BAHADUR'S STOCHASTIC COMPARISON OF COMBINING INDEPENDENT TESTS IN CASE OF CONDITIONAL LOG-LOGISTIC DISTRIBUTION

Abedel-Qader S. Al-Masri[1]
Department of Statistics, Yarmouk University, Irbid-Jordan

ABSTRACT
Bahadur's stochastic comparison of asymptotic relative efficiency of combining infinitely independent tests in case of conditional log-logistic distribution is proposed. Six free-distribution combination producers namely; Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values were studied. Several comparisons among the six procedures using the exact Bahadur's slopes were obtained. Results showed that logistic producer is the best procedure.

KEYWORDS: Asymptotic relative efficiency, Conditional log-logistic distribution, Combining independent tests, Bahadur efficiency, Bahadur slope.

MSC: 62E05

ABSTRACT
Bahadur's stochastic comparison of asymptotic relative efficiency of combining infinitely independent tests in case of conditional log-logistic distribution is proposed. Six free-distribution combination producers namely; Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values were studied. Several comparisons among the six procedures using the exact Bahadur's slopes were obtained. Results showed that logistic producer is the best procedure.

1. INTRODUCTION

Bahadur's stochastic comparison is one of the most common approach in asymptotic relative efficiency for two test procedures in which the $Type\ I$ and $Type\ II$ error probabilities changes with increasing sample size, and the manner of the alternatives are behave.

In comparison of test procedures, let $H_0: F \in \mathcal{F}_0$ is to be tested, where $\mathcal{F}_0$ is a family of distributions, for any test procedure T_n. The function $\gamma_n(T, F) = P_F(T_n \text{ rejects } H_0)$, for distribution functions F, represents the power function of T_n. Under H_0, $\gamma_n(T, F)$ represents the probability of a $Type\ I$ error. The size of the test is $\alpha_n(T, \mathcal{F}_0) = \sup_{F \in \mathcal{F}_0} \gamma_n(T, F)$. For $F \notin \mathcal{F}_0$, the probability of a $Type\ II$ error is $\beta_n(T, F) = 1 - \gamma_n(T, F)$. We are interesting in studying consistent tests, that is for fixed $F \notin \mathcal{F}_0$, $\beta_n(T, F) \to 0$ as $n \to \infty$, and unbiased tests that is $F \notin \mathcal{F}_0$, $\gamma_n(T, F) \geq \alpha_n(T, \mathcal{F}_0)$. To compare two test procedures through their power functions, we will use the asymptotic relative efficiency (ARE) for two test procedures T_A and T_B, with sample sizes n_1 and n_2 respectively, then the ratio n_1/n_2 goes to some limit. This limit is the ARE of T_B relative to T_A. In Bahadur approach, the following behaviors are satisfied: the $Type\ I$ error is $\alpha_n \to 0$, the $Type\ II$ error is $\beta_n \to 0$, and the alternatives is $F^n = F$ fixed. Let T_i be independent one tailed test statistic for testing $H_{i,0}: \theta_i = \theta_{i,0}$ for the independent real parameter θ_i versus $H_{i,0}: \theta_i > \theta_{i,0}$, $i = 1, \dots, k$, where the null hypothesis is rejected for large values of T_i. It is desired to combine be used to test the combined hypothesis $H_0: \theta_i = \theta_{i,0}, i = 1, \dots, k$, versus the alternative $\theta_i \geq \theta_{i,0}, i = 1, \dots, k$, with strict inequality at least one. Asymptotic relative efficiency have been considered by many authors. Kallenberg [10] showed that in testing problems in multivariate exponential families the LR test is deficient in the sense of Bahadur of order $O(log\ n)$. Abu-Dayyeh and El-Masri [2] studied six free-distribution methods (sum of p-values, inverse normal, logistic, Fisher, minimum of p-values and maximum of p-values) of combining infinitely number of independent tests when the p-values are IID rv's distributed with uniform distribution under the null hypothesis versus triangular distribution with essential support $(0,1)$ under the alternative hypothesis. They proved that the sum of p-values method is the best method. Abu-Dayyeh, *et al.* [1] they combined

[1] Corresponding authors: almasri68@yu.edu.jo

infinity number of independent tests for testing simple hypotheses against one-sided alternative for normal and logistic distributions, they used four methods of combining (Fisher, logistic, sum of p-values and inverse normal). Al-Masri [5] studied six methods of combining independent tests. He showed under conditional shifted Exponential distribution that the inverse normal method is the best among six combination methods. AL-Talib, *et al.* [8] considered combining independent tests in case of conditional normal distribution with probability density function $X|\theta \sim N(\gamma\theta, 1)$, $\theta \in [a, \infty], a \geq 0$ when $\theta_1, \theta_2, \ldots$ have a distribution function (DF) F_θ. They concluded that the inverse normal procedure is the best procedure. Al-Masri [6] considered combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of Laplace distribution $\mathbb{L}(\gamma, 1)$. He showed that the sum of *p*-values procedure is better than all other procedures under the null hypothesis, and the inverse normal procedure is better than the other procedures under the alternative hypothesis. Al-Masri and Al-Momani [4] considered combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of log-logistic distribution. They showed that the sum of *p*-values procedure is better than all other procedures under the null hypothesis and under the alternative hypothesis. Al-Masri [7] considered the problem of combining n independent tests as $n \to \infty$ for testing a simple hypothesis in case of log-normal distribution. He showed that as $\xi \to 0$, the maximum of p-values is better than all other methods, followed in decreasing order by the inverse normal, logistic, the sum of p-values, Fisher and Tippett's procedure. Also, as $\xi \to \infty$ the worst method the sum of p-values and the other methods remain the same, since they have the same limit.
The log-logistic distribution ($LL(\vartheta, \sigma)$), which is known also as Fisk distribution in economics, is used to model income data. It is an important right skewed distribution. It is the distribution of the logarithm of a random variable that has logistic distribution, and can have a non-monotonic hazard function for some values of the shape parameter. Also, often used in decision making business, decision making with project management, audio dithering and serves as a parametric model for survival analysis. Let X be a random variable following the log-logistic distribution.
The distribution function (cdf) of X can take the following form

