INFERENCE FOR KUMARASWAMY DISTRIBUTION UNDER TYPE-I GENERALIZED HYBRID CENSORING

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ABSTRACT
In this paper, the estimation of the parameters of Kumaraswamy distribution is carried out in the presence of a Type-I generalized hybrid censoring scheme.Maximum likelihood estimators (MLEs) and Bayes estimators of the parameters are derived. For Bayesian estimation, an importance sampling method is used. Confidence intervals based on MLEs and Bayes credible intervals for the parameters are also obtained. A Simulation study is carried out to check the behavior of the proposed estimators. From the simulation study, we observed that Bayes estimators perform better than the MLE concerning mean squared errors. A real example is also considered to exemplify the results obtained in the paper.

KEYWORDS: Maximum likelihood, Bayes Estimate, gamma priors, credible intervals, importance sampling.

MSC: 62F10, 62F15, 62F30, 62N01

RESUMEN
En este paper, se lleva cabo la estimación de los parámetros de la distribución de of Kumaraswamy en presencia de un esquema de censura generalizada híbridaTipo-I. Se derian estidores de Máxima Verosimiltud (MLEs) y Bayes. Para la estimación Bayesiana , ase usa el método de muestreo por importancia. Son obtenidos intervalos de confianza basados en MLEs e intervalos creibles Bayes de los parámetros. Un estudio ds simulación fue desarrollado para checar el comportamiento de los propuestos estimadores. En el estudio de simulación, observamos que el estimador Bayes se comportó mejor que el MLE concerniente al error cuadrático medio. Un ejemplo real fue considerado tambi-en para ejemplificar los resultados obtenidos en este paper.

PALABRAS CLAVE: Máxima verosimilitud, estimdor Bayes, gamma a-prioris, intervalos creibles, muestreo por importancia

1. INTRODUCTION

There are many situations in life-testing and reliability experiments in which units are lost or removed from experimentation before failure. The experimenter may not obtain complete information on failure times for all experimental units. Data observed from such experiments are called censored data. To save time and cost censored data are used. There are two basic censoring schemes, namely Type-I censoring and Type-II censoring. In Type-I censoring (Time censoring), failures are observed until the pre-determined time T is observed while in Type-II censoring (Failure censoring) the experiment is terminated at the time of r failures, where r is specified before experimenting with n items on the test, $0 < r < n$. Various modified censoring schemes such as progressive censoring, multiply censoring are also available and used to analyze the lifetime data. A mixture of Type-I and Type-II censoring schemes is known as a hybrid censoring scheme. Such a scheme has received considerable attention among practitioners. It can be expressed as follows.
Take into account the following life-testing experiment in which n units are placed on the test. The lifetimes of the sample units are assumed to be (i.i.d) random variables. Let the ordered lifetimes of these units can be marked as $x_{1:n},\ldots x_{n:n}$ respectively. The test is aborted when a number r, $r<n$ out of n items have failed or when a time T has been reached. In other words, the life-test is terminated at a random time $T^{*}=\min\{X_{r:n},T\}$. This time-testing experiment is called Type-I hybrid censoring scheme (Type-I, HCS). Here,r and T are preset. It is also generally assumed that the failed items are not replaced. The basic concept of Type-I hybrid censoring is initiated by Epstein (1954).

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The main disadvantage of Type-I HCS is that most of the inference results are accessed under the situation that the number of observed failures is at least one. Moreover, there may be very few failures occurring up to the pre-chosen time T. So, in that case, the performance of the estimators may be very low. For this reason, childs et al. (2003) suggested another hybrid censoring scheme that would terminate the experiment at the random time $T^*= \max\{X_{r:n}, T\}$, where r and T are preset and r<n. This is called Type-II hybrid censoring scheme (Type-II, HCS) and in this scheme at least r failures to be observed by the end of the test. If the r failures result before time T, the test continues up to time point T. Then again, if the r failures do not result before time T, then the test continues until the r failures take place. Here the duration of the experiment may be too long. To overcome the disadvantages of Type-I hybrid and Type-II hybrid censoring schemes with statistical inference on censoring scheme which can recover the ideal time of the test and the minimum number of failures, Chanrasekar et al. (2004) established the Type-I generalized hybrid censoring scheme(Type-I, GHSC). Type-I generalized hybrid censoring scheme may be described as follows.
Consider a life testing experiment with n items and fixed two positive integers k and r and time epoch T such that $1\leq k\leq r \leq n$. If the k failures occur before time T, the test is terminated at min $\{X_r, T\}$, where X_r is the time of r failures. Then again, if the k failures occur after time T, the test is terminated at the time X_k of k failures. One of the following three types of observations can be verified under a Type-I generalized hybrid censoring scheme.
Define X_i as the time of number of i failures, i = 1,2, … ,n.

Case-I: $X_1 < X_2 < \cdots < X_r$ if $X_k < X_r \leq T$.

Case-II: $X_1 < X_2 < \cdots < X_d$ if $X_k < T < X_r$, $d = k, k+1, \ldots, r-1$.

Case-III: $X_1 < X_2 < \cdots < X_k$ if $T < X_k$.

Let us denote the observed number of failures by m and test termination time as U then we can write

(i) $U=X_r$, m=r when $X_r < T$.

(ii) U=T, m=d when $X_k < T < X_r$.

(iii) $U=X_k$, m=k when $X_k > T$. (1.1)

In this scheme, the data regarding observed failure times will be $\underline{x} = (X_1, X_2, \ldots\ldots, X_m)$.
Later, Sen et al. (2018) under generalized hybrid censoring set-up has used the asymptotic variance minimization approach to determine an optimal life test plan. Most recently, Sayed-Ahmed et al. (2021) furnished Type-I GHSC by using Chen lifetime distribution (CD). In this study, MLE and Bayes methods are discussed. Further, asymptotic confidence intervals, as well as Bayes credible, are constructed.
Various types of lifetime models such as exponential, Rayleigh, Weibull, power function, log normal and many more are available and used by practitioners in life testing experiments.
In the life testing span and reliability tests, much time the data are selected by finite range distribution. This distribution applies to many natural phenomena whose outcome has lower and upper bounds. E.g. Storage pressure in vacuum whose upper bound is Z_{max} and lower is 0. For these types of studies, Kumaraswamy distribution was found appropriate, introduced by Kumaraswamy (1980). Ponnambalam et al. (2001) and Jones (2009) examined the basic properties of Kumaraswamy distribution while some mathematical properties of Kumaraswamy distribution that is a flexible model in analyzing failure time data are considered by Cordeiro et al. (2010). Rayad and Ahmed (2016) gone into the Bayesian and E-Bayesian estimation for the shape parameters of Kumaraswamy distribution based on Type-II censored schemes.
For various types of censoring schemes the work has been considered for the Kumaraswamy distribution by many authors, some of them are Garg (2009), Sindhu et al.(2013), and Sultana et al. (2018). Not much work was observed under Type-I generalized hybrid censoring in the case of Kumaraswamy distribution.
In this paper, the study intends to estimate the parameters of the Kumaraswamy distribution under Type-I generalized hybrid censoring schemes. The rest of the paper is organized as follows:
In section 2 maximum likelihood estimation is carried out. Section 3 covers Bayesian estimation of the parameters of the Kumaraswamy distribution under gamma priors by using importance sampling procedure and Bayes credible for the parameters are derived in Section 4.In Section 5 simulation study is conducted to check the performance of the estimators. At last, we set down the results and conclusions in Section 6.

