EXPONENTIAL ESTIMATORS FOR FINITE POPULATION MEAN USING AUXILIARY ATTRIBUTE: A PREDICTIVE APPROACH

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ABSTRACT
In this paper the problem of estimating the finite population mean using predictive approach has been deliberated when information on auxiliary attribute is available. Singh et al. (2007): ratio and product type exponential estimators are used as a predictor of mean of unobserved elements of the population to propose product and ratio type exponential estimators. The bias and mean square error (MSE): of projected estimators have been acquired up to the first order of approximation and shown that the proposed estimators are less biased and more effective than the standing estimators. A practical analysis is conceded to sustenance the theoretical outcomes.

KEYWORDS: Predictive approach, Study variable, Attribute, Exponential estimators, Point bi-serial correlation, Bias, Mean square error, Efficiency.

MSC: 62D05

RESUMEN
En este paper el problema de estimar la media de una población finita usando un enfoque predictivo es discutido cuando existe la información sobre el atributo auxiliar. Los estimadores exponenciales del tipo razón y producto de Singh et al. (2007): son utilizados como predictores de la media de los elementos de la población no observados. El sesgo y el error cuadrático medio (MSE): de los proyectados estimadores se han obtenido hasta el orden de aproximación de primer orden y se muestra que esos son menos sesgados y más efectivos que los existentes estimadores. Un análisis práctico es desarrollado para sostener los resultados teoréticos.

PALABRAS CLAVE: Enfoque Predictivo, Variable de Estudio, Atributo, Estimadores Exponenciales, Correlación Bi-Serial, Sesgo, Error Cuadrático Medio, Eficiencia.

1. INTRODUCTION

In the existing literature of sampling theory, it is well established fact that proper implementation of auxiliary information gives more efficient estimators when these are correlated with study variable for estimating the population mean. There are several applied situations, where auxiliary information appears in qualitative nature and it is correlated with study variable. The examples of this state are: sex is considered a good auxiliary attribute in estimating the height of persons, in milk and wheat production estimation bread of animal and variety of crop may be considered as auxiliary variables [*see Naik and Gupta (1996):, Jhajj et al. (2006):]* etc.. In all the above mentioned examples bi-serial correlation exists between auxiliary attribute and study variable. A great variety of approaches/techniques is available to construct more efficient design based and model based methods for using auxiliary information (variable/attribute):. For estimation of population mean of study variable using auxiliary information on model based approach is also notorious as predictive approach. Predictive approach for sampling surveys may be deliberated as common structure for statistical inference on the character of finite population. In this methodology, we develop a model for population observations and use it to predict the non-sampled observations. A lot of researcher used many existing estimators as a predictor for various kind of models. Some researcher used ratio, product and regression type estimators under predictive approach for population parameters when auxiliary information is present in from of variables. Srivastava (1983): recommended the predictive approach for product estimator also exploited the prediction criterion given by Basu (1971): and confirms that succeeding and traditional product estimator of the mean of whole population are different if some existing estimator is used as a predictor. Some predictive estimators of variance have been developed by Biradar and Singh (1998):, Agarwal and Roy (1999):, Nayak

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and Sahoo (2012): and Saini (2013): for finite populations. Singh et al. (2014): proposed ratio and product type exponential estimators using predictive methodology in which the estimators developed by Bahl and Tuteja (1991): are used as predictor to estimate the population mean. Yadav and Mishra (2015): suggested enhanced ratio cum product type predictive estimators to estimate the population mean.
Motivated by Singh et al. (2014): and Yadav and Mishra (2015):, here we pursuit to investigate the existing Singh et al. (2007): exponential estimators under information on auxiliary attribute as predictor of the mean of the unobserved units of the population using the information of the observed units in the sample. The paper divided into six sections. Section 1 is introductory in nature and section 2 appended the material and methods. In section 3, the proposed estimators and their properties are developed. Bias and efficiency assessment among suggested and existing estimators are conceded out in section 4 and section 5. In section 6, we observe the performance of various estimators using real data sets. Last section provides some concluding remarks.

2. MATERIAL AND METHODS

Let us consider a finite (survey): population $\Delta = (\Delta_1, \Delta_2, ..., \Delta_N)$ which consists of N identifiable units. Let y and ξ represent the study variable and auxiliary attribute in present work. Where y_i and ξ_i be the observations on y and ξ for the i^{th} unit (i=1, 2, 3,……, N): respectively. Let the population and sample mean of the study variable be $\bar{Y} = \sum_{i=}^{N} y_i / N$ and $\bar{y} = \sum_{i=1}^{n} y_i / n$ respectively. Suppose that there is a broad contradiction in the population with respect to the existence or non-existence of an attribute say ξ which takes only binary values as defined below:

ξ = 1, if i^{th} unit of the population has existence of the specific attribute

= 0, if i^{th} unit of the population has non-existence of the specific attribute.

