(\sum_{i=1}^{n_2} y_i + \beta_2)}{n_2 + \alpha_2 - 1}. \quad (48)$$

5. NUMERICAL STUDY

In this section, a simulation study is conducted to examine the performances of proposed estimators obtained under JRSS and JMRSS schemes. A real example is presented to illustrate the methods described in the paper.

5.1. Simulation Study

We fix the values of the parameters $\theta_1 = 1.5$ and $\theta_2 = 3.5$. Two independent random samples of sizes (m_1, m_2) = (3, 4), (4, 4), and (4, 3) are generated from the exponential distributions with parameters $\theta_1$and θ_2 respectively. The experiment is repeated for $k = 4$, 5, 6 cycles (times). Based on the algorithm described in the earlier sections, joint RSS, joint MRSS and joint SRS are obtained and we compute the MLEs and Bayes estimates under the squared error loss function. The process is repeated for 1000 runs and average values of all the estimators and means squared errors (MSEs) are determined.
The values of hyper parameters of the prior distributions are determined so that the prior means and variances are equal to the values of means and variances obtained by MLE respectively for the parameters of the underlying distribution.
All of the computations are performed using the R language. We tabulate the results of the simulation study in Tables 1-10. In Table 1, we summarize the results based on different values of m_1, m_2 and k. The results under Bayes estimation are presented in Table 2 to Table 10 for different values of m_1, m_2, k and hyper parameters.
In all the tables the first, second and third entry denote estimate, bias and MSE respectively.

Table 1. Results for MLE

m_1	m_2	k	JRSS		JMRSS		JSRS	
			θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
3	4	4	1.533	3.5459	1.5446	3.8243	1.4891	3.4401
			0.033	0.0459	0.0446	0.3243	-0.0109	-0.0599
			0.0743	0.2281	2.8504	4.689	0.1765	0.6194
		5	1.4832	3.5397	1.5496	3.7529	1.4071	3.4312
			-0.017	0.0397	0.0496	0.2529	-0.0929	-0.0688
			0.0442	0.207	2.6739	3.8807	0.1776	0.586
		6	1.5058	3.488	1.528	3.6004	1.521	3.5012
			0.0058	-0.012	0.028	0.1004	0.021	0.0012
			0.0403	0.1445	2.2179	2.914	0.1061	0.576
4	3	4	1.5146	3.5172	1.5506	3.8733	1.4148	3.6382
			0.0146	0.0172	0.0506	0.3733	-0.0852	0.1382
			0.0485	0.2781	2.9988	4.8178	0.1389	1.0424
		5	1.4884	3.6659	1.4956	3.7889	1.4839	3.5893
			-0.012	0.1659	-0.0044	0.2889	-0.0161	0.0893

			0.0267	0.281	2.6894	5.7809	0.0886	0.8074
		6	1.482	3.5044	1.5524	3.9596	1.489	3.5426
			-0.018	0.0044	0.0524	0.4596	-0.011	0.0426
			0.0291	0.2364	3.0346	5.1307	0.1027	0.5575
4	4	4	1.4778	3.5743	1.5078	3.7468	1.4666	3.355
			-0.022	0.0743	0.0078	0.2468	-0.0334	-0.145
			0.0389	0.219	2.5929	3.5217	0.1543	0.6231
		5	1.5081	3.4981	1.4664	3.8776	1.5057	3.4951
			0.0081	-0.002	-0.0336	0.3776	0.0057	-0.0049
			0.025	0.1722	3.0641	5.0489	0.088	0.6951
		6	1.5045	3.4624	1.5089	3.57	1.4607	3.5896
			0.0045	-0.038	0.0089	0.07	-0.0393	0.0896
			0.0276	0.1498	2.2289	2.9091	0.1037	0.513