2. MODEL AND MAXIMUM LIKELIHOOD ESTIMATION

Let lifetime of n items on the test follows Kumaraswamy distribution having probability density function (pdf) and cumulative distribution function (CDF) as

$$f(x;\theta,\lambda) = \lambda\theta x^{\lambda-1}(1-x^{\lambda})^{\theta-1}, \quad (2.1)$$

$F(x;\theta,\lambda) = 1-\left(1-x^{\lambda}\right)^{\theta}, 0<x<1$ and $\lambda,\theta>0$ (2.2)

where λ and θ are shape parameters.

According to Sayed-Ahmed et al. (2021) and Sen et al. (2018), the likelihood function based on observed data $\underline{x}$ observed from (1.1) under Type-I generalized hybrid censoring scheme as discussed in Section 1,is given by

$$L\left(\theta,\lambda|\underline{x}\right)=\frac{n!}{(n-m)!}\prod_{i=1}^{m} f(x_i,\theta,\lambda)[(1-F(U,\theta,\lambda)]^{n-m} \quad (2.3)$$

Substituting pdf and CDF from (2.1) and (2.2) in (2.3) we have

$$\mathrm{L}\left(\theta,\lambda|\underline{x}\right)=\frac{n!}{(n-m)!}\lambda^m\theta^m\prod_{i=1}^{m} x_i^{\lambda-1}\prod_{i=1}^{m}\left(1-x_i^{\lambda}\right)^{\theta-1}(1-U^{\lambda})^{\theta(n-m)} \quad (2.4)$$

The log-likelihood function reduces to

$$\log L\left(\theta,\lambda|\underline{x}\right)=\log \mathrm{C}+\mathrm{m}\log\lambda+\mathrm{m}\log\theta+(\lambda-1)\sum_{\mathrm{I}=1}^{\mathrm{m}}\log \mathrm{x_i}+(\theta-1)\sum_{\mathrm{i}=1}^{\mathrm{m}}\log(1-\mathrm{x_i^{\lambda}})+\theta(\mathrm{n-m})\log(1-\mathrm{U^{\lambda}}) \quad (2.5)$$

Where $C=\frac{n!}{(n-m)!}$

To obtain MLEs of θ and λ,we take partial derivation of equation (2.5) concerning θ and λ

Thus,

$$\frac{\partial\log L\left(\theta,\lambda|\underline{x}\right)}{\partial\theta}=\frac{m}{\theta}+\sum_{i=1}^{m}\log(1-x_i^{\lambda})+(n-m)\log(1-U^{\lambda}) \quad (2.6)$$

and

$$\frac{\partial\log L\left(\theta,\lambda|\underline{x}\right)}{\partial\lambda}=\frac{m}{\lambda}+\sum_{i=1}^{m}\log x_i-(\theta-1)\sum_{i=1}^{m}\frac{x_i^{\lambda}\log x_i}{1-x_i^{\lambda}}-\theta(n-m)\frac{U^{\lambda}\log U}{1-U^{\lambda}} \quad (2.7)$$

Comparing the above equation (2.6) and (2.7) to Zero, We get the equation as

$$\theta=-\frac{m}{\sum_{i=1}^{m}\log(1-x_i^{\lambda})+(n-m)\log(1-U^{\lambda})} \quad (2.8)$$

and

$$\lambda=\frac{m}{\theta(n-m)\left[\frac{U^{\lambda}\,log\,U}{1-U^{\lambda}}\right]-\sum_{i=1}^{m}log\,x_i+(\theta-1)\sum_{i=1}^{m}\frac{x_i^{\lambda}\,log\,x_i}{1-x_i^{\lambda}}} \quad (2.9)$$

As we see that MLE of λ cannot be obtained directly, but we have to solve the equation (2.8) and (2.9) simultaneously. On substituting $\hat{\theta}$ from (2.8) in (2.9), the right-hand side will be a function of only unknown parameter λ. By considering the method of iteration the equation (2.9) can be solved for λ which will be MLE $\hat{\lambda}$ of λ. After substituting $\hat{\lambda}$ in (2.8) we get MLE $\hat{\theta}$ for θ.

In the case of large samples, we can obtain the confidence intervals based on the diagonal elements of the inverse Fisher information matrix $I^{-1}(\hat{\theta},\hat{\lambda})$ which provides the estimated asymptotic variance for the parameters θ and λ respectively.

Fisher information matrix can be estimated by,

$$I\left(\hat{\theta},\hat{\lambda}\right)=\begin{bmatrix}-\frac{\partial^2\,log\,L\left(\theta,\lambda|\underline{x}\right)}{\partial\theta^2} & -\frac{\partial^2\,\log L\left(\theta,\lambda|\underline{x}\right)}{\partial\lambda\partial\theta}\\ -\frac{\partial^2\,\log L\left(\theta,\lambda|\underline{x}\right)}{\partial\lambda\partial\theta} & -\frac{\partial^2\,\log L\left(\theta,\lambda|\underline{x}\right)}{\partial\lambda^2}\end{bmatrix}$$

