A GENERALIZATION OF THE NEW WEIBULL PARETO DISTRIBUTION

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ABSTRACT
In this paper, a transmuted new Weibull Pareto distribution (NWPD) is suggested as a generalization of the new Weibull Pareto distribution. Some mathematical properties of the transmuted NWP distribution are derived, namely; the moments, failure rate and mean residual life functions, order statistics. Also, the maximum likelihood estimators for the transmuted NWP distribution parameters are provided and its Renyi entropy is proved.

KEYWORDS: New Weibull Pareto Distribution; Transformed Distribution; Order Statistics; Reliability; Renyi Entropy.

MSC: 60E05

RESUMEN
En este paper, se sugiere un nueva distribución transmutada Weibull-Pareto (NWPD) como una generalización de la nueva distribución Weibul- Pareto. Se derivan algunas propiedades matemáticas de transmutada distribución NWP, los llamados momentos, tasa de fracasos, las funciones media residual de vida, estadísticos de reden. También se proveen los estimadores de máxima verosimilitud de los parámetros de la distribución transmutada NWP y la entropía de Renyi.

PALABRAS CLAVE : Nueva distribución Weibull Pareto; distribución Transformada; Estadisticos de Orden; Fiabilidad;; Renyi Entropía.

1. INTRODUCTION

A random variable X is said to have a new Weibull Pareto distribution with parameters δ, β and θ if its cumulative distribution function is given by

$$g(x;\delta,\theta,\beta)=\frac{\delta\beta}{\theta^{\beta}}x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, x>0,\ \beta>0, \delta>0,\ \theta>0 \tag{1}$$

The corresponding pdf is

$$G(x;\delta,\theta,\beta)=1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, \quad x>0, \beta>0, \delta>0, \theta>0. \tag{2}$$

The expected value and the variance of the NWP random variable are given by

$$E(X)=\frac{\theta}{\sqrt[\beta]{\delta}}\Gamma\left(\frac{\beta+1}{\beta}\right) \text{ and } \operatorname{Var}(X)=2\frac{\theta}{\sqrt[\beta]{\delta^{2}}}\Gamma\left(\frac{\beta+2}{\beta}\right)-\left[\frac{\theta}{\sqrt[\beta]{\delta}}\Gamma\left(\frac{\beta+1}{\beta}\right)\right]^{2}, \text{ respectively.}$$

The hazard rate function of the NWP random variable is $R(x;\delta,\theta,\beta)=\frac{\delta\beta}{\theta^{\beta}}x^{\beta-1}$. For more about the NWP distribution See Aljarrah at al. (2015) .
Shaw and Buckley (2007) suggested the transmutation map method. The cumulative distribution function (cdf) technique based on the quadratic rank transmutation map satisfies the following general form

$$F_2(x)=(1+\lambda)F_1(x)-\lambda\left[F_1(x)\right]^2, \tag{3}$$

with probability density function (pdf) comes from the first derivative of Equation (1) as

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$$f_2(x) = f_1(x)[1+\lambda-2\lambda F_1(x)], -1\le\lambda\le 1, \tag{4}$$

where $f_1(x)$ and $f_2(x)$ are the corresponding probability density functions of $F_1(x)$ and $F_2(x)$, respectively.

A random variable *X* is said to have a transmuted probability distribution with cdf $F(x)$ if

$$F(x) = (1+\lambda)G(x) - \lambda\left[G(x)\right]^2, \ \ |\lambda|\le 1, \tag{5}$$

where $G(x)$ is the cdf of the base distribution. If $\lambda = 0$, we have the base distribution of *X*.

The transmuted map method is suggested by researchers in the literature, see as an example transmuted additive Weibull distribution which is suggested by Elbatal and Aryal (2003). Elgarhy et al. (2016) suggested transmuted generalized Lindley distribution. Khan and King (2015) proposed transmuted modified inverse Rayleigh distribution. Aryal and Tsokos (2009) suggested transmuted extreme value distribution with an application to climate data. Merovci et al. (2014) proposed the transmuted generalized inverse Weibull distribution. Al-Omari (2017) suggested Transmuted Janardan distribution.