$$F(x; \vartheta) = \Psi_{Logistic}\left(\frac{\ln(x) - \vartheta}{\sigma}\right) I_{\mathbb{R}^+}(x) \tag{1.1}$$

The probability density function (pdf) of X is given by

$$f(x; \vartheta) = \frac{1}{\sigma x} \varphi_{Logistic}\left(\frac{\ln(x) - \vartheta}{\sigma}\right) I_{\mathbb{R}^+}(x), \vartheta \in \mathbb{R}, \sigma \in \mathbb{R}^+. \tag{1.2}$$

Where I_A is the indicator function and $\Psi_{Logistic}(h) = (1 + e^{-h})^{-1}$ is the cdf of the standard logistic distribution with $L(0,1)$. The mean of X is $E(X) = \sigma\, \pi \sin(\pi\sigma)$; if $\sigma > 1$, else undefined. For more details see Ahsanullah and Alzaatreh [3].

2. THE BASIC PROBLEM

Consider testing the hypothesis

$$H_0^{(i)}: \eta_i = \eta_0^i, vs\ , H_1^{(i)}: \eta_i \in \Omega_i - \{\eta_0^i\} \tag{2.1}$$

such that $H_0^{(i)}$ becomes rejected for large values of some real valued continuous random variable $T^{(i)}$, $i = 1,2, \ldots, n$. The n hypotheses are combined into one as

$$H_0^{(i)}: (\eta_1, \ldots, \eta_n) = (\eta_0^1, \ldots, \eta_0^n), vs\ , H_1^{(i)}: (\eta_1, \ldots, \eta_n) \in \{\textstyle\prod_{i=1}^n \Omega_i - \{(\eta_0^1, \ldots, \eta_0^n)\}\} \tag{2.2}$$

For $i = 1,2, \ldots, n$ the p-value of the i-th test is given by

$$P_i(t) = P_{H_0^{(i)}}\left(T^{(i)} > t\right) = 1 - F_{H_0^{(i)}}(t) \tag{2.3}$$

where $F_{H_0^{(i)}}(t)$ is the DF of $T^{(i)}$ under $H_0^{(i)}$. Note that $P_i \sim U(0,1)$ under $H_0^{(i)}$.

In this paper, we will consider the special case where: $\eta_i = \vartheta\Lambda_i$, $i = 1, \ldots, n$. Then our proposed model will be $W|\Lambda \sim LL(\Lambda\vartheta, 1)$, $\Lambda \in \Re\backslash(-\infty, \kappa), \kappa \geq 0$ where $\Lambda_1, \Lambda_2, \ldots$ are independent identically distributed with DF H_Λ with support defined on $\Lambda \in \Re\backslash(-\infty, \kappa), \kappa \geq 0$, assuming that $T^{(1)}, \ldots, T^{(n)}$ are independent,then (2.1) reduces to

$$H_0: \vartheta = 0 \quad vs \quad H_1: \vartheta > 0, \tag{2.4}$$

It follows that the p-values $P_1, \ldots, P_n$ are also iid rv's that have a $U(0,1)$ distribution under H_0, and under H_1 have a distribution whose support is a subset of the interval $(0,1)$ and is not a $U(0,1)$ distribution. Therefore, if f is the probability density function (pdf) of P, then (2.4) is equivalent to

$$H_0: P \sim U(0,1), vs\ , H_1: P \nsim U(0,1) \tag{2.5}$$

where P has a pdf f with support subset of the interval $(0,1)$.
By sufficiency we may assume $n_i = 1$ and $T^{(i)} = X_i$ for $i = 1, \dots, n$. Then we consider the sequence $\{T^{(n)}\}$ of independent test statistics, thus is we will take a random sample $X_1, \dots, X_n$ of size n and let $n \to \infty$ and compare the six non-parametric methods via exact Bahadur slope (EBS).
The producers that we will used in this paper are Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values. These producers are based on p-values of the individual statistics T_i, and reject H_0 if

$$\Psi_{Fisher} = -2\sum_{i=1}^{n} \ln(P_i) > \chi^2_{2n,\alpha},$$

$$\Psi_{logistic} = -\sum_{i=1}^{n} \ln\left(\frac{P_i}{1-P_i}\right) > b_\alpha,$$

$$\Psi_{Normal} = -\sum_{i=1}^{n} \Phi^{-1}(P_i) > \sqrt{n}\Phi^{-1}(1-\alpha),$$

$$\Psi_{Sum} = -\sum_{i=1}^{n} P_i > C_\alpha,$$

$$\Psi_{Max} = -max\, P_i < \alpha^{\frac{1}{n}}, \Psi_T = -min\, P_i < 1-(1-\alpha)^{\frac{1}{n}}.$$

where Φ is the DF of standard normal distribution.