Let $n(< N)$ be the size of the sample selected using SRSWOR from a population. Let $U = \sum_{i=1}^{N} \xi_i$ and $u = \sum_{i=1}^{n} \xi_i$ denoted the aggregate number of observations in the population and sample, respectively, which resumes the attribute ξ. Let the corresponding population and sample proportions be $K = \sum_{i=1}^{N} \xi_i / N = U / N$ and $k = \sum_{i=1}^{n} \xi_i / n = u / n$ respectively. We are interested in estimating the population mean $\bar{Y}$ of the study variable on the basis of observed values of y on units in a sample from Δ.
Let $s \in S$ be any arbitrary element of the set of all possible samples selected from population Δ. Let $\gamma(s)$ represent the actual sample size that is the number of observations in s and $\bar{s}$ represent the assortment of all those items of Δ which are not is s. We label

$$\bar{Y}_s = \frac{1}{\gamma(s)} \sum_{i \in s} y_i \quad \text{and} \quad \bar{Y}_{\bar{s}} = \frac{1}{N - \gamma(s)} \sum_{i \in \bar{s}} y_i \tag{1}$$

Under the usual predictive set-up, it is possible to express $\bar{Y}$ as:

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i = \frac{1}{N} \left[\sum_{i \in s} y_i + \sum_{i \in \bar{s}} y_i \right] \tag{2}$$

For any given $s \in S$, using equation (1):,(2):, can be written as:

$$\bar{Y} = \frac{\gamma(s)}{N}\bar{Y}_s + \frac{[N-\gamma(s)]}{N}\bar{Y}_{\bar{s}} \qquad (3)$$

According to Basu (1971): in the presentation of $\bar{Y}$, the sample mean $\bar{Y}_s$, is known. So, we should always efforts to predict the mean $\bar{Y}_{\bar{s}}$ of the unobserved items of the population using observed units of sample s. He never considered such an approach to represent the "heart of the matter" in estimating the finite population mean due to possible objections from a decision-theorist to making the choice of estimator after observing the data. [*see Cassel et al.(1977,p.110):*].

In SRS procedure, the sample mean for the sample of size n $(i.e\ \gamma(s) = n)$ is

$$\bar{Y}_s = \frac{1}{n}\sum_{i\in s} y_i = \bar{y} \quad \text{and} \quad \bar{Y}_{\bar{s}} = \frac{1}{N-n}\sum_{i\in \bar{s}} y_i \qquad (4)$$

Using equation (4):, $\bar{Y}$ in equation (3): can be written as

$$\bar{Y} = \frac{n}{N}\bar{y} + \frac{(N-n)}{N}\bar{Y}_{\bar{s}} \qquad (5)$$

In the light of the equation (5):, an appropriate estimator of population mean $\bar{Y}$ can be transcribed as

$$p = \left[\frac{n}{N}\bar{y} + \frac{(N-n)}{N}P\right] \qquad (6)$$

where P is painstaking as a predictor of $\bar{Y}_{\bar{s}}$.

If we implement the prediction approach, proposed by Srivastava (1983):, for the population mean $\bar{Y}_{\bar{s}}$ of the unobserved items of the population as follows:

If no additional information is provided on Δ, then an obvious choice of P is $\bar{y}$ i.e.

$$P = \frac{1}{n}\sum_{i=\bar{s}} y_i = \bar{y} \qquad (7)$$

Using equation (7):, we can write equation (6): as

$$p = \left[\frac{n}{N}\bar{y} + \frac{(N-n)}{N}\bar{y}\right] = \bar{y}$$

the routine mean per unit estimator of $\bar{Y}$.