Table 2. Results for Bayes estimation for (m_1, m_2) = (3, 4) and$k$ = 4

α_1	β_1	α_2	β_2	JRSS		JMRSS		JSRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.6825	3.8551	1.7056	3.9229	1.7073	3.9136
				0.1825	0.3551	0.2056	0.4229	0.2073	0.4136
				0.0037	0.0278	0.0013	0.0075	0.0008	0.0068
200	300	160	550	1.5193	3.4954	1.5021	3.461	1.5044	3.4563
				0.0193	-0.0046	0.0021	-0.039	0.0044	-0.044
				0.0019	0.0178	0.0012	0.0047	0.0004	0.0052
225	300	180	550	1.3676	3.1721	1.3536	3.1038	1.3462	3.1093
				-0.132	-0.3279	-0.1464	-0.396	-0.1538	-0.391
				0.0014	0.0115	0.0007	0.003	0.0005	0.0051
200	250	160	500	1.3005	3.2448	1.277	3.1574	1.2669	3.1779
				-0.2	-0.2552	-0.223	-0.343	-0.2331	-0.322
				0.0017	0.0152	0.001	0.0038	0.0006	0.0082
200	300	160	550	1.5193	3.4954	1.5021	3.461	1.5092	3.4639
				0.0193	-0.0046	0.0021	-0.039	0.0092	-0.036
				0.0019	0.0178	0.0012	0.0047	0.0006	0.0053
200	350	160	600	1.7207	3.7463	1.7414	3.7538	1.7433	3.7529
				0.2207	0.2463	0.2414	0.2538	0.2433	0.2529
				0.0024	0.0241	0.0011	0.0047	0.0006	0.0049

Table 3. Results for Bayes estimation for (m_1, m_2) = (3, 4) and k = 5

α_1	β_1	α_2	β_2	JRSS		JMRSS		JSRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.6761	3.8891	1.6996	3.9213	1.7066	3.8911
				0.1761	0.3891	0.1996	0.4213	0.2066	0.3911
				0.0039	0.0213	0.0014	0.01	0.001	0.0109
200	300	160	550	1.5001	3.51	1.5107	3.4579	1.5087	3.4644
				0.0001	0.01	0.0107	-0.042	0.0087	-0.036
				0.0034	0.0245	0.0011	0.0053	0.0007	0.0056
225	300	180	550	1.3704	3.2102	1.3516	3.0963	1.3505	3.1128
				-0.13	-0.2898	-0.1484	-0.404	-0.1495	-0.387

				0.0015	0.0155	0.0008	0.004	0.0006	0.0061
200	250	160	500	1.3081	3.2968	1.2815	3.176	1.2702	3.203
				-0.192	-0.2032	-0.2185	-0.324	-0.2298	-0.297
				0.0023	0.0208	0.0011	0.0052	0.0007	0.0072
200	300	160	550	1.5001	3.51	1.5107	3.4579	1.5065	3.4541
				0.0001	0.01	0.0107	-0.042	0.0065	-0.046
				0.0034	0.0245	0.0011	0.0053	0.0008	0.0086
200	350	160	600	1.7103	3.7661	1.7295	3.7566	1.7351	3.7371
				0.2103	0.2661	0.2295	0.2566	0.2351	0.2371
				0.0026	0.0278	0.0011	0.0054	0.0007	0.0086

