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BAYESIAN INFERENCE OF TYPE - II CENSORED DATA USING MIXTURE OF POWER FUNCTION DISTRIBUTION UNDER UNIFORM AND JEFFREY'S PRIORS

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ABSTRACT
Bayesian estimation in the mixture models under Type – I censored samples have been done by several authors. In this paper the Bayesian estimation of the parameters of the mixture of power function distribution under Type – II censoring is considered. The estimation is carried out using uniform priors and Jeffrey's priors for the parameters of the model under K – loss function and Precautionary loss function. The posterior risk and the root mean square error of the estimators are also obtained and to study the performance of these estimators a simulation study is conducted.

KEYWORDS: K – loss function, Precautionary loss function, Uniform prior, Jeffrey's prior, power function distribution, type – II censoring.

MSC: 62F15, 62N01, 62P30, 65C60.

RESUMEN
La estimación Bayesiana bajo modelos mezclados del Tipo – I de muestras censuradas han sido tratados por varios autores. En este paper tal estimación Bayesiana de los parámetros de la función de potencia bajo el Tipo-II de censura es considerado. La estimación es llevada a cabo usando como a-prioris la Uniformes y la de Jeffrey para los parámetros del modelo bajo funciones de pérdidas K y Precaucionaría. El riesgo a posteriori y la raíz cuadrada de los estimadores también son obtenidos y se estudia el desempeño de estos estimadores en un estudio de simulación desarrollado.

PALABRAS CLAVE: función K – pérdida, función de pérdida Precaucionaría, prior Uniforme, prior de Jeffrey, distribución de la función de potencia , censura tipo – II.

1. INTRODUCTION

Censoring is used in studies which requires conducting the life testing experiments and survival analysis. There are two primary censoring schemes apart from the various types of censoring schemes. In an experiment if we terminate the experiment at a predetermined time then this type of censoring scheme is known as time censoring scheme or Type – I censoring. If the experiment is terminated as soon as the predetermined number of failures are observed then such scheme is known as failure censoring scheme or Type – II censoring.
The analysis of life testing or survival data is done by using different statistical models like Power function, Rayleigh, Weibull, Exponential, etc. Sometimes the failure of object occur due to more than one reason. for e.g. Death of a patient may occur due to high blood pressure or failure of kidney. The mixture model is used in such situations. These mixture models comprises of two or more subpopulations which are known as components of the mixture model. Here, the value of the observation may relate it to the i^{th} sub population with proportion p_i. $0 < p_i < 1; i = 1, 2, \dots, k$ and $\sum_{i=1}^{k} p_i = 1$ where $k =$ no. of subpopulations. Engineering, Medicine, Agriculture and many more fields have vast applications of such mixture distribution.
To analyse crab morphometry data, a statistical model based on finite mixtures of distributions was first introduced by Pearson (1894). Ahsanullah and Kabir (1974) have described characterisation of the power function distribution and Meniconi and Barry (1996) have used power function distribution as a lifetime model for electrical component reliability where as Saleem et al (2010) have considered estimation of

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parameters of the mixture of power function distributions under complete and type – I censored sample. Mendehall and Hader (1958) considered a mixture of exponential life time model. Saleem and Aslam (2008) have used Bayesian procedure for estimating the parameters of a mixture of Rayleigh distribution, Soliman (2006) considered estimation for a mixture of Rayleigh distribution under progressive censoring and Sindhu et al (2014) have considered a mixture distribution of Rayleigh lifetime model to analyse the data under doubly censoring scheme. Kazmi et al (2012) have used Bayesian estimation for a mixture of Maxwell distribution under type-I censoring scheme. Very few works are available in the area of estimation of mixture model of the power function distribution under Bayesian setup.
In this paper we have considered a Bayesian estimation for a mixture of two power function distributions based on type – II censored sample. Uniform prior as well as Jeffrey's Prior are used for the parameters α_1 and α_2 and uniform prior for proportion p of the mixture model. Bayes estimates are obtained for the parameters, considering K – loss function and precautionary loss function. A comparison is done between the efficiency of the Bayes estimators obtained under uniform prior and Jeffrey's prior using simulation. Some interesting conclusions are derived from the simulation results.