3. DEFINITIONS

This section lays out some basic tools to Bahadur's stochastic comparison theory that used in this article
Definition *(Bahadur efficiency and exact Bahadur slope (EBS))* Let $X_1, \dots, X_n$ be i.i.d. from a distribution with a probability density function $f(x,\theta)$, and we want to test $H_0: \theta = \theta_0$ vs. $H_1: \theta \in \Theta - \{\theta_0\}$. Let $\left\{T_n^{(1)}\right\}$ and $\left\{T_n^{(2)}\right\}$ be two sequences of test statistics for testing H_0. Let the significance attained by $T_n^{(i)}$ be $L_n^{(i)} = 1 - F_i\left(T_n^{(i)}\right)$, where $F_i\left(T_n^{(i)}\right) = P_{H_0}\left(T_n^{(i)} \le t_i\right)$, $i = 1,2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^{(i)}\}$ such that

$$C_i(\theta) = \lim_{\theta \to \infty} -2n^{-1}\ln(L_n^i)$$

with probability 1 (w.p.1) under θ and the Bahadur efficiency of $\left\{T_n^{(1)}\right\}$ relative to $\left\{T_n^{(2)}\right\}$ is given by $e_B(T_1, T_2) = C_1(\theta)/C_2(\theta)$. Serfling [11]
Theorem 1 *(Large deviation theorem) Let $X_1, X_2, \dots, X_n$ be IID, with distribution F and put $S_n = \sum_{i=1}^{n} X_i$. Assume existence of the moment generating function (mgf) $M(z) = E_F(e^{zX})$, z real, and put $m(t) = inf_z e^{-z(X-t)} = inf_z e^{-zt}M(z)$. The behavior of large deviation probabilities $P(S_n \ge t_n)$, where $t_n \to \infty$ at rates slower than $O(n)$. The case $t_n = tn$, if $-\infty < t \le EY$, then $P(S_n \le nt) \le [m(t)]^n$, the*

$$-2n^{-1}\ln P_F(S_n \ge nt) \to -2\ln m(t) \quad \text{a.s.} \quad (F_\theta).$$

Serfling [11]
Theorem 2 *(Bahadur theorem) Let $\{T_n\}$ be a sequence of test statistics which satisfies the following:*

1. Under $H_1: \theta \in \Theta - \{\theta_0\}$:

$$n^{-\frac{1}{2}}T_n \to b(\theta) \quad \text{a.s.} \quad (F_\theta),$$

where $b(\theta) \in \Re$.

2. There exists an open interval I containing $\{b(\theta): \theta \in \Theta - \{\theta_0\}\}$, and a function g continuous on I, such that

$$\lim_n -2n^{-1}\log\sup_{\theta\in\Theta_0}\left[1 - F_{\theta_n}(n^{\frac{1}{2}}t)\right] = \lim_n -2n^{-1}\log\left[1 - F_{\theta_n}(n^{\frac{1}{2}}t)\right] = g(t), \quad t \in I.$$

If $\{T_n\}$ satisfied (1)-(2), then for $\theta \in \Theta - \{\theta_0\}$

$$-2n^{-1}\log\sup_{\theta\in\Theta_0}\left[1 - F_{\theta_n}(T_n)\right] \to C(\theta) \quad \text{a.s.} \quad (F_\theta).$$

Bahadur [9]
Theorem 3 *Let $X_1, \dots, X_n$ be i.i.d. with probability density function $f(x,\theta)$, and we want to test $H_0: \theta = 0$*

vs. $H_1: \theta > 0$. *For* $j = 1,2$, *let* $T_{n,j} = \sum_{i=1}^{n} f_i(x_i)/\sqrt{n}$ *be a sequence of statistics such that* H_0 *will be rejected for large values of* $T_{n,j}$ *and let* φ_j *be the test based on* $T_{n,j}$. *Assume* $\mathbb{E}_\theta(f_i(x)) > 0, \forall\theta \in \Theta$, $\mathbb{E}_0(f_i(x)) = 0$, $Var(f_i(x)) > 0$ *for* $j = 1,2$. *Then*
1. If the derivative $b'_j(0)$ is finite for $j = 1,2$, then

$$\lim_{\theta\to 0}\frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))}\left[\frac{b'_1(0)}{b'_2(0)}\right]^2,$$

where $b_i(\theta) = \mathbb{E}_\theta(f_j(x))$, and $C_j(\theta)$ is the EBS of test φ_j at θ.
2. If the derivative $b'_j(0)$ is infinite for $j = 1,2$, then

$$\lim_{\theta\to 0}\frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))}\left[\lim_{\theta\to 0}\frac{b'_1(\theta)}{b'_2(\theta)}\right]^2.$$