Table 4. Results for Bayes estimation for (m_1, m_2) = (3, 4) and k = 6

α_1	β_1	α_2	β_2	JRSS		JMRSS		JSRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.6765	3.9441	1.6894	3.9303	1.7033	3.8953
				0.1765	0.4441	0.1894	0.4303	0.2033	0.3953
				0.0042	0.0485	0.0017	0.0079	0.0013	0.0102
200	300	160	550	1.52	3.5614	1.5074	3.4542	1.5041	3.4599
				0.02	0.0614	0.0074	-0.0458	0.0041	-0.0401
				0.0034	0.0305	0.0012	0.0066	0.0009	0.0064
225	300	180	550	1.38	3.2467	1.3559	3.1145	1.3493	3.1184
				-0.12	-0.2533	-0.144	-0.3855	-0.151	-0.3816
				0.0019	0.0214	0.0009	0.0054	0.0009	0.0076
200	250	160	500	1.3249	3.3529	1.2807	3.1801	1.276	3.203
				-0.175	-0.1471	-0.219	-0.3199	-0.224	-0.297
				0.0031	0.0273	0.0009	0.0065	0.0007	0.0082
200	300	160	550	1.52	3.5614	1.5074	3.4542	1.5008	3.465
				0.02	0.0614	0.0074	-0.0458	0.0008	-0.035
				0.0034	0.0305	0.0012	0.0066	0.0012	0.0076
200	350	160	600	1.7124	3.7752	1.7326	3.748	1.7439	3.743
				0.2124	0.2752	0.2326	0.248	0.2439	0.243
				0.0028	0.0358	0.0014	0.0079	0.0009	0.0073

Table 5. Results for Bayes estimation for(m_1, m_2) = (4, 3) and k = 4

α_1	β_1	α_2	β_2	JRSS		JMRSS		JSRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.6749	3.898	1.7073	3.9316	1.706	3.9257
				0.1749	0.398	0.2073	0.4316	0.206	0.4257
				0.0033	0.0216	0.0014	0.0052	0.0011	0.0084
200	300	160	550	1.5132	3.4864	1.5055	3.4575	1.5064	3.4682
				0.0132	-0.0136	0.0055	-0.0425	0.0064	-0.0318
				0.0029	0.0154	0.0011	0.0027	0.0007	0.0068
225	300	180	550	1.3754	3.1477	1.3495	3.0892	1.3526	3.0911
				-0.125	-0.3523	-0.151	-0.4108	-0.147	-0.4089
				0.0023	0.0073	0.0008	0.0024	0.0006	0.004
200	250	160	500	1.3142	3.2183	1.2797	3.156	1.2732	3.1749
				-0.186	-0.2817	-0.22	-0.344	-0.227	-0.3251

				0.0031	0.0149	0.0011	0.0024	0.0007	0.0057
200	300	160	550	1.5132	3.4864	1.5055	3.4575	1.5064	3.4682
				0.0132	-0.0136	0.0055	-0.0425	0.0064	-0.0318
				0.0029	0.0154	0.0011	0.0027	0.0007	0.0068
200	350	160	600	1.7241	3.7381	1.7263	3.7595	1.7435	3.764
				0.2241	0.2381	0.2263	0.2595	0.2435	0.264
				0.0031	0.0142	0.0014	0.0049	0.0009	0.0054

Table 6. Results for Bayes estimation for (m_1, m_2) = (4, 3) and k = 5

α_1	β_1	α_2	β_2	JRSS		JMRSS		JSRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.6856	3.8514	1.6997	3.9421	1.698	3.9075
				0.1856	0.3514	0.1997	0.4421	0.198	0.4075
				0.0046	0.0238	0.002	0.0093	0.0013	0.0077
200	300	160	550	1.5156	3.4858	1.5107	3.4644	1.5068	3.467
				0.0156	-0.014	0.0107	-0.036	0.0068	-0.033
				0.0039	0.0173	0.0013	0.0046	0.0007	0.0043
225	300	180	550	1.3796	3.1849	1.3571	3.0891	1.3565	3.1062
				-0.1204	-0.315	-0.143	-0.411	-0.1435	-0.3938
				0.0024	0.0133	0.0008	0.0037	0.0008	0.0065
200	250	160	500	1.3283	3.2261	1.2855	3.1641	1.277	3.1754
				-0.1717	-0.274	-0.215	-0.336	-0.223	-0.3246
				0.0033	0.015	0.0013	0.0032	0.0009	0.005
200	300	160	550	1.5156	3.4858	1.5107	3.4644	1.5068	3.467
				0.0156	-0.014	0.0107	-0.036	0.0068	-0.033
				0.0039	0.0173	0.0013	0.0046	0.0007	0.0043
200	350	160	600	1.7103	3.7467	1.7308	3.7629	1.7379	3.7424
				0.2103	0.2467	0.2308	0.2629	0.2379	0.2424
				0.0035	0.0169	0.0012	0.0048	0.0008	0.006