2. MIXTURE MODEL

The two - component mixture model for power function distribution is defined as follows: $f(x) = pf_1(x) + (1-p)f_2(x)$

where

$$f_i(x) = \alpha_i x^{\alpha_i - 1}, \alpha_i > 0, 0 < p < 1, i = 1, 2;\ 0 \le x \le 1 \tag{2.1}$$

The distribution function of the i^{th} component of two - component mixture model for power function distribution is

$$F_i(x) = x^{\alpha_i} \tag{2.2}$$

Here α_1, α_2 are unknown parameters of the power function distributions and p is unknown mixing proportion with mixing weight $p : 1-p$.
Let us suppose that if n units are put on a test and the test is terminated as soon as the r^{th} failure is observed. As per the mixture model object may fail due to cause 1 or cause 2. The failed object can easily be identified whether it is from sub population 1 (which failed due to cause 1) or sub population 2 (which failed due to cause 2).
Thus depending upon the cause of failure, we can identify the number of failures r_1 due to cause 1 and r_2 due to cause 2 from the r observed failures. The remaining $(n-r)$ objects are censored which provide no information about the sub population and survive beyond the time $X_{(r)}$, the observed time of the r^{th} failure. The mixture model must be identifiable to produce precision inferences. In our model we have only shape parameters α_1 and α_2. The model becomes identifiable and we can use it for analysis.
The general form of likelihood function for the two – component mixture distribution under type – II censoring without replacement is given by:

$$L(\alpha_1, \alpha_2, p|x) \propto \prod_{j=1}^{r_1} p.f_1(x_{1j}) \prod_{j=1}^{r_2} (1-p)\, f_2(x_{2j})\ [1 - F(x_r)]^{n-r} \tag{2.3}$$

where x_{ij} = failure time of the j^{th} unit belonging to the i^{th} subpopulation and x_r denotes the r^{th} failure time observed from the n units on the test, $j = 1, 2, 3, \dots, r_i;\ i = 1, 2;\ r_1 + r_2 = r$
Putting the probability density function and cumulative distribution function of power function distribution from (2.1) and (2.2) in (2.3), the likelihood function reduces to,

$$L(\alpha_1, \alpha_2, p|x) \propto \left[\prod_{j=1}^{r_1} p.\alpha_1 {x_{1j}}^{\alpha_1 - 1}\right] \left[\prod_{j=1}^{r_2} (1-p)\alpha_2 {x_{2j}}^{\alpha_2 - 1}\right] \ [(1 - (p.{x_r}^{\alpha_1} + (1-p).{x_r}^{\alpha_2})^{n-r}] \tag{2.4}$$

Simplifying the above equation, we can write it as

$$L = c\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}p^{r_1+s-i}(1-p)^{r_2+i}\,{\alpha_1}^{r_1}{\alpha_2}^{r_2}\exp\left\{\left[-\alpha_1\left(\sum_{j=1}^{r_1}\log\left(\frac{1}{x_{1j}}\right)+(s-i)\log\left(\frac{1}{x_r}\right)\right)\right]\right\}. \exp\left\{-\alpha_2\left[\sum_{j=1}^{r_2}\log\left(\frac{1}{x_{2j}}\right)+i\log\left(\frac{1}{x_r}\right)\right]\right\}.\ e^{-\sum_{j=1}^{r_1}\log x_{1j}}.e^{-\sum_{j=1}^{r_2}\log x_{2j}} \qquad (2.5)$$

where c is constant depending on n, r_1 and r_2.
Consider the uniform prior for the parameters α_1, α_2 and p given as:

$$\pi_1(\alpha_1) = 1, \qquad (2.6)$$
$$\pi_2(\alpha_2) = 1, \quad \text{where } \alpha_i > 0; i = 1, 2. \qquad (2.7)$$
$$\pi_3(p) = 1, \quad \text{where } 0 < p < 1 \qquad (2.8)$$

Using the likelihood function given in (2.5) and prior distributions stated in (2.6), (2.7) and (2.8), the joint distribution of the parameters and sample becomes,

$$\emptyset(\alpha_1, \alpha_2, p, \underline{x}) = c\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}p^{r_1+s-i}(1-p)^{r_2+i}\,{\alpha_1}^{r_1}{\alpha_2}^{r_2}\exp\left\{\left[-\alpha_1\left(\sum_{j=1}^{r_1}\log\left(\frac{1}{x_{1j}}\right)+(s-i)\log\left(\frac{1}{x_r}\right)\right)\right]\right\}. \exp\left\{-\alpha_2\left[\sum_{j=1}^{r_2}\log\left(\frac{1}{x_{2j}}\right)+i\log\left(\frac{1}{x_r}\right)\right]\right\}.\ e^{-\sum_{j=1}^{r_1}\log x_{1j}}.e^{-\sum_{j=1}^{r_2}\log x_{2j}} \qquad (2.9)$$