Al-Masri [5]
Theorem 4 *If* $T_n^{(1)}$ *and* $T_n^{(2)}$ *are two test statistics for testing* $H_0: \theta = 0$ *vs.* $H_1: \theta > 0$ *with distribution functions* $F_0^{(1)}$ *and* $F_0^{(2)}$ *under* H_0, *respectively, and that* $T_n^{(1)}$ *is at least as powerful as* $T_n^{(2)}$ *at* θ *for any* α, *then if* φ_j *is the test based on* $T_n^{(j)}$, $j = 1,2$, *then*

$$C_{\varphi_1}^{(1)}(\theta) \geq C_{\varphi_2}^{(2)}(\theta).$$

Serfling [11]
Corollary 1 *If* T_n *is the uniformly most powerful test for all* α, *then it is the best via EBS.* See [10]
Theorem 5

$$2t \leq m_S(t) \leq et,\ \forall: 0 \leq t \leq 0.5,$$

where

$$m_S(t) = \inf_{z>0} e^{-zt}\,\frac{e^z - 1}{z}.$$

Al-Masri [5]
Theorem 6

1. $m_L(t) \geq 2te^{-t}$, $\forall t \geq 0$,
2. $m_L(t) \leq te^{1-t}$, $\forall t \geq 0.852$,
3. $m_L(t) \leq t\left(\frac{t^2}{1+t^2}\right)^3 e^{1-t}$, $\forall t \geq 4$,

where $m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\, csc(\pi z)$ and csc is an abbreviation for cosecant function.
Al-Masri [5]
Theorem 7 *For* $x > 0$,

$$\phi(x)\left[\frac{1}{x} - \frac{1}{x^3}\right] \leq 1 - \Phi(x) \leq \frac{\phi(x)}{x}.$$

Where ϕ is the pdf of standard normal distribution. See [5]
Theorem 8 *For* $x > 0$,

$$1 - \Phi(x) > \frac{\phi(x)}{x + \sqrt{\frac{\pi}{2}}}.$$

Al-Masri [5]
Lemma 1

1. $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$
2. $m_L(t) \leq \dfrac{e^{-t^2/(t+1)}\left(\frac{\pi t}{t+1}\right)}{sin\left(\frac{\pi t}{t+1}\right)}$
3. $\begin{cases} m_s(t) = \inf\limits_{z>0}\dfrac{e^{-zt}(1-e^{-z})}{z} \leq \inf\limits_{z>0}\dfrac{e^{-zt}}{z} \leq -et, & t < 0 \\ m_s(t) \geq -2t, & -\frac{1}{2} \leq t \leq 0. \end{cases}$
4. $\frac{x-1}{x} \leq \ln x \leq x - 1$, $x > 0$

Al-Talib, *et al.* [8]

4. DERIVATION OF THE EBS WITH GENERAL DF H_Λ

In this section we will study testing problem (2.4). We will compare the six methods Fisher, logistic, sum of p-values,the inverse normal, Tippett's method and maximum of p-values using EBS.

Let $X_1, \dots, X_n$ be IID with probability density function (1.2) and we want to test (2.4). Then by (1.1), the p-value is given by

$$P_n(X_n) = 1 - F^{H_0}(X_n) = 1 - \Psi_{Logistic}(\ln(x_n)). \tag{4.1}$$

The next three lemmas give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), and sum of p-values (C_S), Tippett's method (C_T) and maximum of p-values (C_{max})methods.

Lemma 2 *The exact Bahadur's slope (EBS's) result for the tests, which is given at the end of Section 3, are as follows:*

1. Fisher method. $C_F(\vartheta) = b_F(\vartheta) - 2\ln(b_F(\vartheta)) + 2\ln(2) - 2$,

where

$$b_F(\vartheta) = 2\vartheta \mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{1-e^{-\Lambda\vartheta}}\right).$$

2. Logistic method. $C_L(\vartheta) = -2\ln(m(b_L(\vartheta)))$, where

$$m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\, csc(\pi z)$$

and

$$b_L(\vartheta) = \vartheta \mathbb{E}_{H_\Lambda}(\Lambda).$$

3. Sum of p-values method. $C_S(\vartheta) = -2\ln(m(b_S(\vartheta)))$, where

$$m_S(t) = \inf_{z>0} e^{-zt}\frac{1-e^{-z}}{z}$$

and

$$b_S(\vartheta) = \mathbb{E}_{H_\Lambda}\left(\frac{e^{\Lambda\vartheta} - \Lambda\vartheta e^{\Lambda\vartheta} - 1}{\left(e^{\Lambda\vartheta}-1\right)^2}\right).$$

4. Inverse Normal method. $C_N(\vartheta) = -2\ln(m(b_N(\vartheta))) = b_N^2(\vartheta)$,

where

$$b_N(\vartheta) = -\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}\mathbb{E}_{N(0,1)}\left\{\frac{v}{M_{Binomial(2,\Phi(v))}(\vartheta)}\right\}\right)$$

Proof of B1. For Fisher procedure,

$$T_F = -2\sum_{i=1}^{n} \frac{\ln\left[1-\Psi_{Logistic}(\ln(x))\right]}{\sqrt{n}}.$$

By Theorem 2 (1) and by the strong law of large number (SLLN), we have

$$\frac{T_F}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_F(\vartheta) = -2\mathbb{E}^{H_1}\ln\left[1-\Psi_{Logistic}(\ln(x))\right]$$

then

$$b_F(\vartheta) = -2\mathbb{E}_{H_\Lambda}\mathbb{E}_{LL(\Lambda\vartheta,1)}\left(\ln\left[1-\Psi_{Logistic}(\ln(x))\right]|\Lambda\right).$$

Now, $b_F(\vartheta) = -2\mathbb{E}_{H_\Lambda}\int_0^\infty \frac{1}{x}\ln\left[1-\Psi_{Logistic}(\ln(x))\right]\varphi_{Logistic}(\ln(x) - \Lambda\vartheta)\, dx = 2\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda\vartheta}{1-e^{-\Lambda\vartheta}}\right)$.