Table 7. Results for Bayes estimation for (m_1, m_2) = (4, 3) and k = 6

α_1	β_1	α_2	β_2	JRSS		JMRSS		JSRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
	300	140	550	1.6943	3.8876	1.6863	3.9266	1.7012	3.9016
				0.1943	0.3876	0.1863	0.4266	0.2012	0.4016
				0.0064	0.0219	0.0019	0.0075	0.0015	0.0071
200	300	160	550	1.5331	3.4921	1.5101	3.469	1.5068	3.4717
				0.0331	-0.008	0.0101	-0.031	0.0068	-0.0283
				0.0034	0.0226	0.0016	0.0044	0.0011	0.009
225	300	180	550	1.3964	3.1843	1.3615	3.0944	1.3582	3.1252
				-0.1036	-0.316	-0.139	-0.406	-0.1418	-0.3748
				0.0023	0.0176	0.0011	0.0032	0.0009	0.0059
200	250	160	500	1.3456	3.2628	1.2904	3.1677	1.2836	3.2097
				-0.1544	-0.237	-0.21	-0.332	-0.2164	-0.2903
				0.0025	0.0276	0.0013	0.0037	0.0013	0.008
200	300	160	550	1.5331	3.4921	1.5101	3.469	1.5068	3.4717
				0.0331	-0.008	0.0101	-0.031	0.0068	-0.0283

				0.0034	0.0226	0.0016	0.0044	0.0011	0.009
200	350	160	600	1.7282	3.7484	1.7221	3.7662	1.7367	3.7285
				0.2282	0.2484	0.2221	0.2662	0.2367	0.2285
				0.0049	0.0229	0.0018	0.0056	0.0011	0.0077

Table 8. Results for Bayes estimation for (m_1, m_2) = (4, 4) and k = 4

α_1	β_1	α_2	β_2	RSS		MRSS		SRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.6815	3.898	1.7006	3.9346	1.7034	3.8987
				0.1815	0.398	0.2006	0.4346	0.2034	0.3987
				0.0038	0.0313	0.0014	0.0066	0.001	0.009
200	300	160	550	1.5095	3.525	1.5078	3.4509	1.5084	3.4681
				0.0095	0.025	0.0078	-0.0491	0.0084	-0.0319
				0.0026	0.0258	0.0013	0.0046	0.0008	0.0064
225	300	180	550	1.3755	3.1831	1.3533	3.0836	1.3502	3.108
				-0.1245	-0.3169	-0.1467	-0.4164	-0.1498	-0.392
				0.0021	0.0191	0.0007	0.0036	0.0006	0.0064
200	250	160	500	1.3213	3.2459	1.2798	3.1812	1.2723	3.1747
				-0.1787	-0.2541	-0.2202	-0.3188	-0.2277	-0.3253
				0.0029	0.0184	0.0009	0.005	0.0007	0.0074
200	300	160	550	1.5095	3.525	1.5078	3.4509	1.5084	3.4681
				0.0095	0.025	0.0078	-0.0491	0.0084	-0.0319
				0.0026	0.0258	0.0013	0.0046	0.0008	0.0064
200	350	160	600	1.7142	3.7525	1.729	3.77	1.7345	3.7575
				0.2142	0.2525	0.0014	0.0046	0.2345	0.2575
				0.0035	0.0208	0.229	0.27	0.0006	0.0077