where c is constant depending on n, r_1 and r_2
The marginal distribution of $\underline{x}$ can be derived from the joint distribution (2.9) as,

$$m\left(\underline{x}\right) = c.\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1, r_2+i+1)} \qquad (2.10)$$

where $A_1 = \sum_{j=1}^{r_1}\log\left(\frac{1}{x_{1j}}\right)+(s-i)\log\left(\frac{1}{x_r}\right)$ and $A_2 = \sum_{j=1}^{r_2}\log\left(\frac{1}{x_{2j}}\right)+i\log\left(\frac{1}{x_r}\right)$.
Hence the joint posterior distribution of α_1, α_2 and p can be obtained using (2.9) and (2.10) as,

$$\mathrm{g}\left(\alpha_1, \alpha_2, p, \underline{x}\right) = \frac{\emptyset(\alpha_1,\alpha_2,p,\underline{x})}{m\left(\underline{x}\right)} \qquad (2.11)$$

$$\mathrm{g}\left(\alpha_1, \alpha_2, p, \underline{x}\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}p^{r_1+s-i}(1-p)^{r_2+i}.{\alpha_1}^{r_1}e^{-\alpha_1 A_1}.{\alpha_2}^{r_2}e^{-\alpha_2 A_2}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1, r_2+i+1)}} \qquad (2.12)$$

The marginal posterior distribution of α_i $(i = 1, 2)$ and p can be determined by integrating with respect to other parameters. Thus the marginal posterior distribution of prior α_1 is defined as:

$$h\left(\alpha_1|\underline{x}\right) = \int_{\alpha_2}\int_{p}\mathrm{g}\left(\alpha_1, \alpha_2, p, \underline{x}\right)\,dp\,d\alpha_2 \qquad (2.13)$$

$$h\left(\alpha_1|\underline{x}\right)=\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\alpha_1^{r_1}e^{-\alpha_1A_1}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{2.14}$$

Similarly we have the marginal posterior distributions of α_2 and p as given below:

$$h\left(\alpha_2|\underline{x}\right)=\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\alpha_2^{r_2}e^{-\alpha_2A_2}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{2.15}$$

$$h\left(p|\underline{x}\right)=\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}p^{r_1+s-i}(1-p)^{r_2+i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{2.16}$$

Consider the Jeffrey's prior for the parameters α_1, α_2 and p given as:

$$\pi_1(\alpha_1)=\frac{1}{\alpha_1}, \tag{2.17}$$

$$\pi_2(\alpha_2)=\frac{1}{\alpha_2}, \tag{2.18}$$

where $\alpha_i>0; i=1,2$

$$\pi_3(p)=1, \tag{2.19}$$

where $0<p<1$

Using the likelihood function given in (2.5) and prior distributions stated in (2.17), (2.18) and (2.19), the joint distribution of the parameters and sample becomes,

$$\emptyset\left(\alpha_1,\alpha_2,p,\underline{x}\right)=c.\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}p^{r_1+s-i}(-p)^{r_2+i}.\alpha_1^{r_1-1}e^{-\alpha_1A_1}.\alpha_2^{r_2-1}e^{-\alpha_2A_2} \tag{2.20}$$

where c is constant depending on n, r_1 and r_2 and $A_1=\sum_{j=1}^{r_1}\log\left(\frac{1}{x_{1j}}\right)+(s-i)\log\left(\frac{1}{x_r}\right)$ and $A_2=\sum_{j=1}^{r_2}\log\left(\frac{1}{x_{2j}}\right)+i\log\left(\frac{1}{x_r}\right)$.

The marginal distribution of $\underline{x}$ can be derived from the joint distribution (2.20) as,

$$m\left(\underline{x}\right)=c.\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1^{r_1}}\frac{\Gamma(r_2)}{A_2^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)} \tag{2.21}$$

Hence the joint posterior distribution of α_1,α_2 and p can be obtained using (2.20) and (2.21) as,

$$\mathrm{g}\left(\alpha_1,\alpha_2,p,\underline{x}\right)=\frac{\emptyset\left(\alpha_1,\alpha_2,p,\underline{x}\right)}{m\left(\underline{x}\right)} \tag{2.22}$$

$$g(\alpha_1,\alpha_2,p,\underline{x}) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} p^{r_1+s-i}\,(1-p)^{r_2+i}\,.\,{\alpha_1}^{r_1-1}\, e^{-\alpha_1 A_1}.\,{\alpha_2}^{r_2-1}\, e^{-\alpha_2 A_2}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\,\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{2.23}$$