Captivating the point bi-serial coefficient of correlation between both auxiliary attribute and study variable, Naik and Gupta (1996): proposed ratio estimator of population mean when the former information of population proportion of units, holding the same attribute exists. Ratio estimator proposed by Naik and Gupta (1996): in the form of an obvious choice of P is

$$P = \Sigma_{R\bar{s}} = \left(\frac{\bar{y}}{k}\right)K_{\bar{s}}$$

where $k = \frac{1}{n}\sum_{i=s}\xi_i$, $K = \frac{1}{N}\sum_{i=1}^{N}\xi_i$ and $K_{\bar{s}} = \frac{1}{N-n}\sum_{i\in\bar{s}}\xi_i = \frac{NK-nk}{N-n}$

For this choice of P,

$$p = \frac{n}{N}\bar{y} + \frac{(N-n)}{N}\left(\frac{\bar{y}}{k}\right)K_{\bar{s}}$$

$$p = \frac{n}{N}\bar{y} + \frac{N-n}{N}\frac{NK-nk}{N-n}\left(\frac{\bar{y}}{k}\right)$$
$$= \left(\frac{\bar{y}}{k}\right)K$$

It is proposed by Naik and Gupta (1996): as a ratio estimator of $\bar{Y}$.

If the information on the auxiliary attribute ξ is negatively correlated with study variable y is available and one propose to use this information in the form of product estimator in the form of Naik and Gupta (1996):, an obvious choice of P is

$$P = \Sigma_{\beta\%} = \left(\frac{\bar{y}}{K_{g\%}}\right)k$$

For the obvious choice of P,

$$p = \frac{n}{N}\bar{y} + \frac{N-n}{N}\frac{\bar{y}\,k}{K_{g\%}}$$
$$p = \frac{n}{N}\bar{y} + \frac{N-n}{N}\frac{N-n}{NK-nk}\bar{y}\,k$$
$$p = \bar{y}\,\frac{NK+(N-2n)k}{NK-nk} = \Sigma_{Pp}$$

which is not the existing product estimator $\Sigma_P = \left(\frac{\bar{y}}{K}\right)k$ of $\bar{Y}$ proposed by Naik and Gupta (1996):.

Thus we find that if we adopt the prediction approach proposed by Srivastava (1983):, use of mean per unit, ratio estimator for predicting the mean $\bar{Y}_{g\%}$ of the unobserved units of the population outcomes in the usual estimators of the population mean $\bar{Y}$. However if the product estimator is used with such an approach, the resulting estimator of $\bar{Y}$ is not the customary product estimator Σ_P .

The biases and mean squared errors of the estimators Σ_R , Σ_P and Σ_{Pp} , up to the first order of approximations are found as:

$$Bias(\Sigma_{\mathrm{R}}) = \lambda\bar{Y}C_k^2(1-\tau) \tag{8}$$

$$Bias(\Sigma_P) = \lambda\bar{Y}C_k^2\tau \tag{9}$$

$$Bias(\Sigma_{Pp}) = \lambda\bar{Y}C_k^2(\tau + f(1-f)^{-1}) \tag{10}$$

$$MSE(\Sigma_{\mathrm{R}}) = \lambda\bar{Y}^2\left[C_y^2 + C_k^2(1-2\tau)\right] \tag{11}$$

$$MSE(\Sigma_P) = MSE(\Sigma_{Pp}) = \lambda\bar{Y}^2\left[C_y^2 + C_k^2(1+2\tau)\right] \tag{12}$$

where $\lambda = n^{-1}(1-f)$, $f = ({}^{n}\!/_{N})$ (sample fraction):, $C_y^2 = {}^{S_y^2}\!/_{\bar{Y}^2}$ (population coefficient of variation of y):, $S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})$ (population mean square of y):, $C_k^2 = {}^{S_k^2}\!/_{K^2}$ (population coefficient of variation of ξ):, $S_k^2 = (N-1)^{-1}\sum_{i=1}^{N}(\xi_i - K)^2$ (population mean square of ξ):, $\tau = \rho({}^{C_y}\!/_{C_k})$,

$\rho = S_{yk}/(S_y S_k)$ (bi-serial correlation coefficient between y and ξ):, $S_{yk} = \sum_{i=1}^{N}(y_i - \bar{Y})(\xi_i - K)$ **(**population covariance between y and ξ):.

Following Bahl and Tuteja (1991):, Singh et al. (2007): suggested the following ratio and product type exponential estimators using information on auxiliary attribute

$$\Sigma_{\mathrm{Re}} = \bar{y}\exp\left(\frac{K-k}{K+k}\right) \tag{13}$$

$$\Sigma_{P\mathrm{e}} = \bar{y}\exp\left(\frac{k-K}{k+K}\right) \tag{14}$$

The biases and mean squared errors of Σ_{Re} and $\Sigma_{P\mathrm{e}}$ up to the first degree of approximation are

$$Bias(\Sigma_{\mathrm{Re}}) = \frac{\lambda}{8}\bar{Y}C_k^2(3-4\tau) \tag{15}$$

$$Bias(\Sigma_{P\mathrm{e}}) = \frac{\lambda}{8}\bar{Y}C_k^2(4\tau-1) \tag{16}$$

$$MSE(\Sigma_{\mathrm{Re}}) = \frac{\lambda}{8}\bar{Y}^2\left[C_y^2 + \frac{C_k^2}{4}(1-4\tau)\right] \tag{17}$$

$$MSE(\Sigma_{P\mathrm{e}}) = \frac{\lambda}{8}\bar{Y}^2\left[C_y^2 + \frac{C_k^2}{4}(1+4\tau)\right] \tag{18}$$