Table 9. Results for Bayes estimation for(m_1, m_2) = (4, 4) and k = 5

α_1	β_1	α_2	β_2	RSS		MRSS		SRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.6984	3.9542	1.6915	3.934	1.7031	3.9065
				0.1984	0.4542	0.1915	0.434	0.2031	0.4065
				0.0068	0.0315	0.0017	0.0091	0.0012	0.0085
200	300	160	550	1.5283	3.5575	1.5076	3.464	1.4998	3.4702
				0.0283	0.0575	0.0076	-0.036	-0.0002	-0.0298
				0.0033	0.0232	0.001	0.005	0.0011	0.0087
225	300	180	550	1.3911	3.2498	1.3547	3.0965	1.3514	3.1243
				-0.1089	-0.2502	-0.1453	-0.4035	-0.1486	-0.3757
				0.004	0.0132	0.001	0.0034	0.0008	0.0056
200	250	160	500	1.3387	3.2995	1.2901	3.173	1.2788	3.1877
				0.0031	0.0245	-0.2099	-0.327	-0.2212	-0.3123
				-0.1613	-0.2005	0.0013	0.0046	0.0008	0.0075
200	300	160	550	1.5283	3.5575	1.5076	3.464	1.4998	3.4702
				0.0283	0.0575	0.0076	-0.036	-0.0002	-0.0298
				0.0033	0.0232	0.001	0.005	0.0011	0.0087
200	350	160	600	1.7183	3.7822	1.731	3.7451	1.7398	3.7441
				0.2183	0.2822	0.231	0.2451	0.2398	0.2441

0.0049	0.0242	0.0014	0.0073	0.0009	0.0058

Table 10. Results for Bayes estimation for (m_1, m_2) = (4, 3) and k = 6

α_1	β_1	α_2	β_2	RSS		MRSS		SRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
175	300	140	550	1.7089	3.9603	1.6853	3.9097	1.6983	3.8739
				0.2089	0.4603	0.1853	0.4097	0.1983	0.3739
				0.006	0.0394	0.0015	0.0123	0.0011	0.0119
200	300	160	550	1.5359	3.6025	1.5076	3.4626	1.5078	3.4715
				0.0359	0.1025	0.0076	-0.037	0.0078	-0.029
				0.005	0.0298	0.0018	0.0043	0.001	0.0098
225	300	180	550	1.3988	3.2872	1.3598	3.1032	1.36	3.1332
				-0.101	-0.213	-0.14	-0.397	-0.14	-0.367
				0.0026	0.0287	0.0013	0.004	0.0008	0.0062
200	250	160	500	1.3509	3.3997	1.2912	3.1792	1.2814	3.1825
				-0.149	-0.1	-0.209	-0.321	-0.219	-0.318
				0.0033	0.0327	0.0012	0.0046	0.0008	0.0078
200	300	160	550	1.5359	3.6025	1.5076	3.4626	1.5078	3.4715
				0.0359	0.1025	0.0076	-0.037	0.0078	-0.029
				0.005	0.0298	0.0018	0.0043	0.001	0.0098
200	350	160	600	1.74	3.8534	1.7319	3.7465	1.727	3.7345
				0.24	0.3534	0.2319	0.2465	0.227	0.2345
				0.0055	0.0312	0.0021	0.0084	0.001	0.0081

From these tables, it can be observed that that all of the estimates of the parameters have satisfactory performances. From the simulation results, it is seen that the Bayes estimates of $\theta_1$and $\theta_2$perform better than those obtained by using MLEs in terms of minimum MSEs for both the types of joint RSS as well as for joint SRS schemes. The MLEs obtained under joint RSS have smaller MSE than that of under joint MRSS and joint SRS. In case of Bayes estimation, JMRSS performs well compared to JRSS. We can observe that as the values of hyper parameters α_1 and α_2 increase, the MSE for $\theta_1$and θ_2 decrease in case of JRSS, JMRSS and JSRS. Increase in the values of α_1 and $\alpha_2$has a negative effect on bias of $\theta_1$and θ_2 but increase in the values of β_1 and $\beta_2$has an erratic effect on bias. Increase in number of cycles(k) does not have much effect on MSE or bias of $\theta_1$and θ_2.