The marginal posterior distribution of α_i $(i = 1, 2)$ and p can be determined by integrating with respect to other parameters. Thus the marginal posterior distribution of prior α_1 is defined as:

$$h\left(\alpha_1|\underline{x}\right) = \int_{\alpha_2}\int_p g(\alpha_1,\alpha_2,p,\underline{x})\; dp\, d\alpha_2 \tag{2.24}$$

$$h\left(\alpha_1|\underline{x}\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\, {\alpha_1}^{r_1-1}\, e^{-\alpha_1 A_1}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\,\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\,\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{2.25}$$

Similarly we have the marginal posterior distributions of α_2 and p as given below:

$$h\left(\alpha_2|\underline{x}\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\, {\alpha_2}^{r_2-1}\, e^{-\alpha_2 A_2}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\,\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\,\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{2.26}$$

$$h\left(p|\underline{x}\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\, p^{r_1+s-i}\,(1-p)^{r_2+i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\,\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{2.27}$$

3. BAYES ESTIMATION

In Bayesian estimation theory a loss function gauges the difference of the estimate $\hat{\theta}$ from the parameter θ. There is no fixed set of procedure to select an appropriate loss function. The performance of different Bayes estimators can be compared in terms of posterior risks associated with each estimator. The posterior risk is defined to be the expected value of a loss function. In this paper, we have considered a couple of loss functions for posterior estimation and the description about the loss functions are as follows:

K – loss function (KLF): The K – loss function was proposed by Wasan (1970), is defined as $l(\hat{\theta},\ \theta) = (\hat{\theta}-\theta)^2/\,\hat{\theta}\theta$.

The Bayes estimate and the posterior risk are defined as

$$\hat{\theta} = \sqrt{E(\theta|x)/E(\theta^{-1}|x)} \tag{3.1}$$

and $\rho(\hat{\theta}) = 2\,\{E(\theta|x)E(\theta^{-1}|x) - 1\}$ (3.2)

respectively.

Precautionary loss function (PLF): The Precautionary loss function was proposed by Norstrom (1996), is defined as $l\left(\hat{\theta},\ \theta\right) = (\theta-\hat{\theta})^2\,/\,\hat{\theta}$.

The Bayes estimate and the posterior risk are defined as

$$\hat{\theta} = \{E(\theta^2|x)\}^{\frac{1}{2}} \tag{3.3}$$

and $\rho(\hat{\theta}) = 2[\{E(\theta^2|x)\}^{\frac{1}{2}} - E(\theta|x)]$ (3.4)

respectively.

To estimate the value of the parameter $\alpha_1$and its posterior risk using the K - loss function and precautionary loss function using uniform prior, we will require the posterior expectations like $E_h(\alpha_1|x)$, $E_h\left(\frac{1}{\alpha_1}\middle|x\right)$ and $E_h({\alpha_1}^2|x)$ which will be derived as,

$$E_h(\alpha_1|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+2)}{{A_1}^{r_1+2}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.5}$$

$$E_h\left(\frac{1}{\alpha_1}\middle|x\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.6}$$

$$E_h({\alpha_1}^2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+3)}{{A_1}^{r_1+3}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.7}$$

Similarly for the parameters α_2 and p

$$E_h(\alpha_2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+2)}{{A_2}^{r_2+2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.8}$$

$$E_h\left(\frac{1}{\alpha_2}\middle|x\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.9}$$

$$E_h({\alpha_2}^2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+3)}{{A_2}^{r_2+3}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.10}$$

$$E_h(p|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+2,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.11}$$

$$E_h\left(\frac{1}{p}\middle|x\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.12}$$

$$E_h(p^2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+3,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}} \tag{3.13}$$

Bayes estimate $\widehat{\alpha_1}$ and posterior risk $\rho(\widehat{\alpha_1})$ of parameter α_1 is determined under K – loss function as:

$$\widehat{\alpha_1} = \left[\frac{E_h(\alpha_1|x)}{E_h\left(\frac{1}{\alpha_1}\middle|x\right)}\right]^{\frac{1}{2}} \tag{3.14}$$

$$\rho(\widehat{\alpha_1}) = 2\left[E_h(\alpha_1|x)\,E_h\left(\frac{1}{\alpha_1}\middle|x\right) - 1\right] \tag{3.15}$$