In this paper we projected the ratio and product type exponential estimators of population mean $\bar{Y}$ by using Σ_{Re} and $\Sigma_{P\mathrm{e}}$ as a predictor P of $\bar{Y}_{\%}$ of the unobserved units of the population Δ on the bases of observed units in s. The bias and mean squares error of proposed ratio and product type exponential estimators up to the first order of approximation have obtained.

3. PROPOSED ESTIMATORS AND THEIR PROPERTIES

In the occasion, when both auxiliary attribute and study variable are positively correlated and information on an auxiliary attribute ξ is available, one can plan to use this in the form of Singh et al. (2007): ratio type exponential estimator Σ_{Re}, an obvious choice of P is:

$$\Sigma_{\mathrm{Re}}^{\%} = \bar{y}\exp\left(\frac{K_{\%}-k}{K_{\%}+k}\right)$$

For this choice of P:

$$p = p_{\mathrm{Re}} = \left[\frac{n}{N}\bar{y} + \left(\frac{N-n}{N}\right)\bar{y}\exp\left(\frac{K_{\%}-k}{K_{\%}+k}\right)\right]$$

$$p = p_{\mathrm{Re}} = \left[\frac{n}{N}\bar{y} + \left(\frac{N-n}{N}\right)\bar{y}\exp\left(\frac{N(K-k)}{N(K-k)-2nk}\right)\right] \tag{19}$$

which is not the Singh et al.(2007): ratio type exponential estimator using auxiliary attribute.
If the information of auxiliary attribute ξ is negatively correlated with the study variable y and one wants to use this information in the form of Singh et al (2007): product type exponential estimator $\Sigma_{P\mathrm{e}}$ for an obvious choice of P is:

$$\Sigma_{P\mathrm{e}}^{\%} = \bar{y}\exp\left(\frac{k-K_{\%}}{k+K_{\%}}\right)$$

$$p = p_{P\mathrm{e}} = \left[\frac{n}{N}\bar{y} + \left(\frac{N-n}{N}\right)\bar{y}\exp\left(\frac{k-K_{\%}}{k+K_{\%}}\right)\right] = \left[\frac{n}{N}\bar{y} + \left(\frac{N-n}{N}\right)\bar{y}\exp\left(\frac{N(k-K)}{NK+(N-2n)k}\right)\right] \tag{20}$$

which is not the Singh et al.(2007): product type exponential estimator using auxiliary attribute.
To obtain the bias and mean square error of p_{Re} up to the first order of approximation, we define:

$$\pi_0=\left(\frac{\bar{y}-\bar{Y}}{\bar{Y}}\right) \quad \text{and} \quad \pi_1=\left(\frac{k-K}{K}\right)$$

such that $E(\pi_0)=E(\pi_1)=0$ and up to the first order of approximation

$$E(\pi_0^2)=\lambda C_y^2 \qquad E(\pi_1^2)=\lambda C_k^2 \qquad E(\pi_0\pi_1)=\lambda\tau C_k^2$$

Expressing (19): in terms of $\pi's$, we have

$$p_{\mathrm{Re}}=\bar{Y}(1+\pi_0)\left[\frac{n}{N}+\left(\frac{N-n}{N}\right)\exp\left(\frac{N\pi_1}{2(N-n)+(N-2n)\pi_1}\right)\right]$$

$$p_{\mathrm{Re}}=\bar{Y}(1+\pi_0)\left[f+(1-f)\exp\left(-\frac{\pi_1}{2(1-f)+(1-2f)\pi_1}\right)\right]$$

$$p_{\mathrm{Re}}=\bar{Y}(1+\pi_0)\left[f+(1-f)\exp\left(-\frac{\pi_1}{2(1-f)}\left\{1+\frac{(1-2f)}{2(1-f)}\pi_1\right\}^{-1}\right)\right] \tag{21}$$