Now under H_0, and by Theorem (1), it follows that $M_F(z) = \mathbb{E}_F\left(e^{-2\ln(x)z}\right) = \int_0^1 e^{-2\ln(x)z}\, dx$. Set $t = -\ln(x)$ implies $dt = -e^t dx$. It then follows that $M_F(z) = \int_0^1 e^{-x(1-2t)}\, dx = (1-2z)^{-1}$, $Z < 1/2$. Then, $m_F(t) = \inf_{z>0} e^{-zt}(1-2z)^{-1} = \frac{t}{2}e^{1-t/2}$, now by Bahadur's Theorem (2), we complete the proof, that is

$$C_F(\vartheta) = -2\ln(m_F(b_F(\vartheta))) = -2\ln\left(\frac{b_F(\vartheta)}{2}e^{1-\frac{b_F(\vartheta)}{2}}\right) = b_F(\vartheta) - 2\ln(b_F(\vartheta)) + 2\ln(2) - 2.$$

Proof of B2. Similar to the previous proof.

Proof of B3. For sum of p-values procedure,

$$T_S = -\sum_{i=1}^{n} \frac{\left[1-\Psi_{Logistic}(\ln(x))\right]}{\sqrt{n}}.$$

It follows from Theorem 2 (1) and by the strong law of large number (SLLN) that

$$\frac{T_S}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_S(\theta) = -\mathbb{E}^{H_1}\left(1-\Psi_{Logistic}(\ln(x))\right)$$

then

$$b_S(\vartheta) = -\mathbb{E}_{H_\Lambda}\mathbb{E}_{EV(\Lambda\vartheta,1)}\left\{\left(1-\Psi_{Logistic}(\ln(x))\right)|\Lambda\right\} = \mathbb{E}_{H_\Lambda}\left(\frac{e^{\Lambda\vartheta} - \Lambda\vartheta e^{\Lambda\vartheta} - 1}{\left(e^{\Lambda\vartheta}-1\right)^2}\right).$$

Now, by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt}M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$.

Under H_0: $-\left(1-\Psi_{Logistic}(\ln(x))\right) \sim U(-1,0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 2 we complete the proof, we conclude that $C_S(\vartheta) = -2\ln(m_S(b_S(\vartheta)))$.

For the inverse normal procedure, $T_N = -\sum_{i=1}^{n} \frac{\Phi^{-1}\left(1-\Psi_{Logistic}(\ln(x))\right)}{\sqrt{n}}$. By Theorem 2 (1) and the strong law of large number (SLLN), we have $n^{-\frac{1}{2}}T_N \xrightarrow{\text{w.p.1}} b_N(\vartheta) = -\mathbb{E}^{H_1}\Phi^{-1}\left(1-\Psi_{Logistic}(\ln(x))\right)$, $b_N(\vartheta) = -\mathbb{E}_{H_\Lambda}\mathbb{E}_{EV(\Lambda\vartheta,1)}\left\{\Phi^{-1}\left(1-\Psi_{Logistic}(\ln(x))\right)|\Lambda\right\}$, it follows that $b_N(\vartheta) = -\mathbb{E}_{H_\Lambda}\int_0^\infty \frac{1}{x}\Phi^{-1}(1-\Psi_{Logistic}(\ln(x)))\varphi_{Logistic}(\ln(x)-\Lambda\vartheta)\,dx = -\mathbb{E}_{H_\Lambda}\int_0^\infty \Phi^{-1}\left(\frac{1}{1+x}\right)\frac{e^{\Lambda\vartheta}}{\left(x+e^{\Lambda\vartheta}\right)^2}\,dx$. On substituting $v = \Phi^{-1}\left(\frac{1}{1+x}\right)$, implies $\frac{1}{1+x} = \Phi(v)$, $\frac{dx}{dv} = \frac{\phi(v)}{\Phi^2(v)}$. It follows that $b_N(\vartheta) = \mathbb{E}_{H_\Lambda}\int_{\Re}\frac{-e^{\Lambda\vartheta}v\phi(v)}{\left(1-\Phi(v)+\Phi(v)e^{\Lambda\vartheta}\right)^2}\,dv = -\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}\mathbb{E}_{N(0,1)}\left\{\frac{v}{M_{Binomial(2,\Phi(v))}(\vartheta)}\right\}\right)$. Now under H_0, and by Theorem (1), it follows that $M_N(z) = \mathbb{E}_N\left(e^{-z\Phi^{-1}(X)}\right) = \int_0^1 e^{-z\Phi^{-1}(X)}\,dx$. Set $w = -\Phi^{-1}(x)$ implies $x = 1-\Phi(w)$, then $dx = -\phi(w)dw$. It then follows that $M_N(z) = \int_{\Re} e^{wz}\phi(w)\,dw = M_{N(0,1)}(z) = e^{z^2/2}$. Then, $m_N(t) = \inf_{z>0} e^{-zt}e^{z^2/2} = e^{-t^2/2}$, now by Bahadur's Theorem (2) (2), we complete the proof, that is $C_N(\vartheta) = -2\ln(m_N(b_N(\vartheta))) = -2\ln(e^{-b_N^2(\vartheta)/2}) = b_N^2(\vartheta) = \left[\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}\mathbb{E}_{N(0,1)}\left\{\frac{v}{M_{Binomial(2,\Phi(v))}(\vartheta)}\right\}\right)\right]^2$.