The derivatives in $I(\hat{\theta},\hat{\lambda})$ are

$$\frac{\partial^2\,log\,L\left(\theta,\lambda|\underline{x}\right)}{\partial\theta^2}=-\frac{m}{\theta^2} \quad (2.10)$$

$$\frac{\partial^2\,\log L\left(\theta,\lambda|\underline{x}\right)}{\partial\lambda\partial\theta}=-\sum_{i=1}^{m}\left[\frac{x_{(i)}^{\lambda}\log x_{(i)}}{1-x_{(i)}^{\lambda}}\right]-(n-m)\left[\frac{U^{\lambda}-\log U}{1-U^{\lambda}}\right] \quad (2.11)$$

and

$$\frac{\partial^2\,log\,L\left(\theta,\lambda|\underline{x}\right)}{\partial\lambda^2}=-\frac{m}{\lambda^2}-(\theta-1)\left[\frac{x_i^{\lambda}(log\,x_i)^2\left(1-x_i^{\lambda}\right)+\left(x_i^{\lambda}\,log\,x_i\right)^2}{\left(1-x_i^{\lambda}\right)^2}\right]-\theta(n-m)\left[\frac{U^{\lambda}(\log U)^2(1-U^{\lambda})+\left(U^{\lambda}\log U\right)^2}{\left(1-U^{\lambda}\right)^2}\right] \quad (2.12)$$

Thus, two-sided $100(1-\alpha)\%$ confidence interval of θ can be defined using Goyal et al. (2020) as

$$\hat{\theta}\pm Z_{\alpha/2}\sqrt{V(\hat{\theta})} \quad (2.13)$$

where Z is standard normal variate such that p (Z > $Z_{\alpha/2}$) = $\alpha/2$.

Similarly, it can be obtained for λ also.

3. BAYES ESTIMATION

In this section, we proposed Bayesian inference of the parameters of Kumaraswamy distribution based on Type-I generalized HSC. We obtained a Bayes estimate for parameters θ and λ. We have used proper gamma priors as well as Jeffrey's priors for the parameters of the model. Several authors have used different types of priors for different types of distributions. There is no criterion to say about the particular prior is better than the other one see, Arnold and Press (1983). Such a kind of prior distribution is quite flexible and includes noninformative cases as well. One may refer to Sinha (1998) and Kundu and Pradhan (2009[b]) for further details in this regard. Nadar et al. (2013) have also considered gamma priors for the Kumaraswamy distribution.

Under proper priors:

We take Proper prior distributions for parameter θ and λ respectively as Gamma (b_1, a_1) and Gamma (b_2, a_2), since the range of gamma distribution match with the range of the parameters and it is mathematically tractable to obtain the posterior distribution of the parameters.

$$\Pi_1(\theta) = \frac{e^{-b_1\theta}\theta^{a_1-1}{b_1}^{a_1}}{\Gamma a_1} \qquad \text{where } \theta, a_1, b_1 > 0 \tag{3.1}$$

and

$$\Pi_2(\lambda) = \frac{e^{-b_2\lambda}\lambda^{a_2-1}{b_2}^{a_2}}{\Gamma a_2} \qquad \text{where } \lambda, a_2, b_2 > 0 \tag{3.2}$$

where b_1 and b_2 are scale parameters, a_1 and a_2 are shape parameters and Γa is a gamma function which is define as $\Gamma a = \int_0^\infty e^{-x}\, x^{n-1} dx$.

The joint posterior distribution of (θ, λ) using the likelihood in (2.4) and priors of (3.1) and (3.2) is given by

$$\mathrm{h}(\theta,\lambda|\underline{x}) = \frac{L(\theta,\lambda|\underline{x})\Pi_1(\theta)\Pi_2(\lambda)}{\int_0^\infty\int_0^\infty L(\theta,\lambda|\underline{x})\Pi_1(\theta)\Pi_2(\lambda)\, d\theta d\lambda} \tag{3.3}$$

$$= \frac{1}{D}\theta^{m+a_1-1}e^{-\theta\left\{b_1-\sum_{i=1}^m log\left(1-x_i^\lambda\right)-(n-m)\, log\left(1-U^\lambda\right)\right\}}\lambda^{m+a_2-1}e^{-\lambda\{b_2-\sum_{i=1}^m log\, x_i\}}e^{-\sum_{i=1}^m log\left(1-x_i^\lambda\right)} \tag{3.4}$$

Where $D = \int_0^\infty\int_0^\infty \theta^{m+a_1-1}e^{-\lambda\{b_2-\sum_{i=1}^m \log x_i\}}\lambda^{m+a_2-1}e^{-\theta A(\lambda)}e^{-\sum_{i=1}^m \log(1-x_i^\lambda)}d\theta\, d\lambda$

$$\mathrm{A}(\lambda) = \mathrm{b}_1 - \sum_{\mathrm{i}=1}^{\mathrm{m}} \log\left(1-\mathrm{x}_\mathrm{i}^\lambda\right) - (\mathrm{n}-\mathrm{m})\log\left(1-\mathrm{U}^\lambda\right)$$

The Bayes estimates $\hat{\theta}_B$ and $\hat{\lambda}_B$ of θ and λ under squared error loss function (SELF) are given by

$\hat{\lambda}_B = E_{h_\lambda}(\lambda)$

Where h_λ is the marginal posterior distribution of λ.

$$\hat{\lambda}_B = \int_0^\infty\int_0^\infty \lambda\, h(\theta,\lambda|\underline{x})\, d\theta d\lambda = \frac{1}{D}\int_0^\infty\int_0^\infty \theta^{m+a_1-1}e^{-\lambda\{b_2-\sum_{i=1}^m \log x_i\}}\lambda^{m+a_2}e^{-\theta A(\lambda)}e^{-\sum_{i=1}^m \log(1-x_i^\lambda)}d\theta\, d\lambda$$

$$= \frac{1}{D}\int_0^\infty e^{-\lambda\{b_2-\sum_{i=1}^m \log x_i\}}\lambda^{m+a_2}e^{-\sum_{i=1}^m \log(1-x_i^\lambda)}\frac{\Gamma(m+a_1)}{(A(\lambda))^{m+a_1}}d\lambda \tag{3.5}$$

And $\hat{\theta}_B = E_{h_\theta}(\theta)$

Where h_θ is the marginal posterior distribution of θ.

$$\hat{\theta}_B = \int_0^\infty\int_0^\infty \lambda\, h(\theta,\lambda|\underline{x})\, d\theta d\lambda = \frac{1}{D}\int_0^\infty\int_0^\infty \theta^{m+a_1}e^{-\lambda\{b_2-\sum_{i=1}^m \log x_i\}}\lambda^{m+a_2-1}e^{-\theta A(\lambda)}e^{-\sum_{i=1}^m \log(1-x_i^\lambda)}\, d\theta\, d\lambda$$

$$= \frac{1}{D}\int_0^\infty e^{-\lambda\{b_2-\sum_{i=1}^m \log x_i\}}\lambda^{m+a_2-1}e^{-\sum_{i=1}^m \log(1-x_i^\lambda)}\frac{\Gamma m+a_1+1}{(A(\lambda))^{m+a_1+1}}d\lambda \tag{3.6}$$