Using equation (3.5) and (3.6) in equation (3.14) and (3.15) we get the Bayes estimate and its posterior risk for parameter α_1

$$\widehat{\alpha_1} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+2)}{A_1^{r_1+2}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1^{r_1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.16}$$

$$\rho(\widehat{\alpha_1}) = 2\left[\begin{array}{l}\left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+2)}{A_1^{r_1+2}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)* \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1^{r_1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}\right) - \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+1)}{A_2^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)\end{array}\right] \tag{3.17}$$

Similarly using equation (3.8) and (3.9) we get the Bayes estimate and its posterior risk for parameter α_2 under K – loss function as:

$$\widehat{\alpha_2} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2+2)}{A_2^{r_2+2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1^{r_1+1}}\frac{\Gamma(r_2)}{A_2^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.18}$$

$$\rho(\widehat{\alpha_2}) = 2\left[\begin{array}{l}\left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+2)}{{A_2}^{r_2+2}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)*\\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)-\\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)\end{array}\right] \tag{3.19}$$

Similarly using equation (3.11) and (3.12) we get the Bayes estimate and its posterior risk for parameter p under K – loss function as:

$$\hat{p} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+2,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.20}$$

$$\rho(\hat{p}) = 2\left[\begin{array}{l}\left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+2,r_2+i+1)}\right)*\\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i,r_2+i+1)}\right)-\\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)\end{array}\right] \tag{3.21}$$

Bayes estimate $\widehat{\alpha_1}$ and posterior risk $\rho(\widehat{\alpha_1})$ of parameter α_1 is determined under precautionary loss function as:

$$\widehat{\alpha_1} = [\,E_h({\alpha_1}^2|x)\,]^{\frac{1}{2}} \tag{3.22}$$

$$\rho(\widehat{\alpha_1}) = 2\left[[\,E_h({\alpha_1}^2|x)\,]^{\frac{1}{2}} - E_h(\alpha_1|x)\right] \tag{3.23}$$

Using equation (3.5) and (3.7) in equation (3.22) and (3.23) we get the Bayes estimate and its posterior risk for parameter α_1

$$\widehat{\alpha_1} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+3)}{{A_1}^{r_1+3}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.24}$$

$$\rho(\widehat{\alpha_1}) = 2\left[\left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+3)}{{A_1}^{r_1+3}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)^{\frac{1}{2}} - \left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+2)}{{A_1}^{r_1+2}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)\right] \tag{3.25}$$

Similarly using equation (3.8) and (3.10) we get the Bayes estimate and its posterior risk for parameter α_2 under precautionary loss function as:

$$\widehat{\alpha_2} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+3)}{A_2{}^{r_2+3}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+1, r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.26}$$

$$\begin{aligned}\rho(\widehat{\alpha_2}) = 2\Bigg[&\left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+3)}{A_2{}^{r_2+3}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+1, r_2+i+1)}}\right)^{\frac{1}{2}} \\ &- \left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+2)}{A_2{}^{r_2+2}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+1, r_2+i+1)}}\right)\Bigg]\end{aligned} \tag{3.27}$$

Similarly using equation (3.11) and (3.13) we get the Bayes estimate and its posterior risk for parameter p under precautionary loss function as:

$$\hat{p} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+3, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+1, r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.28}$$

$$\begin{aligned}\rho(\hat{p}) = 2\Bigg[&\left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+3, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+1, r_2+i+1)}}\right)^{\frac{1}{2}} \\ &- \left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+2, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}} \beta_{(r_1+s-i+1, r_2+i+1)}}\right)\Bigg]\end{aligned} \tag{3.29}$$

To estimate the value of the parameter $\alpha_1$and its posterior risk using the K - loss function and precautionary loss function using Jeffrey's prior, we will require the posterior expectations like $E_h(\alpha_1|x)$, $E_h\left(\frac{1}{\alpha_1}\middle|x\right)$ and $E_h({\alpha_1}^2|x)$ which will be derived as,

$$E_h(\alpha_1|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}} \frac{\Gamma(r_2)}{A_2{}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{A_1{}^{r_1}} \frac{\Gamma(r_2)}{A_2{}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.30}$$

$$E_h\left(\frac{1}{\alpha_1}\middle|x\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1-1)}{{A_1}^{r_1-1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.31}$$

$$E_h({\alpha_1}^2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+2)}{{A_1}^{r_1+2}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.32}$$