Theorem 9 *Let $U_1, U_2, \ldots$ be i.i.d. like U with probability density function f and suppose that we want to test $H_0\colon U_i \sim U(0,1)$ vs $H_1\colon U_i \sim f$ on $(0,1)$ but not $U(0,1)$. Then $C_{max}(f) = -2ln\left(ess.sup_f(u)\right)$* where $ess.sup_f(u) = sup\{u\colon f(u) > 0\}$ w.p.1 under f. Al-Masri [5]

Lemma 3

$$C_{max}(\vartheta) = 0.$$

Proof. Assume that $\frac{d}{d\Lambda}H_\Lambda = g_\Lambda$ the probability density function of the DF H_Λ, then the joint probability density function of X and Λ is

$$h(x,\Lambda) = f(x|\Lambda)g_\Lambda$$
$$h(x,\Lambda) = \frac{1}{x}\varphi_{Logistic}(\ln(x)-\Lambda\vartheta)I_{\mathbb{R}^+}(x).$$

The marginal probability density function of X is

$$f(x) = \frac{1}{x}\int_{(\kappa,\infty)} \varphi_{Logistic}(\ln(x)-\Lambda\vartheta)\,d\Lambda, x \in \Re^+, \kappa \geq 0.$$

Now, under ϑ the p-value $P = 1-\Psi_{Logistic}(\ln(x))$, so

$$h(p) = (p-1)^2\int_{(\kappa,\infty)}\frac{e^{\Lambda\vartheta}}{\left(p-(p-1)e^{\Lambda\vartheta}\right)^2}\,d\Lambda, \ \ p \in (0,1). \quad (4.2)$$

Then by Theorem 9 we have $ess.sup_f(p) = 1$. Therefore, $C_{max}(\vartheta) = 0$.

Theorem 10 *If $\pi(ln\pi)^2 f(\pi) \to 0$ as $\pi \to 0$, then $C_T(f) = 0$.* Al-Masri [5]

Lemma 4

$$C_T(\vartheta) = 0.$$

Proof. From (4.2) and by Theorem 10 we have get

$$\lim_{p\to 0} p(\ln p)^2 h(p) = -\lim_{p\to 0} p(\ln p)^2 (p-1)^2\int_{(\kappa,\infty)}\frac{e^{\Lambda\vartheta}}{\left(p-(p-1)e^{\Lambda\vartheta}\right)^2}\,d\Lambda.$$

Clearly, applying by L'Hopital rule twice we have, $\lim_{p\to 0} p(\ln p)^2 = 0$, also, $\lim_{p\to 0}(p-1)^2\int_{(\kappa,\infty)}\frac{e^{\Lambda\vartheta}}{\left(p-(p-1)e^{\Lambda\vartheta}\right)^2}\,d\Lambda = \mathbb{E}_{H_\Lambda}\left(e^{-\Lambda\vartheta}\right)$. Which implies $C_T(\vartheta) = 0$.

5. COMPARISON OF THE EBSS WHEN $\vartheta \to 0$

In this section, we will compare the EBSs that obtained in Section (4). We will find the limit of the ratio of the EBSs of any two methods when $\vartheta \to 0$.

Corollary 2 *The limits of ratios of different tests are as follows:*

C1. $\frac{C_T(\vartheta)}{C_{\mathfrak{D}}(\vartheta)} = \frac{C_{max}(\vartheta)}{C_{\mathfrak{D}}(\vartheta)} = 0$, where $C_{\mathfrak{D}}(\vartheta) \in \{C_F(\vartheta), C_L(\vartheta), C_S(\vartheta), C_N(\vartheta)\}$.

C2. $e_B(T_S, T_F) \to 1.3333$

C3. $e_B(T_L, T_F) \to 1.21585$

C4. $e_B(T_N, T_F) \to 1.27324$

C5. $e_B(T_L, T_N) \to 0.954926$
C6. $e_B(T_N, T_S) \to 0.95493$
C7. $e_B(T_L, T_S) \to 0.911888$
Proof of C2. By Lemma (2) (B1, B3) and Theorem (3)(1) it follows that $b'_F(\vartheta) = 2\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda e^{\Lambda\vartheta}(e^{\Lambda\vartheta}-\Lambda\vartheta-1)}{(e^{\Lambda\vartheta}-1)^2}\right)$. Now, by L'Hopitals rule, we have $\lim_{\vartheta\to 0} b'_F(\vartheta) = \mathbb{E}_{H_\Lambda}(\Lambda) < \infty$. Also $\lim_{\vartheta\to 0} b'_S(\vartheta) = \lim_{\vartheta\to 0}\mathbb{E}_{H_\Lambda}\left[\frac{\Lambda e^{\Lambda\vartheta}(2+e^{\Lambda\vartheta}(\Lambda\vartheta-2)+\Lambda\vartheta)}{(e^{\Lambda\vartheta}-1)^3}\right] = \frac{1}{6}\mathbb{E}_{H_\Lambda}(\Lambda) < \infty$. Now under $H_0: h_F(x) = -2\ln\left[1-\Psi_{Logistic}(\ln(x))\right] \sim \chi_2^2$ and $h_S(x) = -\left(1-\Psi_{Logistic}(\ln(x))\right) \sim U(-1,0)$, so $Var_{\vartheta=0}(h_F(x)) = 4$ and $Var_{\vartheta=0}(h_S(x)) = \frac{1}{12}$. By applying Theorem (3) we get $\lim_{\vartheta\to 0}\frac{C_S(\vartheta)}{C_F(\vartheta)} = \frac{4}{3}$. Similarly we can prove other parts.