Under Jeffrey's priors :

The Jeffrey's priors of θ and λ are given by

$$\Pi_1(\theta) = \frac{1}{\theta}, 0<\theta<\infty, \Pi_2(\lambda) = \frac{1}{\lambda}, 0<\lambda<\infty \tag{3.7}$$

The joint posterior distribution of (θ, λ) using the like-hood in (2.4) and priors in (3.7) can be obtained as

$$h(\theta,\lambda|\underline{x}) = \frac{\theta^{m-1}e^{-\theta\left\{-\sum_{i=1}^m log\left(1-x_i^\lambda\right)-(n-m)\, log\left(1-U^\lambda\right)\right\}}\lambda^{m-1}e^{-\lambda\{-\sum_{i=1}^m log\, x_i\}}e^{-\sum_{i=1}^m log\left(1-x_i^\lambda\right)}}{\int_0^\infty\int_0^\infty \theta^{m-1}e^{-\theta\left\{-\sum_{i=1}^m log\left(1-x_i^\lambda\right)-(n-m)\, log\left(1-U^\lambda\right)\right\}}\lambda^{m-1}e^{-\lambda\{-\sum_{i=1}^m log\, x_i\}}e^{-\sum_{i=1}^m log\left(1-x_i^\lambda\right)}d\theta\, d\lambda} \tag{3.8}$$

Remark:

The above joint posterior will be a particular case of (3.4) under $a_1 = b_1 = a_2 = b_2 = 0$. Hence the Bayes estimates of θ and λ under Jeffery's priors can be directly gained by substituting $a_1 = b_1 = a_2 = b_2 = 0$ in (3.5) and (3.6).

3.1. Importance sampling method

It is difficult to solve the integrals in (3.5) & (3.6) analytically, so we have to use some approximation methods. There are many approximation methods; namely the Lindley approximation method, Tierney and Kadane approximation method, importance sampling method, Metropolis-Hastings methods, Gibbs sampling method, etc. Here we use, the importance sampling method to obtain Bayes estimates of the parameters θ and λ as considered by Kundu and Pradhan (2009[a]). This method has special benefits over some other methods that can be used to construct a confidence interval also. To apply the importance sampling method we need to use the joint posterior distribution of θ and λ given in (3.4), which can be further simplified as,

$$h(\theta,\lambda|\underline{x}) \propto g_\lambda(m+a_2, b_2-\sum_{i=1}^{m}\log x_i)\; g_{\theta|\lambda}(m+a_2, A(\lambda))w(\lambda) \tag{3.9}$$

Where $g_\lambda(m+a_2, b_2-\sum_{i=1}^{m}\log x_i) = \frac{e^{-\lambda\{b_2-\sum_{i=1}^{m}\log x_i\}}\lambda^{m+a_2-1}(b_2-\sum_{i=1}^{m}\log x_i)^{m+a_2}}{\Gamma m+a_2}$ (3.10)

which is gamma distribution with scale parameter $b_2-\sum_{i=1}^{m}\log x_i$ and shape parameter $m+a_2$.

and $g_{\theta|\lambda}(m+a_1, A(\lambda)) = \frac{e^{-\theta A(\lambda)}\theta^{m+a_1-1}(A(\lambda))^{m+a_1}}{\Gamma m+a_1}$ (3.11)

is gamma distribution with scale parameter A (λ) and shape parameter $m+a_1$.

and $w(\lambda) = \frac{e^{-\sum_{i=1}^{m}\log(1-x_i^{\lambda})}}{(A(\lambda))^{m+a_1}}$ is a function of parameter λ. (3.12)

Similarly, we can write (3.8), the joint posterior distribution of (θ, λ) under Jeffery's prior as

$$h(\theta,\lambda|\underline{x}) \propto g_\lambda(m,-\sum_{i=1}^{m}\log x_i)\; g_{\theta|\lambda}(m, A_1(\lambda))v(\lambda) \tag{3.13}$$

$$A_1(\lambda) = -\sum_{i=1}^{m} log(1-x_i^{\lambda}) - (n-m)\, log(1-U^{\lambda}) \text{ of } \lambda .$$

where $g_\lambda(m,-\sum_{i=1}^{m}\log x_i)$is gamma density of λ with scale parameter $-\sum_{i=1}^{m}\log x_i$ and shape parameter m.
where $g_{\theta|\lambda}(m, A_1(\lambda))$is a gamma density of θ given λ with scale parameter A_1 (λ) and shape parameter m.

and $v(\lambda) = \frac{e^{-\sum_{i=1}^{m}\log(1-x_i^{\lambda})}}{(A_1(\lambda))^{m+a_1}}$, a function of λ . (3.14)

The Bayes estimates of θ and λ can be obtained using the importance sampling method described in Sulatana et al.(2018) as follow:
Algorithm 1:

(i) Decide the values of n, r, k, and T.
(ii) Based on the scheme of Type-I generalized hybrid censoring scheme generate the Data and determine U and m as discussed in section 1.
(iii) Using the joint posterior distribution in (3.8).
 (a) Generate N values of λsay ($\lambda_1, \lambda_{2,\ldots,}\lambda_N$) from gamma $(m+a_2, b_2-\sum_{i=1}^{m}\log x_i)$ distribution.
 (b) Generate N values of θ say ($\theta_1, \theta_{2,\ldots,}\theta_N$) from gamma $(m+a_1, A(\lambda))$ distribution based on the N value of λ says ($\lambda_1, \lambda_{2,\ldots,}\lambda_N$) respectively obtain in the earlier step (a).
(iv) Based on the N value of λ and θ compute N values of $w(\lambda)$ and ν(λ).
(v) Under squared error loss function, the Bayes estimate of any function of θ and λ, say Ψ (θ, λ) can be obtained as

 $\hat{\psi}(\theta,\lambda) = E\,[\psi\,(\theta,\lambda)] = \frac{\sum_{i=1}^{N}\psi(\theta_i,\lambda_i)w(\lambda_i)}{\sum_{i=1}^{N}w(\lambda_i)}$ inthe case of gammapriors and

 $\hat{\psi}(\theta,\lambda) = E\,[\psi\,(\theta,\lambda)] = \frac{\sum_{i=1}^{N}\psi(\theta_i,\lambda_i)v(\lambda_i)}{\sum_{i=1}^{N}v(\lambda_i)}$ inthe case of Jeffery's prior,

 The Bayes estimate of θ is obtained by considering Ψ (θ, λ) = θ in the above computation. Similarly, the Bayes estimate of λ can be computed.