Escalating the right hand side of (21):, simplifying after multiplication and neglecting terms of $\pi's$ having power greater than two, we have:

$$p_{\mathrm{Re}}\approx\bar{Y}\left[1+\pi_0-\frac{\pi_1}{2}-\frac{\pi_0\pi_1}{2}+\frac{\pi_1^2}{8}(3-4f)\right]$$

$$(p_{\mathrm{Re}}-\bar{Y})\approx\bar{Y}\left[\pi_0-\frac{\pi_1}{2}-\frac{\pi_0\pi_1}{2}+\frac{\pi_1^2}{8}(3-4f)\right] \tag{22}$$

Taking expectation of both sides, we get the bias of p_{Re} up to the first order of approximation as:

$$Bias(p_{\mathrm{Re}})=E(p_{\mathrm{Re}}-\bar{Y})\approx\bar{Y}\,E\left[\pi_0-\frac{\pi_1}{2}-\frac{\pi_0\pi_1}{2}+\frac{\pi_1^2}{8}(3-4f)\right]$$

$$Bias(p_{\mathrm{Re}})=\frac{\lambda}{8}\bar{Y}C_k^2\left[3-4(\tau+f)\right] \tag{23}$$

Squaring both sides of (22): and neglecting terms of $\pi's$ having power greater than two, we have:

$$(p_{\mathrm{Re}}-\bar{Y})^2\approx\bar{Y}^2\left[\pi_0^2+\frac{\pi_1^2}{4}-\pi_0\pi_1\right]$$

Taking expectation of both sides, we get the MSE of p_{Re} up to the first order of approximation as:

$$MSE(p_{\mathrm{Re}})=E(p_{\mathrm{Re}}-\bar{Y})^2\approx\bar{Y}^2E\left[\pi_0^2+\frac{\pi_1^2}{4}-\pi_0\pi_1\right]$$

$$MSE(p_{\mathrm{Re}})=\lambda\bar{Y}^2\left[C_y^2+\frac{C_k^2}{4}(1-4\tau)\right]$$

(24):

which is equal to the MSE of ratio type exponential estimator proposed by Singh et al. (2007).

Now expressing in terms of $\pi's$, we have

$$p_{Pe}=\bar{Y}(1+\pi_0)\left[\frac{n}{N}+\left(\frac{N-n}{N}\right)\exp\left(\frac{N\pi_1}{2(N-n)+N\pi_1}\right)\right]$$

$$p_{Pe}=\bar{Y}(1+\pi_0)\left[f+(1-f)\exp\left(-\frac{\pi_1}{2(1-f)+\pi_1}\right)\right]$$

$$p_{Pe}=\bar{Y}(1+\pi_0)\left[f+(1-f)\exp\left(\frac{\pi_1}{2(1-f)}\left\{1+\frac{\pi_1}{2(1-f)}\right\}^{-1}\right)\right]$$

(25):

Expanding the right hand side of (25):, multiplying out and neglecting terms of $\pi's$ having power greater than two we have:

$$p_{Pe} \approx \bar{Y}\left[1+\pi_0+\frac{\pi_1}{2}+\frac{\pi_0\pi_1}{2}-\frac{\pi_1^2}{8(1-f)}\right]$$

$$(p_{Pe}-\bar{Y}) \approx \bar{Y}\left[\pi_0+\frac{\pi_1}{2}+\frac{\pi_0\pi_1}{2}-\frac{\pi_1^2}{8(1-f)}\right] \quad (26)$$

Taking expectation of both sides, we get the bias of p_{Pe} up to the first order of approximation as:

$$Bias(p_{Pe}) = E(p_{Pe}-\bar{Y}) \approx \bar{Y}E\left[\pi_0+\frac{\pi_1}{2}+\frac{\pi_0\pi_1}{2}-\frac{\pi_1^2}{8(1-f)}\right]$$

$$Bias(p_{Pe}) = \frac{\lambda}{8}\bar{Y}C_k^2\left[4\tau-\frac{1}{(1-f)}\right] \quad (27)$$

Squaring both sides of (26): and neglecting terms of $\pi's$ having power greater than two, we have:

$$(p_{Pe}-\bar{Y})^2 \approx \bar{Y}^2\left[\pi_0^2+\frac{\pi_1^2}{4}+\pi_0\pi_1\right]$$

Taking expectation on both sides, we get the MSE of p_{Pe} up to the first order of approximation as:

$$MSE(p_{Pe}) = E(p_{Pe}-\bar{Y})^2 \approx \bar{Y}^2E\left[\pi_0^2+\frac{\pi_1^2}{4}+\pi_0\pi_1\right]^2$$

$$MSE(p_{Pe}) = \lambda\bar{Y}^2\left[C_y^2+\frac{C_k^2}{4}(1+4\tau)\right] \quad (28)$$

which is equal to the MSE of Singh et al. (2007): product type exponential estimator .