The rest of this paper is organized as follows. In Section 2 we introduced the transmuted NWP distribution. The reliability analysis and hazard rate function are presented in Section 3. A summary of the distributions of order statistics is given in Section 4. In Section 5 the moment generation function is derived. In Section 6 the maximum likelihood estimates of the suggested TNWP parameters are demonstrated. The Rényi entropy is given in Section 7. Finally, the conclusions are given in Section 8.

2. TRANSMUTED NEW WEIBULL PARETO DISTRIBUTION

The quadratic transmuted map method is used to generalize new Weibull Pareto distribution namely; transmuted new Weibull Pareto (TNWP) distribution. The cdf of the TNWP distribution is

$$G(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right), \ x>0, \ \beta>0, \delta>0, \ \theta>0, \tag{5}$$

Deriving Equation (5) with respect to x will give the pdf of the TNWP distribution as

$$g(x) = \frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]. \tag{6}$$

3. RELIABILITY ANALYSIS

The TNWP distribution can be applied to real data. The reliability and hazard rate functions of the NWP distribution denoted by $R_{TNWP}(x)$ and $H_{TNWP}(x)$, respectively, are defined in the following theorem:

Theorem 1: The reliability and hazard rate functions of the TNWP distribution random variable are, respectively

$$R_{TNWP}(x) = 1 - G_{TNWP}(x) = e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}, \tag{7}$$

and

$$H_{TNWP}(x) = \frac{g_{TNWP}(x)}{1-G_{TNWP}(x)} = \frac{\delta\beta x^{\beta-1}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)}{\theta^{\beta}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)}. \tag{8}$$

4. DISTRIBUTION OF ORDER STATISTICS

Order statistics are essential in many areas of practice and statistical theory. Let $X_1, X_2, ..., X_n$ be a random sample of size n from the pdf $g(x)$ and cdf $G(x)$ defined in (6) and (7), and $X_{[1]}, X_{[2]}, ..., X_{[n]}$ be its order statistics. The pdf of the ith order statistics is defined as

$$g_{(j)}(x) = \frac{n!}{(j-1)!(n-j)!} g(x)[G(x)]^{j-1}[1-G(x)]^{n-j}, \quad i = 1, 2, ..., n. \tag{9}$$

Let, $X_{[1]} = \min\{X_1, X_2, ..., X_n\}$, and $X_{[n]} = \max\{X_1, X_2, ..., X_n\}$ with pdfs $f_{(1)}(x)$ and $f_{(n)}(x)$ defined as follow

$$f_{(1)}(x) = nf(x)[1-F(x)]^{n-1} \text{ and } f_{(n)}(x) = nf(x)[F(x)]^{n-1}.$$

Also, for $0 < x < 1$ $\beta > 0, \delta > 0, \theta > 0.$ As a special case of (10), the pdf of the minimum and maximum order statistics, respectively, are

$$f_{(1)}(x) = n\left[\frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\right]\left(e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{n-1} \tag{10}$$

and

$$f_{(n)}(x) = n\frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\right]^{n-1} \tag{11}$$

Moreover, the *pdf* of the j th order statistic is given as

$$g_{(j)}(x) = \frac{n!}{(j-1)!(n-j)!}\left(\frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]\right)\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{j-1} \tag{12}$$

$$\times\left[e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{n-j}$$

5. THE MOMENTS OF THE TNWP RANDOM VARIABLE

In this section, we will derive the r^{th} r^th moment, the mean and the variance of the TNWP random variable.