3.2. Elicitation of hyperparameters

In Bayes estimation, it is necessary to select the values of prior parameters used in the lifetime model. Many methods are available in the literature. One of these methods we have used is as follows:

Algorithm 2:

(i) Compute MLE $\hat{\theta}$ of parameter say θ and its variance V($\hat{\theta}$).

(ii) Obtain theoretical mean and variance of the prior distribution of θ. In our case for gammaprior distribution of θ as given in (3.1), mean and variance are respectively $E(\theta) = \frac{a_1}{b_1}and(\theta) = \frac{a_1}{b_1^2}$.

(iii) Use MLE $\hat{\theta}$ and V($\hat{\theta}$) as a prior belief of prior mean and variance respectively$\hat{\theta} = \frac{a_1}{b_1}andV(\hat{\theta}) = \frac{a_1}{b_1^2}$

(iv) Solving the equation in (ii),we get $\hat{b}_1 = \frac{\hat{\theta}}{V(\hat{\theta})}and\hat{a}_1 = \hat{b}_1\hat{\theta}$

(v) Similarly, we can get estimates of a parameter of the prior distribution of λ, say $\hat{b}_2$and $\hat{a}_2$.

We have used these values of hyperparameters in Bayes estimation.

4. BAYES CREDIBLE INTERVAL FOR PARAMETER θ AND λ

Here we use the method used by Balakrishnan and Kundu (2013) which is based on importance sampling. For computing the HPD credible interval for θ, let us use $h\left(\theta|\underline{x}\right)$ as the posterior density function of θ and $H\left(\theta|\underline{x}\right)$ as the posterior distribution function of θ.

Let $\theta^{(p)}$ be the p-th quantile of θ, 0<p<1.

Observe that for a given θ^*,

$$H\left(\theta^*|\underline{x}\right) = E[I_{\theta\le\theta^*}|\underline{x}] \tag{4.1}$$

Where $I_{\theta\le\theta^*}$ is the indication function defined as

$$I_{\theta\le\theta^*} = \begin{cases} 1, & if\ \theta \le \theta^* \\ 0, & otherwise \end{cases} \tag{4.2}$$

Then a simulated consistent estimator of $H\left(\theta^*|\underline{x}\right)$ is given by $\frac{\sum_{i=1}^{N} I_{\theta\le\theta^*}(\theta_i,\lambda_i)}{\sum_{i=1}^{N} w(\theta_i,\lambda_i)}$ (4.3)

Let {θ $_{(i)}$} be the set of ordered values of {θ_i} , and let

$$w_i = \frac{w(\lambda_{(i)})}{\sum_{i=1}^{N} w(\lambda_{(i)})}\ \ , i = 1,2, \dots, N. \tag{4.4}$$

Then, we have

$$H\left(\theta^*|\underline{x}\right) = \begin{cases} 0, & if\ \theta^* < \theta_{(1),} \\ \sum_{j=1}^{i} w_j, & if\ \theta_{(i)} \le \theta^* < \theta_{(i+1)}, \\ 1, & if\ \theta^* \ge \theta_{(n)} \end{cases} \tag{4.5}$$

With which $\theta^{(p)}$ can be approximated by

$$\hat{\theta}^{(p)} = \begin{cases} \theta_{(1)}, & if\ p = 0 \\ \theta_{(i)}, & if\ \sum_{j=1}^{i-1} w_j < p < \sum_{j=1}^{i} w_j \end{cases} \tag{4.6}$$

Hence a (1-p)100% credible interval for θ can be obtained as

$$R_j = (\hat{\theta}^{(j/N)}, \hat{\theta}^{(\frac{j+(1-p)N}{N})})\ \ \text{j=1,2,…,[pN]} \tag{4.7}$$

Where [a] stands for largest integer less than or equal to a. Then among all the R_j's the interval with the smallest width becomes the HPD credible interval for θ. Similarly, it can be obtained for parameter λ.

5. SIMULATION

In this section, a simulation study has been organized to check the performance of the estimators obtain under MLE and Bayes estimation (Kundu, 2007; Panahi and Sayyareh, 2014). The samples were generated under a Type-I generalized hybrid censoring scheme for N=1000 times using the following algorithm.

Algorithm 3:

1) Generate n independent uniform random numbers say x_1, x_2… x_n within (0, 1).
2) Fix the values for θ, λ, k, T, r, and n and then generate the value of x based on the uniform random numbers in step 1) using$x = [1-(1-r)^{1/\theta}]^{1/\lambda}$.
3) Compare the value of x_i, i=1,2,……..,n. with T and further check the following conditions.
    a. If x_r< T than put U=$x_{(r)}$ and m=r as well.

b. If x_k>T than put U=$x_{(k)}$ and m=k as well.
c. If x_k<T<x_r than put U=T and m=d.

We have fixed the values as $T = \{0.95, 0.80\}$, $r = \{34,36,37,40\}$, $k = \{25,30,35\}$ and n=40 and n= 120 also the results are occupied for the values of prior parameters (a_1, b_1) and (a_2, b_2) decided according to the method described in section 3 as $a_1 = 11, b_1 = 5, a_2 = 21$ and $b_2 = 5$.
The estimate of the parameters are considered as an average value of the estimates obtained in 1000 simulations and MSE of the estimator is obtained as,

$$MSE(\hat{\theta}_{Bayes}) = \frac{\sum_{i=1}^{1000}(\hat{\theta}_{Bayes} - \theta)^2}{1000}$$

Table 1. Estimates and MSEs for the parameters when T = 0.95, n =40.