4. BIAS COMPARISON OF ESTIMATORS

Here we derive the conditions under which the proposed estimators p_{Re} (ratio-type exponential estimator):, p_{Pe} (product-type exponential estimator): are less biased to Σ_R (ratio estimator):, Σ_P (product estimator): proposed by Naik and Gupta (1996): and Σ_{Re} (ratio-type exponential estimator):, Σ_{Pe} (product-type exponential estimator): proposed by Singh et al.(2007):.

Bias Comparison of Ratio Estimators

From (23):, (15): and (8):, we have:

Condition (i): $\left|Bias(p_{Re})\right| < \left|Bias(\Sigma_R)\right|$ if and only if $\left|1-\tau\right| > \frac{1}{8}\left|3-4(\tau+f)\right|$ (29)

Condition (ii)::

$\left|Bias(p_{Re})\right| < \left|Bias(\Sigma_{Re})\right|$ if and only if $\left|3-4\tau\right| > \left|3-4(\tau+f)\right|$ (30)

Bias Comparison of Product Estimators

From (27):, (16):, (10): and (9):, we have:

Condition (i):: $\left|Bias(p_{Pe})\right| < \left|Bias(\Sigma_P)\right|$ if and only if $\left|1-\tau\right| > \frac{1}{8}\left|4\tau-\frac{1}{1-f}\right|$ (31)

Condition (ii):: $\left|Bias(p_{Pe})\right| < \left|Bias(\Sigma_{Pp})\right|$ if and only if $\left|\tau-\frac{f}{1-f}\right| > \frac{1}{8}\left|4\tau-\frac{1}{1-f}\right|$ (32)

Condition (iii):: $\left|Bias(p_{Pe})\right| < \left|Bias(\Sigma_{Pe})\right|$ if and only if $\left|4\tau-1\right| > \left|4\tau-\frac{1}{1-f}\right|$ (33)

5. Efficiency Comparison of Estimators

In this section we have obtained the conditions under which the proposed estimators p_{Re} (ratio-type exponential estimator):, p_{Pe} (product-type exponential estimator): are better than the Σ_R (ratio estimator):, Σ_P (product estimator): proposed by Naik and Gupta(1996): and Σ_{Re} (ratio-type exponential estimator):, Σ_{Pe} (product-type exponential estimator): proposed by Singh et al.(2007):.

It is well known that sample mean $\bar{y}$ is an unbiased estimator of population mean $\bar{Y}$ and under SRSWOR its variance (or mean square error): is given by

$$Var(\bar{y}) = MSE(\bar{y}) = \lambda \bar{Y} C_y^2 \tag{34}$$

Efficiency Comparison of Ratio Estimators

From (34):, (24):, (17): and (11):, we have:

Condition (i): $MSE(\Sigma_R) < MSE(\bar{y})$ if and only if $\tau > \frac{1}{2}$ (35)

Condition (ii):: $\{MSE(p_{\text{Re}}) = MSE(\Sigma_{\text{Re}})\} < MSE(\bar{y})$ if and only if $\tau > \frac{1}{4}$

(36):

Condition (iii):: $\{MSE(p_{\text{Re}}) = MSE(\Sigma_{\text{Re}})\} < MSE(\Sigma_R)$ if and only if $\tau < \frac{3}{4}$

(37):

Thus from (39): and (37): we conclude that the proposed estimator p_{Re} (ratio-type exponential estimator): and Σ_{Re} (ratio-type exponential estimator): proposed by Singh et al.(2007): is efficient than usual unbiased estimator $\bar{y}$, Σ_R (ratio estimator): proposed by Naik and Gupta(1996): for

$$\frac{1}{4} < \tau < \frac{3}{4}$$

Efficiency Comparison of Product Estimators

From (34):, (28): and (18):, we have:

Condition (i): $\{MSE(\Sigma_P) = MSE(\Sigma_{Pp})\} < MSE(\bar{y})$ if and only if $\tau < -\frac{1}{2}$

(38):

Condition (ii):: $\{MSE(p_{pe}) = MSE(\Sigma_{Pe})\} < MSE(\bar{y})$ if and only if $\tau < -\frac{1}{4}$

(39):

Condition (iii):: $\{MSE(p_{pe}) = MSE(\Sigma_{Pe})\} < MSE(\Sigma_P)$ if and only if $\tau > -\frac{3}{4}$