6. THE LIMITING RATIO OF THE EBS FOR DIFFERENT TESTS WHEN $\vartheta \to \infty$

Now, we will compare the limit of the ratio of EBSs for any two methods when $\vartheta \to \infty$.
Corollary 3 *The limits of ratios for different tests are as follows:*
D1. $e_B(T_L, T_F) \to 1$

D2. $e_B(T_S, T_F) \to 0$
D3. $e_B(T_N, T_S) \to 0$
D4. $\lim_{\vartheta\to\infty}\{C_F(\vartheta) - C_L(\vartheta)\} \le 0$
D5. $e_B(T_N, T_L) \to 0, e_B(T_S, T_L) \to 0$.
Proof of D1. By Lemma (1) part (1) $C_L(\vartheta) \le 2b_L(\vartheta)$. So

$$\frac{C_L(\vartheta)}{C_F(\vartheta)} \le \frac{2b_L(\vartheta)}{b_F(\vartheta)-2\ln(b_F(\vartheta))+2\ln(2)-2}.$$

It is sufficient to obtain $\lim_{\vartheta\to\infty}\frac{2b_L(\vartheta)}{b_F(\vartheta)}$.
Therefore,

$$\lim_{\vartheta\to\infty}\frac{2b_L(\vartheta)}{b_F(\vartheta)} = -\lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}(\Lambda)}{\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{1-e^{-\Lambda\vartheta}}\right)} = 1.$$

So,

$$\lim_{\vartheta\to\infty}\frac{C_L(\vartheta)}{C_F(\vartheta)} \le 1.$$

Also, by Theorem (6) part (2), we have $C_L(\vartheta) \ge 2b_L(\vartheta) - 2\ln(b_L(\vartheta)) - 2$. So

$$\lim_{\vartheta\to\infty}\frac{C_L(\vartheta)}{C_F(\vartheta)} \ge \lim_{\vartheta\to\infty}\frac{2b_L(\vartheta)-2\ln(b_L(\vartheta))-2}{b_F(\vartheta)-2\ln(b_F(\vartheta))+2\ln(2)-2}.$$

It is sufficient to obtain the limit of $\lim_{\vartheta\to\infty}\frac{2b_L(\vartheta)}{b_F(\vartheta)}$.
Therefore,

$$\lim_{\vartheta\to\infty}\frac{2b_L(\vartheta)}{b_F(\vartheta)} = -\lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}(\Lambda)}{\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{1-e^{-\Lambda\vartheta}}\right)} = 1.$$

Then,

$$\lim_{\vartheta\to\infty}\frac{C_L(\vartheta)}{C_F(\vartheta)} \ge 1$$

Thus, by pinching theorem, we have $\lim_{\vartheta\to\infty}\frac{C_L(\vartheta)}{C_F(\vartheta)} = 1$.
Proof of D2. By Lemma (1) part (3) $C_S(\vartheta) \le -2\ln(2) - 2\ln(-b_S(\vartheta))$. So

$$\lim_{\vartheta\to\infty}\frac{C_S(\vartheta)}{C_F(\vartheta)} \le \lim_{\vartheta\to\infty}\frac{-2\ln(2)-2\ln(-b_S(\vartheta))}{b_F(\vartheta)-2\ln(b_F(\vartheta))+2\ln(2)-2}.$$

It is sufficient to obtain the limit of $\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)}$. Then

$$\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)} = \lim_{\vartheta\to\infty}\frac{-\ln\left(\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda\vartheta e^{\Lambda\vartheta}-e^{\Lambda\vartheta}+1}{\left(e^{\Lambda\vartheta}-1\right)^2}\right)\right)}{\vartheta\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{1-e^{-\Lambda\vartheta}}\right)}.$$

Now, by Jensen's inequality where the logarithm is concave function, then $-\ln\left(\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda\vartheta e^{\Lambda\vartheta}-e^{\Lambda\vartheta}+1}{\left(e^{\Lambda\vartheta}-1\right)^2}\right)\right) \le$

$\mathbb{E}_{H_\Lambda}\ln\left(\frac{\left(e^{\Lambda\vartheta}-1\right)^2}{\Lambda\vartheta e^{\Lambda\vartheta}-e^{\Lambda\vartheta}+1}\right)$, so $\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)} \le \lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}\ln\left(\frac{\left(e^{\Lambda\vartheta}-1\right)^2}{\Lambda\vartheta e^{\Lambda\vartheta}-e^{\Lambda\vartheta}+1}\right)}{\vartheta\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{1-e^{-\Lambda\vartheta}}\right)}$. Now by Lemma (1) part (1) we get $\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)} \le \lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}\left(\frac{\left(e^{\Lambda\vartheta}-1\right)^2}{\Lambda\vartheta e^{\Lambda\vartheta}-e^{\Lambda\vartheta}+1}-1\right)}{\vartheta\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{1-e^{-\Lambda\vartheta}}\right)} = 0$. Therefore, $\lim_{\vartheta\to\infty}\frac{C_S(\vartheta)}{C_F(\vartheta)} = 0$.

Proof of D3. From B4 we have $C_N(\vartheta) = \left[\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}\mathbb{E}_{N(0,1)}\left\{\frac{v}{M_{Binomial(2,\Phi(v))}(\vartheta)}\right\}\right)\right]^2$. By Lemma (1) part (3) we get $C_S(\vartheta) \ge -2-2\ln(-b_S(\vartheta))$, now by Theorem (7) and Theorem (8), we have $\lim_{\vartheta\to\infty}\frac{C_N(\vartheta)}{C_S(\vartheta)} \le$

$\lim_{\vartheta\to\infty}\frac{\left[\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}\mathbb{E}_{N(0,1)}\left\{\frac{v}{M_{Binomial(2,\Phi(v))}(\vartheta)}\right\}\right)\right]^2}{-2-2\ln\left[-\mathbb{E}_{H_\Lambda}\left(\frac{e^{\Lambda\vartheta}-\Lambda\vartheta e^{\Lambda\vartheta}-1}{\left(e^{\Lambda\vartheta}-1\right)^2}\right)\right]} = 0$. Therefore, $\lim_{\vartheta\to\infty}\frac{C_N(\vartheta)}{C_S(\vartheta)} = 0$.