r	k	MLE		Bayes- Proper priors		Bayes – Jeffery's Prior	
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
40	25	2.18442 0.46912	4.13642 0.66910	1.95065 0.06808	3.73579 0.18498	1.70599 0.09305	3.38558 0.26143
40	30	2.28954 0.50370	4.31000 0.74471	2.00031 0.09756	3.77943 0.17391	1.77319 0.17494	3.41567 0.25938
40	35	2.11221 0.77269	4.03078 0.52237	1.93047 0.09758	3.64203 0.16455	1.71181 0.07554	3.27980 0.15269
34	25	2.42824 0.79037	4.40758 0.97218	1.93168 0.07014	3.67974 0.14505	1.58400 0.09222	3.22385 0.25847
36	25	2.21156 0.56136	4.03989 0.59029	1.91317 0.07052	3.75387 0.15830	1.59738 0.07110	3.11539 0.17642
37	25	2.30236 0.95245	4.29541 0.83035	1.90801 0.08914	3.73220 0.15130	1.68296 0.07830	3.32527 0.26702

The results are shown in Table 1 to Table 4, the first entry denotes the estimates, and the second denotes the MSE of the estimate. Confidence intervals based on MLE and Bayes credible intervals for parameters θ and λ are respectively displaying in the first and second entry of Table 5 to Table 8. Simulation results for θ = 2 and λ = 4 with $a_1 = 11, b_1 = 5, a_2 = 21$ and $b_2 = 5$.

Table 2. Estimates and MSEs for the parameters when T = 0.80, n = 40.

r	k	MLE		Bayes- Proper priors		Bayes – Jeffery's Prior	
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
40	25	2.42436 1.10386	4.13642 0.66910	1.82167 0.06461	3.45461 0.16180	1.31782 0.06665	2.82403 0.22538
40	30	2.4709 1.00997	4.42384 1.07496	1.86943 0.09446	3.54340 0.15734	1.44187 0.08108	3.03802 0.26967
40	35	2.20720 0.58370	4.02987 0.56938	1.89535 0.06918	3.53557 0.1248	1.56200 0.06565	3.06949 0.17009
34	25	2.35683 0.79264	4.37781 0.92311	1.79096 0.09459	3.48389 0.14692	1.32072 0.07770	2.83721 0.20305
36	25	2.27996 0.50403	4.14679 0.73619	1.75514 0.06693	3.36513 0.13127	1.35454 0.08649	2.87396 0.23052
37	25	2.12468 0.95245	4.09389 0.76770	1.74718 0.08780	3.41235 0.15012	1.30445 0.10284	2.80894 0.23257

Table 3. Estimates and MSEs for the parameters when T = 0.95, n =120.

r	k	MLE		Bayes- Proper priors		Bayes – Jeffery's Prior	
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
40	25	2.32001 0.40076	4.19410 0.62252	1.18804 0.03756	2.61972 0.07799	0.80066 0.01856	2.14581 0.08062
40	30	2.51243 0.41759	4.24893 0.56395	1.15545 0.03678	2.57413 0.05124	0.79104 0.01479	2.11019 0.05817
40	35	2.35990 0.64972	4.18387 0.40228	1.16653 0.03024	2.59826 0.05369	0.81401 0.01894	2.13480 0.06254

34	25	2.59682 0.65012	4.27995 0.57247	1.14745 0.03598	2.59693 0.07902	0.70377 0.01653	2.04911 0.08061
36	25	2.69848 0.31649	4.31098 0.47222	1.14371 0.03438	2.57697 0.05957	0.73975 0.01329	2.04820 0.06186
37	25	2.42906 0.71754	4.22033 0.42166	1.14708 0.03204	2.59353 0.05297	0.77386 0.01428	2.11241 0.06277

Table 4. Estimates and MSEs for the parameters when T = 0.80, n =120.

r	k	MLE		Bayes- Proper priors		Bayes – Jeffery's Prior	
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
40	25	2.32001 0.70075	4.19410 0.62251	1.18804 0.03756	2.61972 0.07799	0.80060 0.01856	2.14581 0.08062
40	30	2.51243 0.41759	4.24893 0.53395	1.15545 0.03678	2.57413 0.05124	0.79104 0.01479	2.11019 0.05817
40	35	2.35991 0.54972	4.18387 0.48228	1.16653 0.03024	2.59826 0.05369	0.81401 0.01894	2.13480 0.06254
34	25	2.59682 0.48301	4.27995 0.57247	1.14745 0.03598	2.59693 0.07902	0.70377 0.01653	2.04911 0.08061
36	25	2.69848 0.61649	4.31098 0.47222	1.14371 0.03438	2.57697 0.05957	0.73975 0.01329	2.04820 0.06186
37	25	2.42906 0.49975	4.22033 0.42166	1.14708 0.03204	2.59353 0.05297	0.77386 0.01428	2.11241 0.06277

Table 5. Confidence intervals and credible intervals when T = 0.95, n= 40.

r	k	MLE	Bayes- Proper priors	Bayes- Jeffery's prior
40	25	(0.84197, 3.52688) (2.53316, 5.73968)	(1.83291, 2.39615) (3.54408, 4.26345)	(1.56267, 2.09618) (3.15663, 4.36354)
40	30	(0.89848, 3.68061) (2.61861, 6.00146)	(1.74557, 2.27029) (3.36020, 4.20350)	(1.54403, 2.02407) (3. 20173, 4.39522)
40	35	(0.91565, 3.30875) (2.61418, 5.44738)	(1.79176, 2.18373) (3.51356, 4.16924)	(1.07046, 2.85355) (2.35819, 4.51927)
34	25	(0.68575, 4.17074) (2.47503, 6.34013)	(1.79285, 2.77098) (3.46239, 4.93397)	(1.36229, 2.81547) (2.80965, 4.61308)
36	25	(0.74305, 3.68007) (2.53401, 5.54577)	(1.75996, 2.33937) (3.75526, 4.50948)	(1.24630, 2.47870) (2.43824, 4.29829)
37	25	(0.38952, 4.21520) (2.50938, 6.08144)	(1.89198, 2.62965) (3.75526, 4.50948)	(1.14303, 2.24790) (2.45572, 4.22218)

Table 6. Confidence intervals and credible intervals when T = 0.80, n = 40.

r	k	MLE	Bayes- Proper priors	Bayes- Jeffery's prior
40	25	(0.36509, 4.48363) (2.53316, 5.73968)	(1.54334,2.36524) (3.04963,4.20753)	(1.19752, 2.68068) (2.67180, 4.32040)
40	30	(0.50114, 4.44065) (2.39170,6.45591)	(1.43840,2.25081) (2.79346,4.21894)	(0.96147,2.92629) (2.12656,4.01773)
40	35	(0.70975, 3.70464) (2.55090, 5.50885)	(1.71613, 2.28662) (3.22675, 4.22206)	(1.20767, 2.73799) (2.44453, 4.39503)
40	25	(0.61184, 4.10183) (2.49466, 6.26096)	(1.41674, 2.39339) (2.95917, 4.25128)	(0.92715,2.29198) (2.19306, 4.38491)