Similarly for the parameters α_2 and p

$$E_h(\alpha_2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.33}$$

$$E_h\left(\frac{1}{\alpha_2}\middle|x\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2-1)}{{A_2}^{r_2-1}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.34}$$

$$E_h({\alpha_2}^2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2+2)}{{A_2}^{r_2+2}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.35}$$

$$E_h(p|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+2, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.36}$$

$$E_h\left(\frac{1}{p}\middle|x\right) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.37}$$

$$E_h(p^2|x) = \frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+3, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1)}{{A_1}^{r_1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}} \tag{3.38}$$

We derive the Bayes estimate $\widehat{\alpha_1}$ and posterior risk $\rho(\widehat{\alpha_1})$ of parameter α_1 under K – loss function using equation (3.30) and (3.31) in equation (3.14) and (3.15) as:

$$\widehat{\alpha_1} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i} \frac{\Gamma(r_1-1)}{{A_1}^{r_1-1}} \frac{\Gamma(r_2)}{{A_2}^{r_2}} \beta_{(r_1+s-i+1, r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.39}$$

$$\rho(\widehat{\alpha_1}) = 2\left[\begin{array}{l}\left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{{A_1}^{r_1+1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)* \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1-1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)- \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)\end{array}\right] \tag{3.40}$$

Similarly using equation (3.33) and (3.34) we get the Bayes estimate and its posterior risk for parameter α_2 under K – loss function as:

$$\widehat{\alpha_2} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2-1)}{{A_2}^{r_2-1}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.41}$$

$$\rho(\widehat{\alpha_2}) = 2\left[\begin{array}{l}\left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2+1)}{{A_2}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)* \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2-1)}{{A_2}^{r_2-1}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)- \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)\end{array}\right] \tag{3.42}$$

Similarly using equation (3.36) and (3.37) we get the Bayes estimate and its posterior risk for parameter p under K – loss function as:

$$\hat{p} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+2,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.43}$$

$$\rho(\hat{p}) = 2\left[\begin{array}{l}\left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+2,r_2+i+1)}\right)* \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i,r_2+i+1)}\right)- \\ \left(\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s \binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{{A_1}^{r_1}}\frac{\Gamma(r_2)}{{A_2}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}\right)\end{array}\right] \tag{3.44}$$

We derive the Bayes estimate $\widehat{\alpha_1}$ and posterior risk $\rho(\widehat{\alpha_1})$ of parameter α_1 under precautionary loss function using equation (3.30) and (3.32) in equation (3.22) and (3.23) as:

$$\widehat{\alpha_1} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+2)}{A_1{}^{r_1+2}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.45}$$

$$\rho(\widehat{\alpha_1}) = 2\left[\left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+2)}{A_1{}^{r_1+2}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)^{\frac{1}{2}} - \left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1+1)}{A_1{}^{r_1+1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)\right] \tag{3.46}$$

Similarly using equation (3.33) and (3.35) we get the Bayes estimate and its posterior risk for parameter α_2 under precautionary loss function as:

$$\widehat{\alpha_2} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2+2)}{A_2{}^{r_2+2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.47}$$

$$\rho(\widehat{\alpha_2}) = 2\left[\left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2+2)}{A_2{}^{r_2+2}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)^{\frac{1}{2}} - \left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2+1)}{A_2{}^{r_2+1}}\beta_{(r_1+s-i+1,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)\right] \tag{3.48}$$

Similarly using equation (3.36) and (3.38) we get the Bayes estimate and its posterior risk for parameter p under precautionary loss function as:

$$\hat{p} = \left[\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+3,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right]^{\frac{1}{2}} \tag{3.49}$$

$$\rho(\hat{p}) = 2\left[\left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+3,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)^{\frac{1}{2}} - \left(\frac{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+2,r_2+i+1)}}{\sum_{s=0}^{n-r}\sum_{i=0}^{s}(-1)^s\binom{n-r}{s}\binom{s}{i}\frac{\Gamma(r_1)}{A_1{}^{r_1}}\frac{\Gamma(r_2)}{A_2{}^{r_2}}\beta_{(r_1+s-i+1,r_2+i+1)}}\right)\right] \quad (3.50)$$

4. SIMULATION STUDY

A simulation study was carried out to check the performance of Bayes estimators obtained in Section 3 using R software (v. 3.4). To simulate samples from two - component mixture of power function distributions we have used the following algorithm.