Theorem 2: If *X* has a TNWP distribution, then the r^{th} moment is defined as

$$E(X^r) = \left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r \Gamma\left(\frac{r}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^r\right] \tag{14}$$

Proof:

Assume that *X* is a random variable follows the TNWP distribution, then the r^{th} moment for this random variable is defined as:

$$E(X^r) = \int_0^{\infty} x^r \left[\frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \frac{\lambda\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \frac{2\lambda\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right] dx$$

$$= \int_0^{\infty} \frac{\delta\beta}{\theta^{\beta}} x^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} dx - \int_0^{\infty} \frac{\lambda\delta\beta}{\theta^{\beta}} x^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} dx + \int_0^{\infty} \frac{2\lambda\delta\beta}{\theta^{\beta}} x^{r+\beta-1} e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}} dx$$

The first term of $\int_0^\infty \frac{\delta\beta}{\theta^\beta} x^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx$ can be calculated as

$$u=\delta\left(\frac{x}{\theta}\right)^\beta=\frac{\delta}{\theta^\beta}x^\beta;\ x=\frac{\theta}{\delta^{\frac{1}{\beta}}}u^{\frac{1}{\beta}};\ du=\frac{\delta\beta}{\theta^\beta}x^{\beta-1}dx,\ \text{then}\ dx=\frac{du}{\frac{\delta\beta}{\theta^\beta}x^{\beta-1}}$$

Then

$$\int_0^\infty \frac{\delta\beta}{\theta^\beta} x^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx=\int_0^\infty \frac{\delta\beta}{\theta^\beta}\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}u^{\frac{1}{\beta}}\right)^{r+\beta-1} e^{-u}\left[\frac{\delta\beta}{\theta^\beta}\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}u^{\frac{1}{\beta}}\right)^{\beta-1}\right]^{-1} du=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\int_0^\infty u^{\frac{r}{\beta}}e^{-u}du=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)$$

$$\int_0^\infty \frac{\delta\beta}{\theta^\beta} x^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\int_0^\infty u^{\frac{r}{\beta}}e^{-u}du=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right),$$

Similarly, the other two terms can be done as:

$$-\int_0^\infty \frac{\lambda\delta\beta}{\theta^\beta} x^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx=-\lambda\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\int_0^\infty u^{\frac{r}{\beta}}e^{-u}du=-\lambda\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right),$$

and

$$\int_0^\infty \frac{2\lambda\delta\beta}{\theta^\beta} x^{r+\beta-1} e^{-2\delta\left(\frac{x}{\theta}\right)^\beta} dx=2\lambda\left(\frac{\theta}{(2\delta)^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right).$$

Therefore,

$$E(X^r)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)-\lambda\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)-2\lambda\left(\frac{\theta}{(2\delta)^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)$$

$$=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^r\right].$$

Remark: Let *X* has a TNWPD, then as special cases of the r^{th} moment the first and second moments are defined as

$$E(X)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)\Gamma\left(\frac{1}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)\right]\ \text{and}\ E(X^2)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^2\Gamma\left(\frac{2}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^2\right].$$

Therefore, the variance of *X* is

$$\sigma^2_{TNWP}=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^2\left\{\Gamma\left(\frac{2}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^2\right]-\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)\right]^2\right\}$$

The third and fourth moments are

$$E(X^3)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^3\Gamma\left(\frac{3}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^3\right],$$

and

$$E(X^4)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^4\Gamma\left(\frac{4}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^4\right].$$