36	25	(0.61179, 3.94812) (2.46507, 5.82850)	(1.19973, 2.38471) (2.73034, 4.50189)	(0.85838, 2.62401) (2.23894, 4.35397)
37	25	(0.73316, 3.51619) (2.37656, 5.81122)	(1.26407, 2.14956) (2.66176, 3.85305)	(1.16648, 2.48693) (2.73229, 4.33356)

Table 7. Confidence intervals and credible intervals when T = 0.95, n = 120.

r	k	MLE	Bayes- Proper priors	Bayes- Jeffery's prior
40	25	(0.85705, 4.03307) (2.85070, 6.05584)	(1.77248, 2.54293) (3.44487, 4.20237)	(1.06507,2.79861) (2.10601,4.07924)
40	30	(0.86712, 3.65479) (2.72825, 5.74551)	(1.55643, 2.20513) (3.44487, 4.20237)	(1.06507,2.79861) (2.10601,4.07924)
40	35	(0.65981, 3.86125) (2.64177, 5.83010)	(1.60505, 2.17821) (3.22741, 4.18274)	(1.34141, 2.74885) (2.68719,4.06182)
34	25	(0.65981, 3.86125) (2.64177, 5.83010)	(1.60505, 2.70882) (3.22741, 4.41310)	(1.13096, 2.06171) (2.46594, 4.58341)
36	25	(0.72139, 3.71088) (2.67280, 5.52892)	(1.53897, 2.37289) (3.01628, 4.17536)	(1.17903,2.42171) (2.53286,4.17412)
37	25	(0.55278, 3.89866) (2.50154, 5.85258)	(1.57748, 2.65303) (3.21536, 4.33088)	(0.94885, 2.40843) (2.06059, 4.18653)

Table 8.Confidence intervals and credible intervals when T = 0.80, n= 120.

r	k	MLE	Bayes- Proper priors	Bayes- Jeffery's prior
40	25	(0.46660, 3.82261) (2.47700, 5.68802)	(1.48526, 2.13667) (3.13991, 4.27169)	(1.51620, 3.08651) (2.34185,4.06537)
40	30	(0.85037, 3.39019) (2.59627, 5.46357)	(1.52985, 2.19236) (3.05102, 4.05864)	(1.27377, 2.59151) (2.69560, 4.02698)
40	35	(0.91351, 4.08140) (2.86102, 6.12877)	(1.49139, 2.04396) (3.06828, 3.69372)	(1.16006, 2.51751) (2.47051, 4.05922)
34	25	(0.76924, 3.53323) (2.59625, 5.58796)	(1.12740, 2.62190) (2.71431, 4.21318)	(0.99492, 2.12613) (2.20125, 4.19555)
36	25	(0.58353, 3.73242) (2.63826, 5.38910)	(1.12528, 2.34460) (2.54232, 4.12295)	(0.82836, 2.44163) (2.55920, 4.09252)
37	25	(0.83641, 3.79801) (2.82078, 5.81973)	(1.31079, 2.29273) (2.87812, 4.37771)	(1.31079, 2.54273) (2.63819, 4.01133)

6. REAL DATA

Example 1. (Real Data) In this example, we analyze a real data set, considered by Sulatana et al. (2018), which represents the monthly water capacity from the Shasta reservoir in California, USA, and data are recorded for the month of February from 1991 to 2010. For further details about the data, one may visit http://cdec.water.ca.gov/reservoir_map.html. The maximum capacity of the reservoir is 4552000 AF. The data points are listed below as follows:

0.338936, 0.431915, 0.759932, 0.724626, 0.757583, 0.811556, 0.785339, 0.783660, 0.815627, 0.847413, 0.768007, 0.843485, 0.787408, 0.849868, 0.695970, 0.842316, 0.828689, 0.580194, 0.430681, 0.742563

They have shown that the Kumaraswamy distribution fits the data set well.

We compute MLEs, Bayes estimates using informative as well as non-informative priors based on the results derived in this paper.

In Table 9, we have reported estimates of θ and λ of the proposed estimators as well as Table 10 and Table 11 to reveal confidence intervals and credible intervals for the parameters θ and λ in the two entries of each cell respectively for T = 0.80 and 0.82.

Table 9. Maximum Likelihood and Bayes Estimates.

T	r	k	MLE		Bayes- Proper priors		Bayes- Jeffrey's priors	
			$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
0.80	13	10	2.30800	4.76859	1.89551	4.58570	1.49263	3.38354
	15	10	2.05892	4.54667	1.84609	3.79850	1.19288	3.52303
	17	10	2.05892	4.54667	1.88287	4.23318	1.53248	3.69812
	17	12	2.05892	4.54667	3.03159	4.57821	2.07009	3.88257
	17	15	2.57580	5.00753	1.95977	3.84646	1.40896	3.06031
0.82	13	10	2.30800	4.76859	1.93151	4.58570	1.38022	2.96578
	15	10	2.57580	5.00753	1.95977	3.84646	2.07637	4.31318
	17	10	2.48981	4.93572	1.98839	4.27362	1.56178	3.73311
	17	12	2.48981	4.93572	3.13131	4.61304	2.39814	4.14339
	17	15	2.48981	4.93572	1.94581	3.84433	1.39161	3.05131

Table10. Confidence intervals and credible intervals when T = 0.80.

r	k	MLE	Bayes- Proper priors	Bayes-Jeffery's prior
17	10	(0.26413,4.51635) (2.33612,5.78991)	(1.34347,2.32465) (3.14736,4.23578)	(1.27852,2.47829) (2.43165,4.35241)
17	12	(0.41712,4.50171) (2.13671,6.29147)	(1.24710,2.18331) (2.89617,4.25637)	(0.99741,2.67273) (2.23948,4.01463)
17	15	(0.52826,3.86121) (2.33716,5.19321)	(1.54625,2.24573) (3.34418,4.25691)	(1.28993,2.52236) (2.54489,4.39647)
13	10	(0.42723,3.66781) (2.03891,4.79864)	(1.34108,2.23812) (3.12716,4.50981)	(0.91328,2.56891) (2.43712,4.32811)
15	10	(0.57128,3.56892) (2.00876,4.77468)	(1.54987,2.13995) (2.76136,3.88305)	(1.17842,2.36997) (2.73886,4.33705)