Proof of D4. By Theorem 6 (2), we have

$$C_F(\vartheta) - C_L(\vartheta) \le b_F(\vartheta) - 2\ln b_F(\vartheta) + 2\ln(2) + 2\ln b_L(\vartheta) - 2b_L(\vartheta)$$

$$= b_F(\vartheta) - 2b_L(\vartheta) + 2\ln\left(\frac{b_L(\vartheta)}{b_F(\vartheta)}\right) + 2\ln(2).$$

Now, $$b_F(\vartheta) - 2b_L(\vartheta) = 2\vartheta\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{e^{\Lambda\vartheta}-1}\right).$$

Also, $$\lim_{\vartheta\to\infty}\frac{b_L(\vartheta)}{b_F(\vartheta)} = \lim_{\vartheta\to\infty}\frac{\vartheta\mathbb{E}_{H_\Lambda}(\Lambda)}{2\vartheta\mathbb{E}_{H_\Lambda}\left(\frac{\Lambda}{1-e^{-\Lambda\vartheta}}\right)} = \frac{1}{2}.$$

Then,

$$\lim_{\vartheta\to\infty}(C_F(\vartheta) - C_L(\vartheta)) \le \lim_{\vartheta\to\infty}(b_F(\vartheta) - 2\ln b_F(\vartheta)) + 2\lim_{\vartheta\to\infty}\ln\left(\frac{b_L(\vartheta)}{b_F(\vartheta)}\right) + 2\ln(2) = 0 - 2\ln(2) + 2ln(2) = 0.$$

So, $C_F(\vartheta) \le C_L(\vartheta)$ for large ϑ.

Proof of D5. Straight forward by using D1 to D3.

7. CONCLUSION

In this section we will compare the EBS for the six combination producers. From the relations in section (6) we conclude that locally as $\vartheta \to 0$, the sum of p-values procedure is better than all other procedures since it has the highest EBS, followed in decreasing order by the inverse normal, logistic procedure and the Fisher's procedure. The worst two are the Tippett's and the maximum of p-values procedures, i.e,

$$C_S(\vartheta) > C_N(\vartheta) > C_L(\vartheta) > C_F(\vartheta) > C_T(\vartheta) = C_{max}(\vartheta).$$

Whereas, from result of Section (6.1) as $\vartheta \to \infty$ the worst methods are Tippett's and the maximum of p-values. The logistic is better than all other procedures, followed in decreasing order by Fisher's procedure, sum of p-values and the inverse normal procedures, i.e,

$$C_L(\vartheta) > C_F(\vartheta) > C_S(\vartheta) > C_N(\vartheta) > C_T(\vartheta) = C_{max}(\vartheta).$$

RECEIVED: MARCH, 2021.
REVISED: DECEMBER, 2021.

REFERENCES

[1] ABU-DAYYEH, W. A., AL-MOMANI, M. A. AND MUTTLAK, H. A. (2003): Exact bahadur slope for combining independent tests for normal and logistic distributions. **Applied mathematics and computation**, 135, 345-360.
[2] ABU-DAYYEH, W. A. AND EL-MASRI, A. E.-Q. (1994): Combining independent tests of triangular distribution. **Statistics & Probability Letters,** 21, 195-202.
[3] AHSANULLAH, M. AND ALZAATREH, A. (2018): Some characterizations of the log-logistic distribution. **Stochastics and Quality Control**, 33, 23-29.
[4] AL-MASRI, A.-Q. AND AL-MOMANI, N. (2021): On combining independent tests in case of log-logistic distribution. **Electronic Journal of Applied Statistical Analysis**, 14, 217-229.

[5] AL-MASRI, A.-Q. S. (2010): Combining independent tests of conditional shifted exponential distribution. **Journal of Modern Applied Statistical Methods**, 9, 221-226.
[6] AL-MASRI, A.-Q. S. (2021): Exact Bahadur slope for combining independent tests in case of Laplace distribution. **Jordan Journal of Mathematics and Statistics**, 14, 163-176.
[7] AL-MASRI, A.-Q. S. (2021): On combining independent tests in case of log-normal distribution. **American**
Journal of Mathematical and Management Sciences, 0(0):1-12 DOI: 10.1080/01966324.2021.1997676
[8] AL-TALIB, M., KADIRI, M. A. AND AL-MASRI, A.-Q. (2020): On combining independent tests in case of conditional normal distribution. **Communications in Statistics - Theory and Methods**, 49, 5627-5638.
[9] BAHADUR, R. R. (1960): Stochastic comparison of tests. **Annals of Mathematical Statistics**, 31, 276-295.
[10] KALLENBERG, W. C. (1981): Bahadur deficiency of likelihood ratio tests in exponential families. **Journal of Multivariate Analysis**, 11, 506-531.
[11] SERFLING, R. J. (2009): Approximation theorems of mathematical statistics. **John Wiley & Sons**, 162.