a. A uniform $U\ (0, 1)$ random number (u) is generated and if $u \leq p$ (mixture proportion parameter) then draw an observation from the 1st sub population $f_1(x)$ having parameter α_1, otherwise from the 2nd sub population $f_2(x)$ having parameter α_2. Here we have used two sets for (α_1, α_2, p) as $(0.5, 0.9, 0.4)$ and $(1.5, 1.2, 0.6)$.
b. Repeat the above step n times to generate a sample of size n from the mixture distribution. Here the value of n is taken as (30, 60).
c. Arrange the above generated n values in ascending order and take the 1st r values as observed values and $(n - r)$ are considered as censored values.
d. Identify the observations belongs to 1st sub population, say r_1 and 2nd sub population, say r_2.
e. Calculate Bayes estimates of parameters p, α_1 and α_2 using the respective formulas from section 3.
f. Repeat the above steps $N = 5000$ times, thus we have $\widehat{\alpha_{1\iota}}$, $\widehat{\alpha_{2\iota}}$ and $\widehat{p_\iota}$, $i = 1,2, \dots, 5000$.
g. Calculate Posterior Risk (PR) and Bayes estimates of α_1, α_2, and p by taking average of the 5000 values in step f.
h. Calculate Root Mean Square Error, using the formula,

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(\widehat{\theta_\iota} - \theta)^2}{N}}$$

The outputs obtained from the simulations are presented in Table 1 to Table 4. Table – 1: Bayes estimates, posterior risk and RMSE for Uniform Prior with values of $\alpha_1 = (0.5, 1.5), \alpha_2 = (0.9, 1.2)$ and $p = (0.4, 0.6)$ under the K loss function are given below:

KLF										
	α1	α2	p	n	10% Censoring			20% Censoring		
					α1	α2	p	α1	α2	p
Estimate	0.5	0.9	0.4	30	0.5712	0.9763	0.3971	0.5746	0.9671	0.4007
PR					4.98E-39	3.80E-40	4.53E-39	2.41E-40	2.03E-41	2.24E-40
RMSE					0.2070	0.2648	0.0885	0.2104	0.2641	0.0926
Estimate				60	0.5354	0.9338	0.3991	0.5398	0.9387	0.3944
PR					1.65E-91	5.80E-92	1.23E-91	1.27E-93	5.39E-94	9.65E-94
RMSE					0.1236	0.1678	0.0635	0.1294	0.1977	0.0711
Estimate	1.5	1.2	0.6	30	1.6119	1.3759	0.5806	1.5942	1.3767	0.5765

PR					5.60E-23	1.29E-22	2.16E-23	8.34E-23	9.93E-22	1.08E-23
RMSE					0.4224	0.5116	0.0907	0.4146	0.5228	0.0947
Estimate				60	1.5574	1.2763	0.5907	1.5533	1.2810	0.5829
PR					1.49E-56	6.90E-56	3.23E-57	2.11E-61	5.39E-61	7.76E-62
RMSE					0.2785	0.2892	0.0653	0.2832	0.3038	0.0741

Table – 2: Bayes estimates, posterior risk and RMSE for Uniform Prior with values of $\alpha_1 = (0.5, 1.5), \alpha_2 = (0.9, 1.2)$ and $p = (0.4, 0.6)$ under the precautionary loss function are given below:

PLF										
	α1	**α2**	**p**	**n**	**10% Censoring**			**20% Censoring**		
					α1	**α2**	**p**	**α1**	**α2**	**p**
Estimate	0.5	0.9	0.4	30	0.6224	1.0324	0.4173	0.6285	1.0254	0.4231
PR					4.99E-02	5.52E-02	1.93E-02	5.24E-02	5.72E-02	2.13E-02
RMSE					0.2474	0.2994	0.0873	0.2534	0.2980	0.0924
Estimate				60	0.5590	0.9606	0.4096	0.5642	0.9619	0.4281
PR					2.33E-02	2.66E-02	1.03E-02	2.43E-02	2.57E-02	3.44E-02
RMSE					0.1372	0.1796	0.0631	0.1434	0.1849	0.1019
Estimate	1.5	1.2	0.6	30	1.7056	1.4979	0.5949	1.6914	1.5050	0.5928
PR					9.22E-02	1.19E-01	1.36E-02	9.54E-02	1.24E-01	1.55E-02
RMSE					0.4785	0.6105	0.0857	0.4696	0.6263	0.0885
Estimate				60	1.6024	1.3326	0.5980	1.5988	1.3396	0.6067
PR					4.47E-02	5.57E-02	7.12E-03	4.55E-02	5.82E-02	2.14E-02
RMSE					0.2984	0.3205	0.0635	0.3005	0.3330	0.0874