Therefore, the skewness and kurtosis of a random variable, respectively, are defined as

$$Sk=\frac{E(X^3)-3\mu\sigma^2-\mu^3}{\sigma^3}\text{ and }Ku=\frac{E(X^4)-4\mu E(X^3)+6E(X^2)\sigma^2+3E(X^4)}{\sigma^8}.$$

$$CV=\frac{\frac{(2\delta)^{\frac{1}{\beta}}}{\theta}\sqrt{\begin{matrix}-2^{-2/\beta}\left[\delta\left(\frac{1}{\theta}\right)^{\beta}\right]^{-2/\beta}\left[-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right]^2\Gamma\left[1+\frac{1}{\beta}\right]^2\\+4^{-1/\beta}\left[\delta\left(\frac{1}{\theta}\right)^{\beta}\right]^{-2/\beta}\left[-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right]\Gamma\left[\frac{2+\beta}{\beta}\right]\end{matrix}}}{\left[-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right]\Gamma\left[1+\frac{1}{\beta}\right]},$$

$$Sk=e^{-0.69/\beta}\left[\delta\theta^{-\beta}\right]^{-\frac{1}{\beta}}\frac{\left\{\begin{matrix}-\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^3\Gamma\left[1+\frac{1}{\beta}\right]^3-3(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda)\Gamma\left[1+\frac{1}{\beta}\right]\\\left[\begin{matrix}-1\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^2\Gamma\left[1+\frac{1}{\beta}\right]^2+\left(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{2+\beta}{\beta}\right]\\+\left(-8^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{3+\beta}{\beta}\right]\end{matrix}\right]\end{matrix}\right\}}{\left\{-1\left[\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^2\Gamma\left[1+\frac{1}{\beta}\right]^2+\left(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{2+\beta}{\beta}\right]\right]\right\}^{3/2}}$$

,

$$Ku=\frac{e^{2.8/\beta}\left(\delta(\theta)^{\beta}\right)^{4/\beta}\left\{\begin{matrix}3\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^4\Gamma[1+\frac{1}{\beta}]^4+\left(-16^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{4+\beta}{\beta}\right]\\+6\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^2\Gamma\left[1+\frac{1}{\beta}\right]^2\left\{\begin{matrix}-\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^2\Gamma\left[1+\frac{1}{\beta}\right]^2\\+\left(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{2+\beta}{\beta}\right]\end{matrix}\right\}\\-4\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\left(-8^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[1+\frac{1}{\beta}\right]\Gamma\left[\frac{3+\beta}{\beta}\right]\end{matrix}\right\}}{\left(-(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda)^2 Gamma\left[1+\frac{1}{\beta}\right]^2+(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda)Gamma\left[\frac{2+\beta}{\beta}\right]\right)^4}$$

6. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimates (MLEs) of the parameters of the *TNWPD* are derived from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size *n* from the *TNWPD* with parameters σ, β, θ and λ, and a *pdf* of *TNWPD*, then, the likelihood function is given by
then, the likelihood function is given by

$$Lg(x;\delta,\beta,\theta,\lambda)=\prod_{i=1}^{n}\left(\frac{\delta\beta}{\theta^{\beta}}x_i^{\beta-1}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}-\frac{\lambda\delta\beta}{\theta^{\beta}}x^{\beta-1}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}+\frac{2\lambda\delta\beta}{\theta^{\beta}}x_i^{\beta-1}e^{-2\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right)$$

$$=\prod_{i=1}^{n}\frac{\delta\beta}{\theta^{\beta}}x_i^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]$$

The log-likelihood function is

$$\ln Lg\,(x,x,...,x;d,b,q,/)=\ln\left\{\prod_{i=1}^{n}\frac{db}{q^{b}}x_i^{b-1}e^{-d\left(\frac{x_i}{q}\right)^{b}}\left[1-/+2/e^{-d\left(\frac{x_i}{q}\right)^{b}}\right]\right\}$$

$$=\ln\left\{\left(\frac{db}{q^{b}}\right)^{n}\prod_{i=1}^{n}x_i^{b-1}e^{-d\sum_{i=1}^{n}\left(\frac{x_i}{q}\right)^{b}}\prod_{i=1}^{n}\left[1-/+2/e^{-d\left(\frac{x_i}{q}\right)^{b}}\right]\right\}$$

$$\ln Lg=\ln\left(\frac{\delta\beta}{\theta^{\beta}}\right)^{n}+\ln\prod_{i=1}^{n}x_i^{\beta-1}+\ln e^{-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}}+\ln\prod_{i=1}^{n}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]$$