5.2. Real Example

To exemplify the results obtained in the paper, we use the data presented by Proschan [18] on intervals between failures (in hours) of the air-conditioning system of a fleet of 13 Boeing 720 jet airplanes. He has shown that the failure time distribution of the air-conditioning system for each of the planes was well fitted by exponential distributions. For the purpose of illustration, we chose the planes "7913" and "7914," and the corresponding failure time data are presented in Table 11.This data set is also used by Rasouli and Balakrishnan[19]. We assume two exponential distributions with means θ_1 and θ_2 for the 24 and 27 observations in the data of planes "7914" and "7913" respectively.

Table11. Failure times of air-conditioning systems in two airplanes

Plane	Failure times
7914	3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139,188, 197, 210
7913	1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106,111, 141, 142, 163, 191, 206, 216

Now, suppose that two independent random samples of sizesm_1= 3 andm_2 = 4 are generated from the data set of planes "7914" and "7913" respectively. The experiment is repeated for k = 6 cycles (times). A joint rank set sample and joint modified rank set sample are obtained by combining both the samples observations using the algorithm described in the appropriate sections. The process is repeated for 1000 runs and average values of all the estimators and mean squared errors (MSEs) are calculated. The results in case of proposed estimators are shown in the Table 12 and Table 13 respectively.

Table 12. Results for MLE for (m_1, m_2) = (3, 4) and k = 6

	JRSS		JMRSS		JSRS	
	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
Estimate	62.2123	80.5885	63.9823	85.9069	64.0602	76.8497
Bias	-1.9127	3.7737	-0.1427	9.0921	-0.0648	0.0349
MSE	3.9931	14.5941	1.2331	86.0181	0.7231	0.2243

Table 13. Results for Bayes estimation for (m_1, m_2) = (3, 4) and k = 6

α_1	β_1	α_2	β_2	JRSS		JMRSS		JSRS	
				θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
5710	347443	26478	1933022	60.8559	73.0352	60.8534	73.0220	60.8661	73.0116
				-3.2691	-3.7796	-3.2716	-3.7928	-3.2589	-3.8032
				0.0197	0.0027	0.0084	0.0019	0.0006	0.0000
5710	365730	26478	2034760	64.0256	76.8489	64.0666	76.8507	64.0621	76.8502
				-0.0994	0.0341	-0.0584	0.0359	-0.0629	0.0354
				0.0153	0.0035	0.0106	0.0028	0.0005	0.0000
5710	384016	26478	2136498	67.2092	80.6872	67.2421	80.6964	67.2581	80.6894
				3.0842	3.8724	3.1171	3.8815	3.1331	3.8746
				0.0154	0.0034	0.0100	0.0023	0.0006	0.0000
5425	365730	25154	2034760	67.3677	80.9012	67.4082	80.9032	67.4162	80.8912
				3.2427	4.0864	3.2832	4.0883	3.2912	4.0764
				0.0163	0.0033	0.0106	0.0031	0.0007	0.0000
5996	365730	27802	2034760	61.0111	73.2095	60.9985	73.1975	61.0165	73.1934
				-3.1139	-3.6053	-3.1265	-3.6174	-3.1085	-3.6214
				0.0122	0.0024	0.0084	0.0021	0.0006	0.0000

In case of real data, from Table 12 and Table 13, It can be seen that we have similar conclusions as observed in a simulation study.

6. CONCLUSIONS

In this paper, joint RSS and joint MRSS are proposed for estimating the unknown parameters of two the exponential distributions. Maximum likelihood estimation and Bayes estimation under squared error loss function has been considered. Performances of the proposed estimators have been compared with that of based on joint SRS in terms of their MSEs. The computational results show that the Bayesian estimation based on the squared error loss function performs better than the ML estimation. Performance of the Bayes estimators depend on the values of the hyper parameter α_1 and α_2, but not on the values of β_1 and β_2. Although we have provided the results mainly for squared error loss function in case of Bayes estimation, our method can be applied for asymmetric loss functions too, namely linear exponential loss function, general entropy loss function. More work is needed along this direction.

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