Table – 3: Bayes estimates, posterior risk and RMSE for Jeffrey's Prior with values of $\alpha_1 = (0.5, 1.5), \alpha_2 = (0.9, 1.2)$ and $p = (0.4, 0.6)$ under the K loss function are given below:

KLF										
	α1	**α2**	**p**	**n**	**10% Censoring**			**20% Censoring**		
					α1	**α2**	**p**	**α1**	**α2**	**p**
Estimate	0.5	0.9	0.4	30	0.5214	0.9226	0.3966	0.5234	0.9139	0.3995
PR					1.11E-38	4.21E-40	5.05E-39	5.83E-40	2.35E-41	2.73E-40
RMSE					0.1769	0.2405	0.0890	0.1795	0.2422	0.0937
Estimate				60	0.5125	0.9082	0.3988	0.5176	0.9191	0.3925
PR					1.99E-91	6.47E-92	1.39E-91	1.22E-93	4.95E-94	8.75E-94
RMSE					0.1141	0.1601	0.0636	0.1243	0.2446	0.0722
Estimate	1.5	1.2	0.6	30	1.5224	1.2571	0.5812	1.5058	1.2550	0.5778
PR					1.39E-24	1.70E-23	3.52E-25	1.56E-23	3.61E-22	1.83E-24
RMSE					0.3848	0.4361	0.0911	0.3815	0.4465	0.0957
Estimate				60	1.5147	1.2218	0.5911	1.5162	1.2280	0.5817

PR	1.13E-57	5.70E-57	2.37E-58	1.78E-62	5.20E-62	6.01E-63
RMSE	0.2657	0.2680	0.0655	0.2836	0.2885	0.0774

Table – 4: Bayes estimates, posterior risk and RMSE for Jeffrey's Prior with values of $\alpha_1 = (0.5, 1.5), \alpha_2 = (0.9, 1.2)$ and $p = (0.4, 0.6)$ under the precautionary loss function are given below:

PLF										
	α1	α2	p	n	10% Censoring			20% Censoring		
					α1	α2	p	α1	α2	p
Estimate	0.5	0.9	0.4	30	0.5728	0.9789	0.4168	0.5780	0.9725	0.4220
PR					5.00E-02	5.53E-02	1.93E-02	5.28E-02	5.74E-02	2.13E-02
RMSE					0.2082	0.2660	0.0877	0.2131	0.2663	0.0932
Estimate				60	0.5361	0.9351	0.4093	0.5417	0.9383	0.4312
PR					2.33E-02	2.66E-02	1.03E-02	2.41E-02	2.50E-02	3.99E-02
RMSE					0.1241	0.1682	0.0633	0.1307	0.1758	0.1139
Estimate	1.5	1.2	0.6	30	1.6164	1.3799	0.5954	1.6036	1.3851	0.5940
PR					9.23E-02	1.19E-01	1.36E-02	9.58E-02	1.26E-01	1.55E-02
RMSE					0.4244	0.5146	0.0862	0.4186	0.5291	0.0896
Estimate				60	1.5598	1.2783	0.5983	1.5594	1.2857	0.6133
PR					4.47E-02	5.58E-02	7.12E-03	4.43E-02	5.78E-02	2.89E-02
RMSE					0.2793	0.2903	0.0637	0.2853	0.3046	0.1130

5. CONCLUSION

From Table 1 to 4 of uniform prior and Jeffrey's prior we observe the following conclusions:

i. For any set of selected value of the parameters (α_1, α_2, p) the values of PR and RMSE remain almost same in 10% as well as 20% censoring.
ii. As n increases RMSE and PR decreases in case of KLF & PLF for 10% and 20% censoring both.
iii. The values of PR and RMSE remains smaller in case of KLF compared to the results in PLF for any sample size and censoring (10% & 20%).
iv. The estimates of the parameters remain close to the actual values in case of KLF compared to PLF for sample for n, 10% and 20% censoring.
v. The estimates of the parameters remain close to the actual values in case of Jeffrey's prior compared to the Uniform prior for sample for n, 10% and 20% censoring.

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