$$=\ln\left(\frac{\delta\beta}{\theta^{\beta}}\right)^{n}+\sum_{i=1}^{n}\ln\left(x_i^{\beta-1}\right)-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}+\sum_{i=1}^{n}\ln\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]$$

$$=n\ln\delta+n\ln\beta-n\beta\ln\theta+(\beta-1)\sum_{i=1}^{n}x_i-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}+\sum_{i=1}^{n}\ln\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]$$

$$\frac{\partial\ln Lg}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}x_i-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)-\delta\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}$$

Setting this derivative to zero, we get

$$\frac{n}{\beta}=\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)-\sum_{i=1}^{n}x_i+\delta\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}$$

$$\frac{\partial\ln Lg}{\partial\theta}=\frac{n\beta}{\theta}-\delta\beta\sum_{i=0}^{n}\frac{x_i^{\beta}}{\theta^{\beta+1}}-\sum_{i=0}^{n}\frac{2\lambda\beta\delta x_i^{\beta}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right)\theta^{\beta+1}}$$

Equating this derivative to zero, we get

$$n\theta^{\beta}=\delta\sum_{i=0}^{n}x_i^{\beta}+\sum_{i=0}^{n}\frac{2\lambda\delta x_i^{\beta}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}$$

$$\frac{\partial\ln Lg}{\partial\delta}=\frac{n}{\delta}-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}-\sum_{i=1}^{n}\left[\frac{2\lambda\left(\frac{x_i}{\theta}\right)^{\beta}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\right]$$

Setting this derivative to zero, we get

$$\frac{n}{\delta}=\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\beta}+\sum_{i=1}^{n}\left[\frac{2\lambda\left(\frac{x_i}{\theta}\right)^{\beta}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\right]$$

$\frac{\partial \ln Lg}{\partial\lambda}=\sum_{i=0}^{n}\frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}-1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}$.

Equating to zero, we have

$\sum_{i=0}^{n}\frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}-1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}=0$

Therefore, the maximum likelihood estimates of the distribution parameters, β,θ,δ and λ are the solution of the following system of equations:

$$\begin{cases}\frac{n}{\beta}=\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)-\sum_{i=1}^{n}x_i+\delta\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\\ n\theta^{\beta}=\delta\sum_{i=0}^{n}x_i^{\beta}+\sum_{i=0}^{n}\frac{2\lambda\delta x_i^{\beta}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\\ \frac{n}{\delta}=\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\beta}+\sum_{i=1}^{n}\left[\frac{2\lambda\left(\frac{x_i}{\theta}\right)^{\beta}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\right]\\ \sum_{i=0}^{n}\frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}-1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\end{cases}$$

Which does not have a close form, hence; it must be solved numerically.

7. THE RÉNYI ENTROPY

The entropy of a random variable *X* with density (x) is a measure of variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. Rényi entropy is defined as

$$Entropy=E_R(\rho)=\frac{1}{1-\rho}\log\int_0^{\infty}(f(x))^{\rho}dx \tag{24}$$

where $\rho>0$ and $\rho\neq 1$. The Rényi entropy for the TNWP variable *X* with probability density (x) is given by the following theorem.

Theorem 4: If a random variable *X* has the *TNWPD* distribution, then the Rényi entropy of *X*, is given by

$$E_R(\rho)=\frac{1}{1-\rho}\left[\left\{\log\left(\frac{\delta\beta(\lambda-1)}{\theta^{\beta}}\right)^{\rho}+\log\Gamma\left(\frac{\rho(\beta-1)+1}{\beta}\right)+\log\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^{i}\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\right\}\right]$$