(40):

Thus from (39): and (40): we conclude that the proposed estimator p_{Pe} (product-type exponential estimator): and Σ_{Pe} (product-type exponential estimator): proposed by Singh et al. (2007): is efficient than usual unbiased estimator $\bar{y}$, Σ_P (product estimator): proposed by Naik and Gupta (1996): when

$$-\frac{3}{4} < \tau < -\frac{1}{4}$$

6. EMPIRICAL STUDY

To measure the performance of the suggested estimators p_{Re} and p_{Pe} over the existing estimators $\bar{y}, \bar{y}_R$, $\bar{y}_P, \Sigma_{Pp}, \Sigma_{\text{Re}}$ and Σ_{Pe} of estimation of population mean $\bar{Y}$ we have considered the following real datasets:

Data 1: [Source: Gujrati and Saneetha (2007):, pp. 601]

The variables are defined as:

y=Income (thousands of dollars):

ξ =Home ownership

For this dataset, we have

N=40, $\bar{Y}$ =14.4, K= 0.124, ρ =0.897 C_y =0.308, C_k =0.963, n=11

Data 2: [Source: Sukhatme and Sukhatme (1970):, pp. 279]

The variables are defined as:

y=Area (in acres): of the circles under wheat crop

ξ =Set of five villages is termed as a circle

For this data, we have

N=89, $\bar{Y}$ =3.360, K= 0.1236, ρ =0.766, C_y =0.60400, C_k =2.19012, n=23

Data 3: [Source: Mukhopadhyaya (2000):, PP. 44]

The variables are defined as:

y= Family size

ξ = A Family that got an agricultural finance from a bank

For this data, we have N=25, $\bar{Y}$ =9.44, K= 0.400, ρ = -0.387, C_y =0.17028, C_k =1.27478, n=7

To equate the biases of diverse estimators of the population mean $\bar{Y}$, we have calculated the quantities by exhausting the following ways:

$$Q_{R1}=\left|\frac{Bias(\Sigma_R)}{\lambda\bar{Y}C_k^2}\right|= |1-\tau|;\quad Q_{R2}=\left|\frac{Bias(\Sigma_{\mathrm{Re}})}{\lambda\bar{Y}C_k^2}\right|= \frac{1}{8}|3-4\tau|;\quad Q_{R3}=\left|\frac{Bias(p_{\mathrm{Re}})}{\lambda\bar{Y}C_k^2}\right|=\frac{1}{8}|3-4(\tau+f)|;$$

$$Q_{P1}=\left|\frac{Bias(\Sigma_P)}{\lambda\bar{Y}C_k^2}\right|= |\tau|;\quad Q_{P2}=\left|\frac{Bias(\Sigma_{Pp})}{\lambda\bar{Y}C_k^2}\right|=\frac{1}{8}\left|\tau-\frac{f}{1-f}\right|; Q_{P3}=\left|\frac{Bias(\Sigma_{P\mathrm{e}})}{\lambda\bar{Y}C_k^2}\right|= \frac{1}{8}|4\tau-1|;$$

$$Q_{P4}=\left|\frac{Bias(p_{P\mathrm{e}})}{\lambda\bar{Y}C_k^2}\right|=\frac{1}{8}\left|4\tau-\frac{1}{1-f}\right|$$

To equate the efficiency of various estimators of the population mean $\bar{Y}$, we have figured the percentage relative efficiency (PREs): of various estimators with deference to $\bar{y}$ by using the following way:

$$PRE(\Sigma_R,\bar{y})=\frac{MSE(\bar{y})}{MSE(\Sigma_R)}\times 100=\frac{C_y^2}{[C_y^2+C_k^2(1-2\tau)]}\times 100$$

$$PRE(\Sigma_P,\bar{y})=\frac{MSE(\bar{y})}{MSE(\Sigma_P)}\times 100=\frac{C_y^2}{[C_y^2+C_k^2(1+2\tau)]}\times 100=PRE(\Sigma_{Pp},\bar{y})$$

$$PRE(\Sigma_{\mathrm{Re}},\bar{y})=\frac{MSE(\bar{y})}{MSE(\Sigma_{\mathrm{Re}})}\times 100=\frac{C_y^2}{[C_y^2+\frac{C_k^2}{4}(1-4\tau)]}\times 100 = PRE(p_{\mathrm{Re}},\bar{y})$$

$$PRE(\Sigma_{P\mathrm{e}},\bar{y})=\frac{MSE(\bar{y})}{MSE(\Sigma_{P\mathrm{e}})}\times 100=\frac{C_y^2}{[C_y^2+\frac{C_k^2}{4}(1+4\tau)]}\times 100=PRE(p_{P\mathrm{e}},\bar{y})$$