Table11 . Confidence intervals and credible intervals when T = 0.82.

r	k	MLE	Bayes	Bayes-Jeffery's prior
17	10	(0.76781,4.71882) (2.35621,5.68989)	(1.14080,2.30117) (3.04631,4.21981)	(1.07899,3.01135) (2.73789,4.31478)
17	12	(0.58761,4.70781) (2.10885,6.57893)	(1.78741,2.39817) (3.17867,4.21779)	(1.57869,2.19974) (3.20368,4.36955)
17	15	(0.79994,3.83771) (2.19447,5.33998)	(1.79166,2.18337) (3.67663,4.18944)	(1.08871,2.59776) (2.35866,4.52417)
13	10	(0.74557,3.68119) (2.49116,5.11709)	(1.75677,2.33744) (3.75652,4.08441)	(1.00121,2.68171) (2.15595,4.29881)
15	10	(0.47889,3.48297) (2.18993,5.81443)	(1.89978,2.48996) (3.75512,4.50375)	(1.15247,2.18714) (2.38775,4.22911)

7. CONCLUSION

In this article, we have obtained MLEs and Bayes estimates of unknown parameters θ and λ of a Kumaraswamy distribution based on Type-I generalized hybrid censoring scheme. Bayes estimates are computed by using importance sampling under the squared error loss function. The asymptotic confidence intervals and credible intervals are also obtained. A numerical example is also analyzed using the proposed methods of estimation. The proposed methodology may be useful to the practitioners for a practical purpose where such type of situation arises.

From Tables 1 to 4, we observed that as n increases, MSEs decrease in the case of MLEs, as well as Bayes estimators but Bayes estimators, perform well compared to MLEs concerning their MSEs. The Bayes estimators under Jeffery's prior produce an underestimate compared to MLEs and Bayes estimates under proper prior in case of both the parameters.

From Tables 5 to 8 we found that Bayes credible intervals give a smaller length of the parameters compared to confidence intervals based on MLEs. In the Bayesian setup, credible intervals based on gamma priors compete quite well with Jeffery's priors. As k increases for a fixed value of n, T, and r, the length of the confidence interval under Bayesian estimation with a gamma prior decreases, but the length of confidence interval get fluctuated when r increases for a fixed value of n,T,k in case of Bayesian estimation as well as MLEs.

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REFERENCES

[1] BALAKRISHNAN, N. AND KUNDU, D. (2013): Hybrid censoring: Models, inferential results and applications. **Computational Statistics & Data Analysis,** 57,166-209.

[2] CHANDRASEKAR, B., CHILDS, A. AND BALAKRISHNAN, N. (2004): Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring. **Naval Research Logistics,** 51, 994 – 1004.

[3] CHILDS, A., CHANDRASEKAR, B., BALAKRISHNAN, N. AND KUNDU, D.(2003): Exact likelihood inference based on an unified hybrid censored sample from the exponential distribution. **Annals of the institute of statistics mathematic**, 55, 319-330.

[4] CORDEIRO, G. M., ORTEGA, E. M. AND NADARAJAH, S. (2010): The Kumaraswamy Weibull distribution with application to failure data. **Journal of the Franklin Institute**, 347, 1399-1429.

[5] EPSTEIN, B.(1954): Truncated life test in the exponential case. **Annals of the institute of statistics mathematics,** 25, 555-564.

[6] GARG, M.(2009): On generalized order statistics from Kumaraswamy distribution. **Tamsui Oxford Journal of Mathematical Sciences**, 25, 153- 166.

[7] JONES, M. C.(2009): Kumaraswamy's distribution: A beta-type distribution with some tractability advantages.**Statistical Methodology**, 6,70-81.

[8] KUMARASWAMY, P.(1980): A generalized probability density function for double-bounded random processes. **Journal of Hydrology**,46,79-88.

[9] KUNDU, D. (2007): On hybrid censored Weibull distribution. **Journal of Statistical Planning and Inference**, 137,2127-2142.

[10] KUNDU, D. AND PRADHAN, B. (2009[a]): Estimating the Parameters of the Generalized Exponential Distribution in Presence of Hybrid Censoring.**Communications in Statistics-Theory and Methods**, 38 , 2030-2041.

[11] KUNDU, D. AND PRADHAN, B. (2009[b]): Bayesian inference and life testing plans for generalized exponential distribution. **Sci China Ser A Math**, 52,1373–1388.

[12] NADAR, M., PAPADOPOULOS, A. AND KIZILASLAN, F.(2013): Statistical analysis for Kumaraswamy's distribution based on record data. **Stat Papers** ,54, 355–369.

[13] PANAHI, H. AND SAYYAREH, A. (2014): Parameter estimation and prediction of order statistics for the Burr Type XII distribution with Type II censoring. **Journal of Applied Statistics**, 41,215-232.

[14] PONNAMBALAM, K., SEIFI, A. AND VLACH, J. (2001): Probabilistic design of systems with general distribution of parameters. **International Journals on Circuit Theory Applications**, 29, 527-536.

[15] RAYAD, H. M. AND AHMED, S. (2016): Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on type-II censoring. **International Journal of Advanced Mathematical Sciences**,4, 10-17.

[16] SAYED-AHMED. N., JAWA, T. M., ALOAFI, T. A., BAYONES, F.S., ELHAG, A.A., BOUSLIMI, J. AND ABD-ELMOUGOD, G.A. (2021): Generalized Type-I Hybrid Censoring Scheme in Estimation Competing Risks Chen Lifetime Populations. **Mathematical Problems in Engineering,** Hindawi, vol. 2021, 1-12.

[17] SINDHU, T. N., FEROZE, N. AND ASLAM, M. (2013): Bayesian analysis of Kumaraswamy distribution under failure censoring sample scheme. **International Journal of Advanced Science and Technology**, 51, 39-58.

[18] SULTANA, F., MANI, Y., TRIPATHI, Y.M., RASTOGI M. K. AND WU, S. (2018): Parameter Estimation for the Kumaraswamy Distribution Based on Hybrid Censoring. **American Journal of Mathematical and Management Sciences**, 37, 243-261.

[19] GOYAL, T., RAI, P.K. AND MAURYA, S. (2020): Bayesian Estimation for GDUS Exponential Distribution Under Type-I Progressive Hybrid Censoring. **Annals of Data Science**, 7, 307–345.