Proof:

$$E_R(\rho)=\frac{1}{1-\rho}\log\int_0^{\infty}[f(x)]^{\rho}dx$$

$$=\frac{1}{1-\rho}\log\int_0^{\infty}\left[\frac{\delta\beta(\lambda-1)}{\theta^{\beta}}x^{\beta-1}e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[\frac{2\lambda}{\lambda-1}-e^{\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]\right]^{\rho}dx$$

$$=\frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^{\beta}}\right)^{\rho}\int_0^{\infty}x^{\rho(\beta-1)}\,e^{-2\rho\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[\frac{2\lambda}{\lambda-1}-e^{\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\rho}dx\right\}$$

$$\left[\frac{2\lambda}{(\lambda-1)}-e^{\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\rho}=\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}\left(-e^{\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{i}$$

$$E_R(\rho)=\frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho \int_0^\infty x^{\rho(\beta-1)}\ e^{-2\rho\delta\left(\frac{x}{\theta}\right)^\beta}\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}\left(-e^{\delta\left(\frac{x}{\theta}\right)^\beta}\right)^i dx\right\}$$

$$=\frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho \sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^i\int_0^\infty x^{\rho(\beta-1)}\ e^{-\left(\frac{x}{\theta}\right)^\beta(2\rho\delta-i\delta)}dx\right\}.$$

$$\int_0^\infty x^{\rho(\beta-1)}\ e^{-\left(\frac{x^\beta}{\theta^\beta}\right)(2\rho\delta-i\delta)}dx=\int_0^\infty x^{\rho(\beta-1)}\ e^{-x^\beta\left(\frac{2\rho\delta-i\delta}{\theta^\beta}\right)}dx$$

$$let\quad u=x^\beta\left(\frac{2\rho\delta-i\delta}{\theta^\beta}\right)\quad;\ \frac{u}{\frac{2\rho\delta-i\delta}{\theta^\beta}}=x^\beta;\ \ \frac{\theta^\beta u}{2\rho\delta-i\delta}=x^\beta;x=\sqrt[\beta]{\frac{\theta^\beta u}{2\rho\delta-i\delta}}=\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}u^{\frac{1}{\beta}}$$

$$dx=\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}u^{\frac{1}{\beta}-1}du$$

$$\int_0^\infty x^{\rho(\beta-1)}\ e^{-x^\beta\left(\frac{2\rho\delta-i\delta}{\theta^\beta}\right)}dx=\int_0^\infty\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}u^{\frac{1}{\beta}}\right)^{\rho(\beta-1)}\ e^{-u}\ \frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}u^{\frac{1}{\beta}-1}du$$

$$\int_0^\infty x^{\rho(\beta-1)}\ e^{-x^\beta\left(\frac{2\rho\delta-i\delta}{\theta^\beta}\right)}dx=\int_0^\infty\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}u^{\frac{1}{\beta}}\right)^{\rho(\beta-1)}\ e^{-u}\ \frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}u^{\frac{1}{\beta}-1}du$$

$$=\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\int_0^\infty u^{\frac{\rho(\beta-1)+1-\beta}{\beta}}e^{-u}du$$

$$=\frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^i\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\Gamma\left(\frac{\rho(\beta-1)+1-\beta}{\beta}+1\right)\right\}$$

$$=\frac{1}{1-\rho}\left[\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho\Gamma\left(\frac{\rho(\beta-1)+1}{\beta}\right)\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^i\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\right\}\right]$$

$$=\frac{1}{1-\rho}\left[\left\{\log\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho+\log\Gamma\left(\frac{\rho(\beta-1)+1}{\beta}\right)+\log\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^i\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\right\}\right]$$

8. CONCLUSIONS

We proposed new distribution, named the transmuted new Weibull Pareto distribution, which is an extension of the new Weibull Pareto distribution. The parameter λ gives more flexibility in modeling reliability data. The moments, the rth moment, and the order statistics function of the TNWP distribution are derived. Also, we define its hazard rate and reliability functions as well as Rényi entropy.

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