Table: 1. Quantities Values

Population	Q_{R1}	Q_{R2}	Q_{R3}	Q_{P1}	Q_{P2}	Q_{P3}	Q_{P4}
Data 1	0.713	0.233	**0.095**	0.287	0.183	0.180	0.110
Data 2	0.789	0.269	**0.157**	0.211	0.163	0.190	0.162

Data 3	1.016	0.383	0.243	0.215	0.147	0.113	**0.081**

Note: Intrepid number indicate the least biased value in the relevant data set

Table: 2. Percentage Relative Efficiency (PREs):

Population	$PRE(\Sigma_R, \bar{y})$	$PRE(\Sigma_P, \bar{y}) = PRE(\Sigma_{Pp}, \bar{y})$	$PRE(\Sigma_{\mathrm{Re}}, \bar{y}) = PRE(p_{\mathrm{Re}}, \bar{y})$	$PRE(\Sigma_{P\mathrm{e}}, \bar{y}) = PRE(p_{P\mathrm{e}}, \bar{y})$
Data 1	62.864	31.434	**105.389**	57.335
Data 2	18.657	8.518	**77.367**	22.310
Data 3	32.975	37.712	64.278	**73.244**

Note: Intrepid number indicate the highest percentage relative efficiency in the relevant data set

From table 1, we have following observations:

(i): The proposed ratio type exponential estimator p_{Re} is less biased than the ratio type exponential estimator Σ_{Re} proposed by Singh et al. (2007): and ratio type estimator Σ_R proposed by Naik and Gupta (1991): ($Q_{R3} < Q_{R2} < Q_{R3}$): for all data sets.

(ii): The proposed product type exponential estimator p_{Pe} is less biased than the product estimator Σ_{Pp} , product type exponential estimator $\Sigma_{P\mathrm{e}}$ proposed by Singh et al. (2007): and ratio type estimator Σ_P proposed by Naik and Gupta (1991): ($Q_{P4} < Q_{P3} < Q_{P2} < Q_{P1}$): for all data sets.

From table 2, we have following observations:

(i): The suggested ratio type exponential estimator p_{Re} and Singh et al. (2007): ratio type exponential estimator Σ_{Re} both have the largest percentage relative efficiency than the traditional unbiased estimator $\bar{y}$, ratio estimator Σ_R and product estimator Σ_P proposed by Naik and Gupta (1991): and Singh et al. (2007): product type exponential estimator Σ_{Pe} in population data set 1 and 2 when positive bi-serial correlation coefficient is exist.

(ii): The proposed product type exponential estimator p_{Pe} and Singh et al. (2007): product type exponential estimator Σ_{Pe} both have the largest percentage relative efficiency than the traditional unbiased estimator $\bar{y}$, ratio estimator Σ_R and product estimator Σ_P proposed by Naik and Gupta (1991): and Singh et al. (2007): ratio type exponential estimator Σ_{Re} in population data set 3 when negative bi-serial correlation coefficient is exist.

(iii): The projected ratio type exponential estimator p_{Re} has the uppermost percentage relative efficiency and least bias when positive bi-serial correlation coefficient is occur and proposed product type exponential estimator p_{Pe} is highest percentage relative efficiency and less biased when negative bi-serial correlation coefficient exist. Therefore the proposed ratio and product type exponential estimator utilizing auxiliary attribute seems to best having highest percentage relative efficiency and less bias when positive and negative bi-serial correlation are present.

Concluding Remarks

In this paper we projected the ratio and product type exponential estimators when auxiliary information is accessible in form of auxiliary attribute by exploited estimators from Singh et al (2007):. Here, we pragmatic that the proposed estimators are different from the Singh et al. (2007): ratio and product type exponential estimators. The biases and mean squared errors (MSE): of the projected estimators are derived up to first order of approximation. The mean squared errors of the projected estimators are equal to the Singh et al. (2007): ratio and product type exponential estimators when auxiliary attribute is used. We derived the theoretical conditions under which the projected estimators are less biased and more efficient than the traditional unbiased estimator, ratio and product estimators proposed by Naik and Gupta (1991): and Singh et al. (2007): ratio and product type exponential estimators. This fact has been also supported through an empirical study using real data sets. The results of this article is quite informative both academically and empirically. Thus we endorse the use of projected estimators in practice.

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